NTNU – Trondheim
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Department of Physics

Examination paper for TFY4345 - (Classical Mechanics)

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Examination date: December 11, 2015

Examination time (from-to): 9-13

Permitted examination support material:

- **Approved, simple calculator**
- **K. Rottmann (matematisk formelsamling)**
- **Barnett & Cronin: Mathematical Formulae**
- **Notice:** ***Supplementary Information*** **on the last page of the exam contains useful formulas**

Other information:

Grading: Problem 1 (30%), Problem 2 (30%), Problem 3 (30%), Problem 4 (10%)

Language: English

Number of pages (front page excluded): 5

Number of pages enclosed: 5

Informasjon om trykking av eksamensoppgave

Originalen er:

1-sidig □ **2-sidig** □

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Problem 1. Bead on a rotating rod (30%)

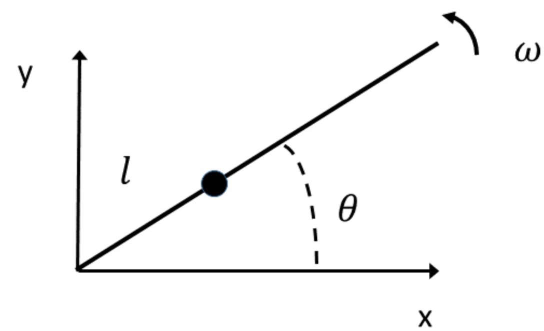

Fig 1. A bead is attached to a rotating rod. The position of the bead is determined by its distance to the origin: $l = \sqrt{x^2 + y^2}$, and by the angle θ. Notice that $\theta = \omega t$.

A bead of mass M is attached to a rod, and slides without friction along the rod. The rod is in the xy-plane and is rotating with a constant angular velocity ω. The distance between the bead and the origin is l. We assume that the rod is very long, such that the distance l can take any positive value in the following calculations.
At time $t=0$ we have: $l = l_0$ and $\frac{dl}{dt} = 0$.

1a) Assume first that there are no gravitational forces on the bead. Write down the Lagrangian L for the bead.

1b) Write down the Lagrange equations and find the solution $l(t)$. Show that when $t \gg \frac{1}{\omega}$ the solution is approximately:

$$l(t) \approx \frac{l_0}{2} e^{\omega t}$$

1c) We now assume that the bead is in a uniform gravitational field $\vec{g} = -g\vec{e}_y$. Write down the Lagrangian L for the bead.

1d) Write down the Lagrange equation associated with the Lagrangian in point 1c). Look for a solution of the form:

$$l(t) = C_1 e^{\omega t} + C_2\, e^{-\omega t} + C_3 \sin(\omega t)$$

Determine the constants C_1, C_2, C_3.

1e) If the rod is rotating fast, the bead will move away from the origin (x=0,y=0). However if the rotation is slow, the bead will fall to the origin, which correspond to $l = 0$. Argue that when $\omega^2 < \frac{g}{2l_0}$, the bead must reach the origin after some time.

Problem 2. Particle in a harmonic central force potential (30%)

Consider a particle of mass M in two dimensions. The particle is in a harmonic central force potential:

$$V = \frac{K}{2}(x^2 + y^2)$$

where K is a constant. In polar coordinates ($x = r\cos(\theta)$ and $y = r\sin(\theta)$) the potential is:

$$V = \frac{K}{2}r^2$$

2a) Write down the Lagrangian $L=T - V$ for the particle (in polar coordinates).

2b) Show that the Lagrangian L has a corresponding conservation law. What is the conserved quantity?

2c) Write down the total energy. Use the fact that the energy is conserved, and find an expression for $\frac{d\theta}{dr}$ as function of r.
Hint: Use the conservation law found in 2b)

2d) Solve the differential equation in 2c) and show that there exist particle trajectories (bound orbit solutions) of the form:

$$r = \frac{A}{\sqrt{1 + B\,cos(2\theta)}}$$

where A and B are constants (to be determined).
Hint: Make a substitution $u = \frac{1}{r^2}$.
Useful integral:

$$\int \frac{du}{\sqrt{\alpha + \beta u + \gamma u^2}} = \frac{1}{\sqrt{-\gamma}}\arccos[-\frac{\beta + 2\gamma u}{\sqrt{\beta^2 - 4\alpha\gamma}}\ ]$$

2e) Show that the particle trajectory found in 2d) is in an ellipse:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

where a and b are constants (to be determined).

Problem 3. Rotation of a free body using the Euler angles (30%)

We consider a free rigid body (no external forces acting on the body). The laboratory frame coordinates are (x, y, z), and the coordinates of the rotating body frame are (x', y', z'). The body coordinate system is fixed in the body (by definition). We shall use the Euler angles to describe the rotational motion of the body.

3a) The angular momentum in the laboratory frame is $\vec{L} = L\vec{e}_z$. Use Euler angle rotation to show that the components of the angular momentum in the body frame is:

$$L_{x'} = L\sin(\theta)\sin(\psi)$$
$$L_{y'} = L\sin(\theta)\cos(\psi)$$
$$L_{z'} = L\cos(\theta)$$

3b) We now fix the body coordinate system such that (x', y', z') are aligned with the principal axis of the body. This implies that moment of inertia tensor (matrix) is diagonal:

$$L_{x'} = I_1\omega_{x'}$$
$$L_{y'} = I_2\omega_{y'}$$
$$L_{z'} = I_3\omega_{z'}$$

where I_1, I_2, I_3 are the principal moment of inertia of the body. Consider the special case $I_1 = I_2 \neq I_3$ (axi-symmetric body) and show that the solution for the Euler angles of the rotating body has the form:

$$\frac{d\theta}{dt} = c_1$$
$$\frac{d\varphi}{dt} = c_2$$
$$\frac{d\psi}{dt} = c_3\cos(\theta)$$

where c_1, c_2, c_3 are constants. Determine these constants.

3c) Show that the vector: $\vec{w} = \omega_{x'}\vec{e}_{x'} + \omega_{y'}\vec{e}_{y'}$ rotates with a constant angular velocity Ω. Express Ω as a function of I_1 and I_3

3d) Show that the solution in 3b) is a solution of the Euler equation for a free body.

Problem 4. **Light in a moving medium (10%)**

Light that goes through water has a lower speed than the vacuum light speed c. If water is set into motion relative to the lab frame, the speed of light will change relative to the lab frame.

4a) Consider a uniform water flow with velocity v in the z-direction. Let S be the lab coordinate system, and S' be the coordinate system moving with the same speed as the water flow. The speed of light in the S' system is u' (in z-direction), and in the S system the speed of light is u (in z-direction).

Use the Lorentz transformations to find u as a function of u'.

4b) The speed of light in S' is $u' = \frac{c}{n}$, where c is the light speed in vacuum and n the refractive index of water.

Show that for $v \ll c$ the light speed measured in the laboratory frame is:

$$u \approx \frac{c}{n} + \left(1 - \frac{1}{n^2}\right) v$$

Supplementary information

Rotation around Euler angle φ (rotation around z-axis):

$$D = \begin{bmatrix} \cos(\varphi) & \sin(\varphi) & 0 \\ -\sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation around Euler angle θ :

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

Rotation around Euler angle ψ :

$$B = \begin{bmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Transformation from laboratory frame to body frame by Euler angle rotations:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = BCD \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Angular velocities in body frame:

$$\omega_{x'} = \dot{\varphi}\sin(\theta)\sin(\psi) + \dot{\theta}\cos(\psi)$$
$$\omega_{y'} = \dot{\varphi}\sin(\theta)\cos(\psi) - \dot{\theta}\sin(\psi)$$
$$\omega_{z'} = \dot{\varphi}\cos(\theta) + \dot{\psi}$$

Euler equation for a free body (zero external torque) :

$$\left(\frac{d\vec{L}}{dt}\right)_{body} + \vec{\omega} \times \vec{L} = 0$$

Lorentz transforms between a reference system S' moving with a constant velocity v in the z-direction, with respect to a reference system S:

$$\boxed{\begin{aligned} x' &= x \\ y' &= y \\ z' &= \gamma(z - vt) \\ t' &= \gamma\left(t - \frac{vz}{c^2}\right) \end{aligned}}$$

$$\boxed{\begin{aligned} x &= x' \\ y &= y' \\ z &= \gamma(z' + vt') \\ t &= \gamma\left(t' + \frac{vz'}{c^2}\right) \end{aligned}}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$