Det skapende universitet

Department of physics

Examination paper for TFY4345 Classical mechanics

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Allowed to use: Calculator, mathematical tables.

All subproblems are given the same weight in the grading.

Problem 1:

The constraint equation

$$z = b\sqrt{x^2 + y^2}$$

with b constant defines a conical surface. A point particle of mass m is constrained to move on this surface. Its potential energy is

$$V(z) = mgz\ ,$$

where g is the acceleration of gravity, assumed to be constant.

a) Write down a Lagrangian for the system, and write also the corresponding Euler–Lagrange equations.

b) Find the constants of motion, and use them to simplify the equations of motion.

c) Show that the equations of motion have solutions that are circular orbits with arbitrary constant distance to the origin.

What is the period of such a circular orbit as a function of the distance to the origin?

Problem 2:

The potential energy of a small point mass m at position $\vec{r}$ at time t in a gravitational field is $V(\vec{r},t) = m\phi(\vec{r},t)$ where $\phi(\vec{r},t)$ is the gravitational potential. The gravitational potential from a spherically symmetric planet of radius R and constant mass density ρ is a function only of the distance r from the centre of the planet:

$$\phi(r) = \begin{cases} \frac{2\pi}{3}\, G\rho(r^2 - 3R^2) & \text{for} \quad r \le R\,, \\ -\dfrac{4\pi G\rho R^3}{3r} & \text{for} \quad r \ge R\,. \end{cases}$$

G is Newton's gravitational constant, $G = 6.6743 \times 10^{-11}$ m^3/(kg s^2).

a) Show that both $\phi(r)$ and its derivative $\phi'(r)$ are continuous at $r = R$.

b) Show that the force on the mass m at radius r depends only on the mass $M(r)$ inside the radius r, and is the same as if all of this mass $M(r)$ were located at the centre of the planet. This holds both outside and inside the planet.

c) If the planet has mass M and a moon of mass m, then the period T of the orbit of the moon around the planet is related to the semimajor axis a of the elliptical orbit by Kepler's third law:

$$T^2 = \frac{4\pi^2 a^3}{G(M+m)}\ .$$

If the two masses are both pointlike and start at rest, then they will fall towards each other until they collide at distance zero.

How long time does it take from they start at rest at a distance a until they collide?

Compare this so called free fall time to the period of a circular orbit of radius a.

Hint: you may solve this problem either by solving the equation of motion or by using Kepler's third law (or both).

d) When a star explodes as a type II supernova the explosion starts with the gravitational collapse of the core of the star. The core starts to collapse because the gas pressure is no longer sufficient to prevent it from collapsing. It collapses nearly in free fall.

The core has a mass roughly equal to the mass of the Sun, $M = 2 \times 10^{30}$ kg, and a radius of roughly $R = 1000$ km. As an approximation we assume that the mass density inside the core is constant.

What is the free fall time of a particle starting its fall a distance a from the centre, with $a \le R$?

Find a numerical value for the free fall time.

How does the density profile inside the core (the mass density as a function of the radius r) vary with time during the collapse?

Problem 3:

The finite element method is a numerical method for solving differential equations based on variational principles. As a simple example we consider here the equation of motion of a one dimensional harmonic oscillator with angular frequency ω,

$$\ddot{x} + \omega^2 x = 0 \ ,$$

which follows from the Lagrangian

$$L = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} m\omega^2 x^2 \ .$$

We want to compute numerically the solution within a time interval $[t_a, t_b]$ with boundary conditions $x(t_a) = x_a$, $x(t_b) = x_b$. We divide the whole interval into N intervals of equal size

$$\tau = \frac{t_b - t_a}{N} \ ,$$

and define

$$t_k = t_a + k\tau \ , \qquad x_k = x(t_k) \qquad \text{for} \qquad k = 0, 1, 2, \ldots, N \ .$$

The time interval from t_k to $t_{k+1} = t_k + \tau$ is a finite element, inside which we interpolate linearly. That is, we make the approximation that

$$x(t) = x_k + v_k(t - t_k) \tag{1}$$

for $t_k \leq t \leq t_{k+1}$ with v_k constant.

a) Show that with this approximation the action integral from t_k to t_{k+1} is

$$S_k = \int_{t_k}^{t_{k+1}} \mathrm{d}t \ L = \frac{1}{2} m \left(\frac{(x_{k+1} - x_k)^2}{\tau} - \frac{\omega^2 \tau}{3} \left(x_k^2 + x_k x_{k+1} + x_{k+1}^2 \right) \right) .$$

The total action integral, in the finite element approximation, is

$$S = \int_{t_a}^{t_b} \mathrm{d}t \ L = \sum_{k=0}^{N-1} S_k \ .$$

b) We have fixed $x_0 = x_a$ and $x_N = x_b$, but we are free to vary x_k for $k = 1, 2, \ldots, N-1$. Hamilton's principle, in the finite element method, requires that the variation of S is zero, that $\delta S = 0$, for arbitrary variations δx_k.

Write down the $N - 1$ equations to be satisfied by the unknown positions x_k for $k = 1, 2, \ldots, N-1$.

c) Show that these equations have two solutions of the form

$$x(t) = \mathrm{e}^{\pm \mathrm{i}\lambda t} \ ,$$

with λ constant, $\lambda > 0$.

Derive an equation for λ.

d) We say that this approximate numerical method for solving a differential equation is of order n if the error of an approximate answer, as compared to the corresponding exact answer, is proportional to τ^n in the limit $\tau \to 0$.

What is the order of this method when we compare the approximate angular frequency λ to the exact angular frequency ω?

What is the order of the linear approximation in equation (1)?

Problem 4:

The Lorentz transformation

$$x' = \gamma(x - Vt) , \qquad t' = \gamma\left(t - \frac{V}{c^2}\, x\right) , \qquad \gamma = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} ,$$

relates the time and space coordinates t, x and t', x' measured by two observers moving with a constant relative velocity V in the x direction.

We imagine a space ship travelling to Sirius at a speed of $0.6\,c$. We ignore the time it takes to accelerate to this speed, or to decelerate on arrival at Sirius.

The distance to Sirius is 9 light years.
$c = 299\,792\,458$ m/s is the speed of light in vacuum.

A correct answer gives 50 % score. The remaining 50 % are earned for a careful argument. It is not enough to answer "because of time dilatation" or "because of Lorentz contraction".

a) How long time does the trip to Sirius take, as measured by observers on the Earth and on the space ship?

b) What is the distance to Sirius, as measured by an observer on the space ship?

What is the length of a meter stick carried on the space ship, oriented along the direction between the Earth and Sirius, as measured by observers at rest relative to the Earth?

Comment?

c) How long time does the light from Sirius take to reach the Earth, as measured by observers on the Earth and on the space ship?

How long time does the light from the Sun take to reach Sirius, as measured by observers on the Earth and on the space ship?