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Exam TFY4345: Classical Mechanics
Tuesday June 7th 2011
09.00-13.00
English

The exam consists of 4 problems. Each problem counts for in total 25% of the total weight of the exam, but each sub-exercise **(a)**, **(b)**, etc. does not necessarily count equally.

Read each problem carefully in order to avoid unnecessary mistakes.

Allowed material to use at exam: C.

- Approved, simple calculator.
- K. Rottmann: Matematisk formelsamling.
- K. Rottmann: Mathematische Formelsammlung. Barnett & Cronin: Mathematical Formulae.

Also consider the Supplementary Material on the last page of this exam.

PROBLEM 1

A massive particle moves along the z-axis and is subject to a potential $V(z) = -Fz$, where F is a constant.

(a) Describe the force acting on this particle and the resulting motion. You may assume that the particle is initially at rest.

The particle is now seen to move from $z(t=0) = 0$ to $z(t=t_0) = a$, in effect covering a distance a in a time interval t_0. Assume that the motion of the particle can be parametrized as $z(t) = \mathcal{A} + \mathcal{B}t + \mathcal{C}t^2$.

(b) Identify the values of $\mathcal{A}, \mathcal{B}, \mathcal{C}$ such that the action of the particle is a minimum.

Consider now a different problem. It is known that if a Lagrangian can be written like $L = L(q, \dot{q}, t)$, the Lagrange-equation may be derived by Hamilton's principle $\delta I = 0$, where I is the action of the system given by:

$$I = \int_{t_1}^{t_2} L(q, \dot{q}, t)\mathrm{d}t. \tag{1}$$

From this equation, one obtains:

$$\delta I = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q}\right]\mathrm{d}t. \tag{2}$$

To finally derive the Lagrange-equation, one makes use of the fact that δq has no variation at the end-points t_1 and t_2, and that any variations are performed at a fixed time t such that $\delta t = 0$.

(c) Consider now a Lagrangian which can be written like $L = L(q, \dot{q}, \ddot{q}, t)$. Derive in detail, by means of Hamilton's principle, that the resulting equation reads:

$$\frac{\mathrm{d}^{j+1}}{\mathrm{d}t^{j+1}}\left(\frac{\partial L}{\partial \ddot{q}}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{\partial^n L}{\partial q^n} = 0. \tag{3}$$

Identify the value of the integer quantities j and n. Use that $\delta \dot{q} = 0$ at the end-points t_1 and t_2.

(d) Apply this result to the Lagrangian $L = -mq\ddot{q}/2 - kq^2/2$ and identify what type of system the equation of motion describes.

Consider now a different problem.

(e) Explain in detail the relation between symmetries of the Lagrangian describing a given system and the possibility of having conserved quantities. What can you immediately conclude about the momentum and energy of a particle described by the following Lagrangian: $L = m(\mathrm{d}\mathbf{r}/\mathrm{d}t)^2/2 + E_0 \sin(r/r_0)$? Here, $r = |\mathbf{r}|$ is the norm of the position vector while E_0 and r_0 are normalization constants.

PROBLEM 2

(a) A massive particle moves in a 2D plane and is subject to a force $F(r) = -(k/r^2)$ where k is a real and positive constant. Derive the resulting equations of motion and reduce them to an equivalent one-dimensional problem with an effective potential that depends on the angular momentum of the particle. Moreover, use this effective potential to discuss qualitatively the resulting motion of the particle in this force field and how the motion depends on the energy E and the angular momentum l.

(b) Define in words, and in detail, what the differential scattering cross section gives information about physically. Also define in words the difference between the differential scattering cross section and the total scattering cross section.

(c) Consider the total scattering cross section of a particle moving in a Coulomb potential: $\sigma \to \infty$. What does this result mean physically?

(d) Consider a particle incident towards a hard sphere, such that $V(r) \to \infty$ for $r \leq a$ whereas $V(r) = 0$ for $r > a$. What is the total scattering cross section for this process? Explain how you arrived at this result.

Consider now a different problem. Imagine that you have a physical system which is described by a set of generalized coordinates q_i, $i = 1, 2, 3, \ldots$.

(e) What is the condition on the generalized forces $Q_i = -\frac{\partial V}{\partial q_i}$ in order for the system to be in an equilibrium state? Imagine now that you slightly perturb the system from its equilibrium position. What kind of motion will the system undergo?

PROBLEM 3

Consider the collision of two particles with masses m_1 and m_2. Assume that this collision produces n new particles with masses that may be different from both m_1 and m_2.

(a) Define in words the concept of "threshold energy" in the context of particle collisions.

(b) Provide a detailed explanation in words for why the minimum threshold energy is obtained in the center-of-mass (COM) system of the original particles m_1 and m_2.

(c) Prove by means of explicit analytical calculations that the threshold energy in the COM system is always *smaller* than the threshold energy in a reference system where one of the particles m_1 and m_2 is initially at rest (for instance, $\mathbf{p}_1 \neq 0$ and $\mathbf{p_2} = 0$). You may assume that the total rest mass M_{tot} of the n new particles is greater or equal to the mass of the original particles m_1 and m_2. In effect, you may assume that:

$$M_{\text{tot}} = k(m_1 + m_2) \text{ where } k \geq 1. \tag{4}$$

Hint: You might find the following identity useful

$$k^2 - 2k + 1 \geq 0. \tag{5}$$

PROBLEM 4

Consider the three situations (A), (B), and (C) sketched in the figure below. In situation (A), a capacitor consisting of two conducting plates with charge Q and $-Q$ is at rest in the observer's frame of reference $\mathcal{S}$. The plates have an area A and a negligible thickness. They are separated by a distance d. In situations (B) and (C), the capacitor moves with a constant velocity v along a specific direction (indicated in the figure) relative the observer in $\mathcal{S}$. The velocity is assumed to be comparable to the speed of light c in magnitude.

(a) In situation (A), the electric field observed between the two conducting plates is constant and equal to $E = Q/(\varepsilon_0 A)$

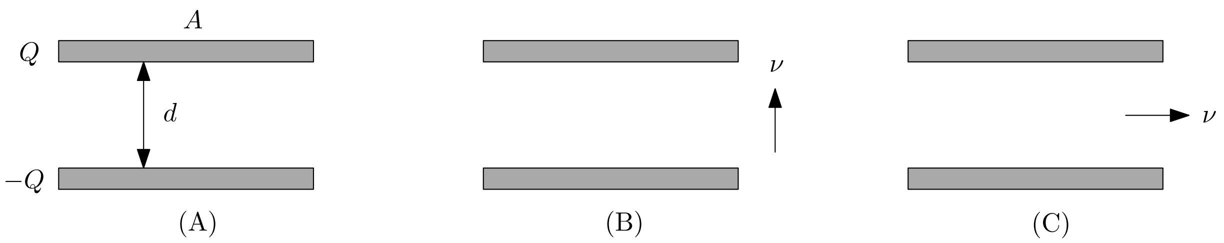

FIG. 1: (Color online). The system under consideration.

in magnitude whereas the magnetic field is zero ($B = 0$). Here, ε_0 is the vacuum permittivity constant. Using a Lorentz transformation, derive analytically in detail the electric field E and magnetic field B observed in $\mathcal{S}$ for the scenarios (B) and (C). In both these cases, find the magnitude and direction of the fields.

(b) Explain how you could have found the result for the electric field E in situation (B) and (C) simply by properly accounting for Lorentz contraction in the equation $E = Q/(\varepsilon_0 A)$ valid for scenario (A).

Consider now a different problem. The definition of Poisson-brackets is given in the Supplementary Information to this exam.

(c) Show by an explicit calculation that a quantity F is conserved, i.e. $\mathrm{d}F/\mathrm{d}t = 0$, if the following two criteria are satisfied: 1) F has no explicit time-dependence and 2) $[F, H]_{q,p} = 0$ where $\{q, p\}$ are canonical variables and H is the Hamiltonian of the system.

Supplementary Information

The regime of validity and the meaning of the symbols below are assumed to be known by the reader.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}. \tag{6}$$

$$[u,v]_{q,p} = \sum_{i=1}^{n}\left(\frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}\right) \tag{7}$$

$$\begin{aligned} x_\mu &= (\mathbf{r}, \mathrm{i}ct), \\ p_\mu &= (\mathbf{p}, \mathrm{i}E/c) \end{aligned} \tag{8}$$

$$A_\mu = (\mathbf{A}, \mathrm{i}\phi/c),\ \mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t,\ \mathbf{B} = \nabla\times\mathbf{A} \tag{9}$$

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \tag{10}$$

From the above equations, it follows that the general form of $F_{\mu\nu}$ in a given reference system is:

$$F_{\mu\nu} = \begin{pmatrix} 0 & B_z & -B_y & -\mathrm{i}E_x/c \\ -B_z & 0 & B_x & -\mathrm{i}E_y/c \\ B_y & -B_x & 0 & -\mathrm{i}E_z/c \\ \mathrm{i}E_x/c & \mathrm{i}E_y/c & \mathrm{i}E_z/c & 0 \end{pmatrix} \tag{11}$$

$$F'_{\mu\nu} = L_{\mu\alpha}L_{\nu\beta}F_{\alpha\beta}. \tag{12}$$

The Lorentz-transformation matrix for the situation in Fig. 2 is given by:

$$L_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & \mathrm{i}\beta\gamma \\ 0 & 0 & -\mathrm{i}\beta\gamma & \gamma \end{pmatrix} \tag{13}$$

where $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$.

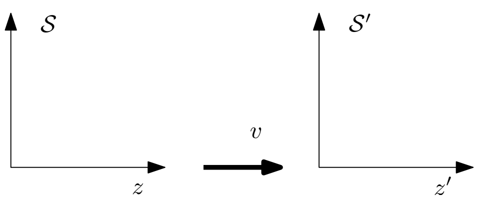

FIG. 2: Lorentz-transformation along the z-axis.