1. The CNOT gate.

(a) We find straightforwardly (from implementing the CNOT truth table)

$$U_{\text{CNOT}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

The "inverted" gate follows from implementing

$$\begin{aligned} |00\rangle &\to |00\rangle\,, \\ |01\rangle &\to |11\rangle\,, \\ |10\rangle &\to |10\rangle\,, \\ |11\rangle &\to |01\rangle\,, \end{aligned}$$

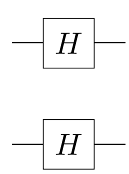
and thus reads

$$\tilde{U}_{\text{CNOT}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

(b) As a first step we explicitly write the Hadamard gate,

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

The operation



on the two qubits can then be represented by the 4×4 matrix

$$H_1 \otimes H_2 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

We now simply evaluate the matrix product $(H_1 \otimes H_2) U_{\text{CNOT}} (H_1 \otimes H_2)$ and find that it is indeed equal to $\tilde{U}_{\text{CNOT}}$.

2. Scattering matrix and transfer matrix.

(a) From writing out $\hat{s}\hat{s}^\dagger = 1$ explicitly, we get

$$\begin{aligned} rr^\dagger + t'(t')^\dagger &= 1, \\ rt^\dagger + t'(r')^\dagger &= 0, \\ tr^\dagger + r'(t')^\dagger &= 0, \\ tt^\dagger + r'(r')^\dagger &= 1, \end{aligned}$$

where we see that the second and third equations are actually the same.

From the last equation we see that

$$tt^\dagger = 1 - r'(r')^\dagger.$$

We now insert 1 on the right,

$$tt^\dagger = 1 - r'(t')^{-1}t'(r')^\dagger,$$

and then use the second equation to write

$$tt^\dagger = 1 + r'(t')^{-1}rt^\dagger.$$

Multiplying from the right with $(t^\dagger)^{-1}$ gives the result asked for.

(b) Straightforward solving of the four equations and using the result found at (a) yields

$$\hat{m} = \begin{pmatrix} (t^\dagger)^{-1} & r'(t')^{-1} \\ -(t')^{-1}r & (t')^{-1} \end{pmatrix}.$$

3. Transmission and Landauer-Büttiker conductance.

(a) The stationary Schrödinger equation is

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_0\delta(x)\right]\psi(x) = \epsilon\psi(x)\,. \tag{1}$$

Let us first consider an incoming electron from the left. The wave function is

$$\psi(x) = \begin{cases} \frac{1}{\sqrt{2\pi\hbar v}}\left[e^{ikx} + r\,e^{-ikx}\right], & \text{for } x < 0, \\ \frac{1}{\sqrt{2\pi\hbar v}}t\,e^{ikx}, & \text{for } x > 0, \end{cases}$$

where $v = \sqrt{2\epsilon/m}$ is the electron's velocity.

Continuity of the wave function at $x = 0$ gives

$$\psi(0^+) \quad = \quad \psi(0^-)$$

$$1 + r = t\,.$$

By integrating the Schrödinger equation (1) across the scatterer located at $x = 0$, we find

$$-\frac{\hbar^2}{2m}\left[\left(\frac{d\psi}{dx}\right)_{x=0^+} - \left(\frac{d\psi}{dx}\right)_{x=0^-}\right] + V_0\,\psi(0) = 0,$$
$$-\frac{\hbar}{2}iv\,[t-(1-r)] + V_0\,t = 0.$$

From these two equations we find that

$$t = \frac{i\hbar v}{i\hbar v - V_0}, \tag{2}$$

and

$$r = \frac{V_0}{i\hbar v - V_0}. \tag{3}$$

Furthermore, since the system is mirror symmetric around $x = 0$, the reflection and transmission amplitudes associated with an incoming electron from the right are identical to the ones for an incoming electron from the left, $r = r'$ and $t = t'$.

(b) We use that the transfer matrix reads in terms of the elements of the scattering matrix

$$\hat{m} = \begin{pmatrix} 1/t^* & r'/t' \\ -r/t' & 1/t' \end{pmatrix},$$

see the previous problem *Scattering matrix and transfer matrix*. This yields straightforwardly for the two transfer matrices associated to the potential barriers

$$\hat{m}_{1,3} = \begin{pmatrix} 1+\alpha & \alpha \\ -\alpha & 1-\alpha \end{pmatrix},$$

where

$$\alpha = \frac{V_0}{i\hbar v}.$$

For the middle region we find

$$\hat{m}_2 = \begin{pmatrix} e^{ika} & 0 \\ 0 & e^{-ika} \end{pmatrix}.$$

The top-left element of the total transfer matrix, $\hat{m}_{\rm tot} = \hat{m}_1\hat{m}_2\hat{m}_3$, equals $1/t^*$ of the total scattering matrix. The matrices are easily multiplied, yielding

$$t_{\rm tot} = \frac{t^2 e^{ika}}{1 - r^2 e^{2ika}},$$

in terms of the single-barrier t and r found at (a).

Alternative:

We first consider the total transmission amplitude t_{12} through two scatterers connected in series. We sum over all possible ways to get through the total scattering region from the left to the right,

$$\begin{aligned} t_{12} &= t_1 t_2 + t_1 r_2 r_1' t_2 + t_1 r_2 r_1' r_2 r_1' t_2 + \dots \\ &= t_1 [1 - r_2 r_1']^{-1} t_2, \end{aligned}$$

where we have used that $(1-x)^{-1} = \sum_{i=0}^{\infty} x^i$.

Similarly, we can find the reflection amplitude for an imcoming electron from the left

$$\begin{aligned} r_{12} &= r_1 + t_1 r_2 t_1' + t_1 r_2 r_1' r_2 t_1' + \dots \\ &= r_1 + t_1 r_2 [1 - r_2 r_1']^{-1} t_1'. \end{aligned}$$

In the same way, we can find the reflection and transmission amplitudes associated with an incoming electron from the right (r_{12}' and t_{12}') by interchanging $r_1 \leftrightarrow r_2'$, $r_2 \leftrightarrow r_1'$, $t_1 \leftrightarrow t_2'$, and $t_2 \leftrightarrow t_1'$ with the result:

$$\begin{aligned} t_{12}' &= t_2' [1 - r_1' r_2]^{-1} t_1', \\ r_{12}' &= r_2' + t_2' r_1' [1 - r_1' r_2]^{-1} t_2. \end{aligned}$$

Let us first concatenate the scattering matrices S_1 and S_2. Since $r_2 = r_2' = 0$, we find

$$\begin{aligned} t_{12} &= t e^{ika}, \\ r_{12} &= r, \\ t_{12}' &= t e^{ika}, \\ r_{12}' &= r e^{2ika}, \end{aligned}$$

where the transmission t and reflection amplitudes r are the ones found at (a).

Next, we concatenate the combined scattering matrix S_{12} with scattering matrix S_3 to find the total transmission amplitude t_{tot},

$$\begin{aligned} t_{\text{tot}} &= t_{12} [1 - r_3 r_{12}']^{-1} t_3 \\ &= \frac{t^2 e^{ika}}{1 - r^2 e^{2ika}}. \end{aligned}$$

(c) Writing $r = \sqrt{R} \exp i\theta$, with $\theta = \arctan \hbar v / V_0$, we find the transmission probability

$$\begin{aligned} T_{\text{tot}} &= |t_{\text{tot}}|^2 \\ &= \frac{|t^2 e^{ika}|^2}{[1 - R e^{2ika} e^{2i\theta}][1 - R e^{-2ika} e^{-2i\theta}]} \\ &= \frac{T^2}{1 - 2R \cos \xi + R^2}, \end{aligned}$$

with $\xi = 2ka + 2\theta$.

(d) The maximum T_{tot} is achieved when the denominator attains its minimum, which is when $\xi = 0$. In this case, we have

$$T_{\text{tot}} = \frac{T^2}{1 - 2R + R^2} = \frac{T^2}{(1-R)^2} = 1.$$

Despite the fact that the transparancy of each scatterer is low, the total transmission probability $T_{\text{tot}} = 1$, with an associated Landauer-Büttiker conductance that equals the conductance quantum $G = 2e^2/h$.

This is resonant tunneling and occors when the energy of the particle equals the resonant state between the scatterers. For instance, in the limit of an infinite strength of the scatterers $V_0 \to \infty$, the condition $\xi = 0$ implies that

$$ka = n\pi,$$

so that the wave function is a standing wave between the scatterers. When the energy of the particle $\epsilon = \hbar^2 k^2/2m$ is in resonance with one of the bound states between the two scatterers

$$\epsilon_n = \frac{\hbar^2}{2m}\frac{n^2\pi^2}{a^2}\,,$$

the transmission probability becomes equal to one.

4. Quantum Hall Effect.

(a) We have

$$\begin{aligned} T_{21} &= N, \\ T_{12} &= N - n, \\ T_{42} &= n, \\ T_{14} &= n - 1, \\ T_{64} = T_{16} &= 1. \end{aligned}$$

This yields the matrix

$$G = \begin{pmatrix} N & -N+n & -n+1 & -1 \\ -N & N & 0 & 0 \\ 0 & -n & n & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

(b) We set $V_1 = 0$ and disregard the equation for i_1 (we can use that $i_1 \equiv i = -i_4$). This yields the equations

$$\begin{aligned} i_2 &= NV_2, \\ i_4 &= n(V_4 - V_2), \end{aligned}$$

$$i_6 = V_6 - V_4.$$

We then make use of the fact that terminal 2 and 6 are voltage probes, causing $i_2 = i_6 = 0$, and we easily find

$$\begin{aligned} V_2 &= 0, \\ V_4 &= -i/n, \\ V_6 &= V_4 = -i/n. \end{aligned}$$

The Hall resistance thus reads

$$R_{\mathrm{GH}} = \frac{V_2 - V_6}{I} = \frac{h}{2e^2}\frac{1}{n}.$$