NTNU – Trondheim
Norwegian University of
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Department of (name)

Examination paper for (TFY4345) (Classical Mechanics)

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Permitted examination support material:

- **Approved, simple calculator**
- **K. Rottmann (matematisk formelsamling)**
- **Barnett & Cronin: Mathematical Formulae**

Other information: **Grading: Problem 1 (30%), Problem 2 (35%), Problem 3 (20%), Problem 4 (15%)**

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Informasjon om trykking av eksamensoppgave

Originalen er:

1-sidig ☐ **2-sidig** ☐

sort/hvit ☐ **farger** ☐

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Problem 1 **Mass on a rolling wheel** **(30 %)**

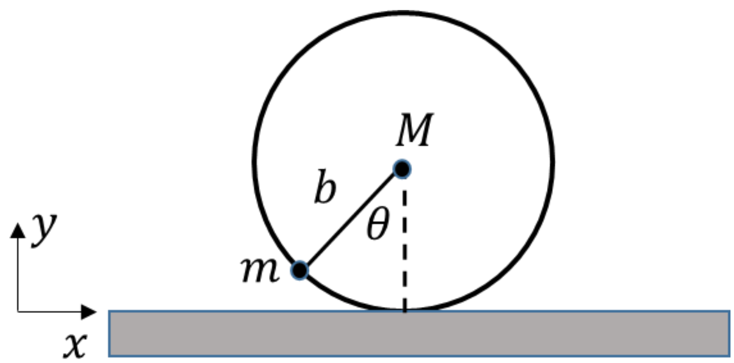

Figure 1. Mass m attached to wheel rolling on a surface (no slip).

A mass m is attached to the perimeter of wheel of radius b. The wheel is massless, except for a point mass M at its center. The radius of the wheel is b. The system is subject to a uniform gravitational field: $\vec{g} = -g\,\vec{e}_y$. When $\theta = 0$ the position of the mass m is $x = 0, y = 0$ in Cartesian coordinates.

(1a) The velocity of the mass m is $\vec{v}$, and the velocity of the mass M is $\vec{V}$. Show that:

$$v^2 = 2b^2\dot{\theta}^2\,[1 - \cos(\theta)]$$

$$V^2 = b^2\dot{\theta}^2$$

where $\dot{\theta}$ denotes the time derivative of the angle.

(1b) Write down the Lagrangian for this system.

(1c) Find the equation of motion for the angle θ.

(1d) Linearize the equation of motion (valid for small θ) and show that the wheel can oscillate. Calculate the frequency of oscillation as a function of m, M, b and g.

Problem 2   Central force problem producing a lemniscate orbit   (35 %)

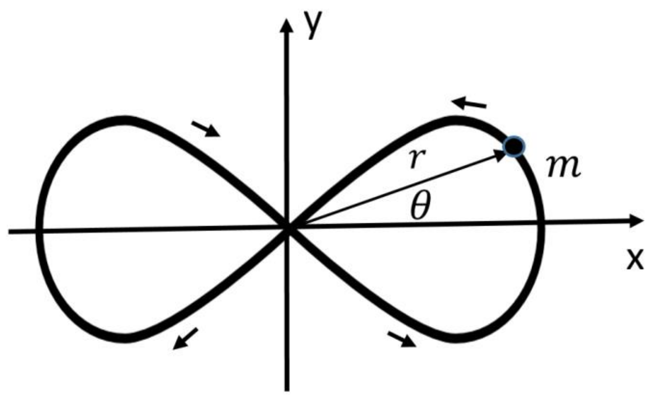

Figure 2. Particle moving in periodic orbit shaped like a "lemniscate curve".

A particle of mass *m* is subject to an attractive central force potential:

$$V = -\frac{K}{r^6}$$

where r is the distance between the particle and the force centre (x=0,y=0).

(2a) Write down the Lagrangian for this system in polar coordinates *r* and θ.

(2b) Determine the conservation law associated with this Lagrangian. What is the conserved quantity?

(2c) Derive the equation of motion of the particle.

(2d) Show that there exist a solution of the form:

$$r = c\sqrt{\cos(2\theta)}$$

Derive an expression for c as a function of K, m and the conserved quantity calculated in question (2b). What is the total energy of the particle?

(2e) Solve the differential equation for $\theta(t)$ when $0 < \theta < \frac{\pi}{4}$. The initial condition is $\theta(t = 0) = 0$.

(2f) Calculate the orbital period as a function of *m*,*K* and *c*.

Problem 3 Precession of a frisbee (20 %)

(3a) Consider an axi-symmetric body with moment of inertia: $I_1 = I_2 \neq I_3$. The angular momentum in the laboratory frame is $\vec{L} = L\vec{e}_z$. Derive the equation of motion for the body, using the Euler equation and the angles θ, ψ, φ.

(3b) Find the expression for the Euler angles θ, ψ, φ as a function of time.

(3c) For a Frisbee $I_1 = I_2$ and $I_3 = 2\, I_1$. The precession (wobble) of the frisbee is given by $\dot{\varphi}$. Show that the precession of the frisbee is twice as fast as the rotation frequency of the frisbee, assuming that the angle θ is small $(i.e.\ \cos(\theta) \approx 1)$.

Problem 4. Special relativity: length contraction and rotation (15%)

(4a) A rod is moving along the z-axis with a uniform speed *v*. Assume first that the rod is parallel with the z-axis. The length of the rod in the moving reference system *(x',y',z')* is *L'*. Use the Lorentz transformations to calculate the length *L* of the rod in the stationary reference system *(x,y,z)*.

(4b) Assume now that the same rod is rotated so that it makes an angle θ_0 with the z' -axis. Calculate the length of the rod *L* as seen from the stationary reference system.

(4c) Show that the angle the rod makes with the z-axis is given by $\tan(\theta) = \frac{\tan(\theta_0)}{\sqrt{1-\frac{v^2}{c^2}}}$. In other words the rod appears to have rotated.

Useful Formula and Equations

Rotation around Euler angle φ (rotation around z-axis):

$$D = \begin{bmatrix} \cos(\varphi) & \sin(\varphi) & 0 \\ -\sin(\varphi) & \cos(\varphi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation around Euler angle θ :

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

Rotation around Euler angle ψ :

$$B = \begin{bmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Transformation from laboratory frame to body frame by Euler angle rotations:

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = BCD \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Angular velocities in body frame:

$$\omega_{x'} = \dot{\varphi}\sin(\theta)\sin(\psi) + \dot{\theta}\cos(\psi)$$
$$\omega_{y'} = \dot{\varphi}\sin(\theta)\cos(\psi) - \dot{\theta}\sin(\psi)$$
$$\omega_{z'} = \dot{\varphi}\cos(\theta) + \dot{\psi}$$

Euler equation for a free body (zero external torque) :

$$\left(\frac{d\vec{L}}{dt}\right)_{body} + \vec{\omega} \times \vec{L} = 0$$

The angular momentum in a body frame:

$$L_{x'} = L\sin(\theta)\sin(\psi)$$
$$L_{y'} = L\sin(\theta)\cos(\psi)$$
$$L_{z'} = L\cos(\theta)$$

where the angular momentum in the laboratory frame is : $\vec{L} = L\vec{e}_z$.

Moment of inertia when body frame (x', y', z') is aligned with the principal axis of the body:

$$L_{x'} = I_1\omega_{x'}$$
$$L_{y'} = I_2\omega_{y'}$$
$$L_{z'} = I_3\omega_{z'}$$

Lorentz transforms between a reference system S' moving with a constant velocity v in the z-direction, with respect to a reference system S:

$$\boxed{\begin{aligned} x' &= x \\ y' &= y \\ z' &= \gamma(z - vt) \\ t' &= \gamma\left(t - \frac{vz}{c^2}\right) \end{aligned}}$$

$$\boxed{\begin{aligned} x &= x' \\ y &= y' \\ z &= \gamma(z' + vt') \\ t &= \gamma\left(t' + \frac{vz'}{c^2}\right) \end{aligned}}$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$