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Continuation Exam TFY4345: Classical Mechanics
Thursday August 11th 2011
09.00-13.00
English

The exam consists of 4 problems. Each problem counts for in total 25% of the total weight of the exam, but each sub-exercise **(a)**, **(b)**, etc. does not necessarily count equally.

Read each problem carefully in order to avoid unnecessary mistakes.

Allowed material to use at exam: C.

- Approved, simple calculator.
- K. Rottmann: Matematisk formelsamling.
- K. Rottmann: Mathematische Formelsammlung. Barnett & Cronin: Mathematical Formulae.

Also consider the Supplementary Material on the last page of this exam.

PROBLEM 1

The brachistochrone problem consists of finding the curve between two points such that the time required for a particle to move between them becomes minimal.

(a) Solve the brachistochrone problem where the coordinate axes are laid as in Fig. 1. The particle starts from the origin, at rest, when $t = 0$. Find a closed analytical form for the coordinates x and y.

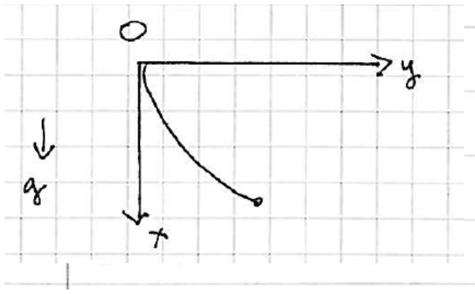

FIG. 1: (Color online). The system under consideration in a).

(b) Assume that the initial velocity is now $\mathbf{v_0}$, making an angle $\pi/4$ with the y-axis at $t = 0$. Show that the brachistochrone curve is determined from the equation

$$[y'(x)]^2 = f(v_0, g, x) \tag{1}$$

and identify the function $f(v_0, g, x)$ where $v_0 = |\mathbf{v_0}|$.

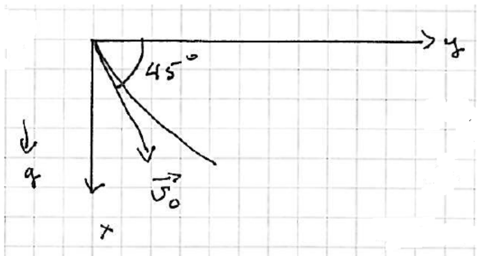

FIG. 2: (Color online). The system under consideration in b).

Consider now a different problem.

(c) Consider a rigid body rotating around a fixed point of the body. Find an expression for the time variation of the kinetic energy T as a function of the angular velocity vector ω and the total torque vector τ acting on the body. In effect, find dT/dt as a function of ω and τ.

PROBLEM 2

(a) The CO_2 molecule is a linear molecule with two oxygen atoms at each side of a carbon atom, with respective masses m and M. The deviation from the equilibrium positions of each atom can be written as $\eta_i = \sum_\alpha \Delta_{i\alpha}\Theta_\alpha$, where $\theta_\alpha = \mathrm{Re}\{C_\alpha e^{-i\omega_\alpha t}\}$ are the normal coordinates. The subindex i represents the three different atoms. Firstly, write V_{ij} and T_{ij} on matrix form. Secondly, identify the eigenfrequencies ω_1, ω_s, and ω_a. Thirdly, find the cofactors $\Delta_{i\alpha}$, for $i = 1,2,3$, $\alpha = 1,s,a$ by using the normalization condition $\sum_{i,j=1}^{3} T_{ij}\Delta_{i\alpha}\Delta_{j\beta} = \delta_{\alpha\beta}$ and the equation of motion $\sum_{j=1}^{3}(V_{ij} - \omega_\alpha^2 T_{ij})\Delta_{j\alpha} = 0$.

Consider now a different problem.

(b) The Hamiltonian for a particle with mass m is in cylindrical coordinates (r,θ,z) given by:

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_z^2}{2m} + V. \tag{2}$$

Assume that the potential is separable in the following way: $V = a(r) + b(z)$, where $a(r)$ and $b(z)$ are known functions. Put $S(q,\alpha,t) = W(q,\alpha) - \alpha t$ into the Hamilton-Jacobi equation:

$$H(q, \frac{\partial S}{\partial q}, t) + \frac{\partial S}{\partial t} = 0. \tag{3}$$

Assume now that W is separable in the following manner: $W = p_\theta\theta + S_1(r) + S_2(z)$, and show that the quantity

$$\beta = b(z) + \frac{1}{2m}[S_2'(z)]^2 \tag{4}$$

has to be a constant. Finally, write down the solution for Hamilton's principal function on integral form.

PROBLEM 3

(a) A particle with mass m moves in an attractive potential $V(r)$. Show, on the basis of energy conservation, how the problem can be looked upon as a one-dimensional problem with effective potential $\tilde{V}(r)$. What is the condition for the particle to reach the scattering centre, $r = 0$?

(b) A hard sphere has radius a. For $r > a$, the sphere yields a Kepler potential $V = -k/r$, where $k > 0$. Particles coming in from infinity have mass m and original velocity v_0. The part of the particles having impact parameter $s \leq s_{\max}$, will hit the sphere's surface. Find $s_{\max}$ and the corresponding "effective" scattering cross section $\sigma_{\text{eff}} = \pi s_{\max}^2$.

Consider now a different problem.

(c) Two particles with rest masses m_1 and m_2 are observed to move along an observer's z-axis toward each other with speeds v_1 and v_2, respectively. Upon collision, they form a particle with mass m_3 with speed v_3 relative the observer. Find m_3 and v_3 in terms of m_1, m_2, v_1, v_2. Would it be possible for the resultant particle to be a photon ($m_3 = 0$) if neither m_1 or m_2 are zero?

PROBLEM 4

Consider the three situations (A), (B), and (C) sketched in the figure below. In situation (A), a capacitor consisting of two conducting plates with charge Q and $-Q$ is at rest in the observer's frame of reference $\mathcal{S}$. The plates have an area A and a negligible thickness. They are separated by a distance d. In situations (B) and (C), the capacitor moves with a constant velocity v along a specific direction (indicated in the figure) relative the observer in $\mathcal{S}$. The velocity is assumed to be comparable to the speed of light c in magnitude.

(a) In situation (A), the electric field observed between the two conducting plates is constant and equal to $E = Q/(\varepsilon_0 A)$ in mag-

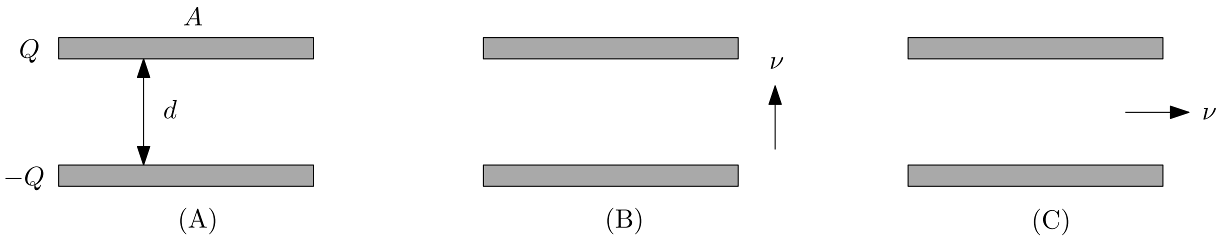

FIG. 3: (Color online). The system under consideration.

nitude whereas the magnetic field is zero ($B = 0$). Here, ε_0 is the vacuum permittivity constant. Using a Lorentz transformation, derive analytically in detail the electric field E and magnetic field B observed in $\mathcal{S}$ for the scenarios (B) and (C). In both these cases, find the magnitude and direction of the fields.

(b) Explain how you could have found the result for the electric field E in situation (B) and (C) simply by properly accounting for Lorentz contraction in the equation $E = Q/(\varepsilon_0 A)$ valid for scenario (A).

Consider now a different problem.

(c) A charged particle in an electromagnetic field has the Hamiltonian

$$H = \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 + q\varphi \tag{5}$$

Find Hamilton's equations. Use these to find the expression for the Lorentz force

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B} \tag{6}$$

Use, for instance, the Levi-Civita symbol to rewrite $[\mathbf{v} \times (\nabla \times \mathbf{A})]_i$.

Supplementary Information

The regime of validity and the meaning of the symbols below are assumed to be known by the reader.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}. \tag{7}$$

$$[u,v]_{q,p} = \sum_{i=1}^{n}\left(\frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i}\frac{\partial v}{\partial q_i}\right) \tag{8}$$

$$\begin{aligned} x_\mu &= (\mathbf{r}, \mathrm{i}ct), \\ p_\mu &= (\mathbf{p}, \mathrm{i}E/c) \end{aligned} \tag{9}$$

$$A_\mu = (\mathbf{A}, \mathrm{i}\phi/c),\ \mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t,\ \mathbf{B} = \nabla\times\mathbf{A} \tag{10}$$

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \tag{11}$$

From the above equations, it follows that the general form of $F_{\mu\nu}$ in a given reference system is:

$$F_{\mu\nu} = \begin{pmatrix} 0 & B_z & -B_y & -\mathrm{i}E_x/c \\ -B_z & 0 & B_x & -\mathrm{i}E_y/c \\ B_y & -B_x & 0 & -\mathrm{i}E_z/c \\ \mathrm{i}E_x/c & \mathrm{i}E_y/c & \mathrm{i}E_z/c & 0 \end{pmatrix} \tag{12}$$

$$F'_{\mu\nu} = L_{\mu\alpha}L_{\nu\beta}F_{\alpha\beta}. \tag{13}$$

The Lorentz-transformation matrix for the situation in Fig. 4 is given by:

$$L_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & \mathrm{i}\beta\gamma \\ 0 & 0 & -\mathrm{i}\beta\gamma & \gamma \end{pmatrix} \tag{14}$$

where $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$.

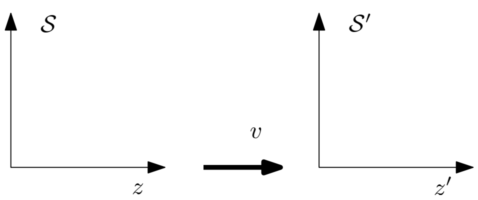

FIG. 4: Lorentz-transformation along the z-axis.

The Levi-Civita symbol ε_{ijk} is 1 if (i,j,k) is an even permutation of (1,2,3), -1 if it is an odd permutation, and 0 if any index is repeated.