Problem 1. Set of questions (12p). The answers must be provided digitally.

A. Explain briefly by few sentences [10p]:

(i) What are *holonomic constraints*?

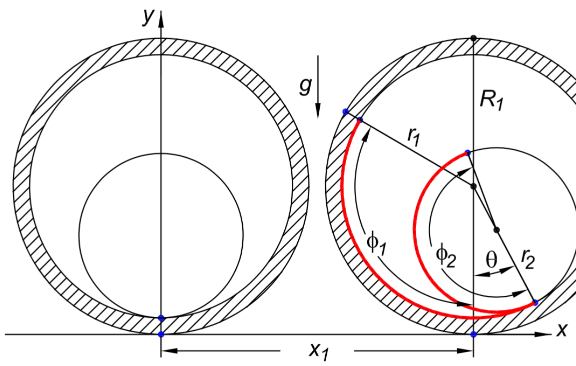
(ii) What is meant by *monogenic forces* and how are they associated with the Hamilton's principle?

(iii) What are *Euler angles* and why are they used?

(iv) What is the *Hamilton's principal function* and how is it related with the Hamilton-Jacobi equation.

(v) What is the *Coriolis effect*? How can one observe it in practice?

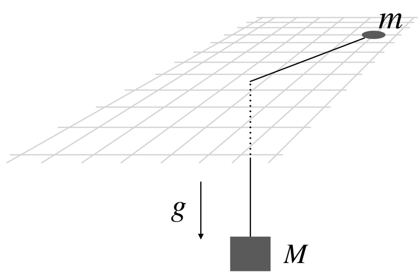
B. Consider the motion of a cylinder that rolls without slipping inside another cylinder, when the latter rolls without slipping on a horizontal plane [2p].



What are the associated constraints and what is their type? How many degrees of freedom are there? What generalized coordinates would you choose for solving the mechanical problem? (No use for undetermined multipliers here.)

Problem 2. Sliding buck attached to a weight [10p]

A buck with mass m slides frictionless on an (x, y)-plane (see the figure). The buck is attached to a mass M via a massless wire (length l_0) which penetrates the plane in the origin and is pulled down by gravity along the z-direction. Let us choose the polar coordinates (r, θ) as the generalized coordinates.



(a) Solve the Lagrangian function and derive the Lagrange´s equations of motion [4p].

(b) One equation of motion indicates that angular momentum is conserved. Let us denote its value by $\mathcal{L}$. This results in that the other equation will describe one-dimensional motion. Derive the associated *effective potential* for that motion [2p].

(c) The system has one solution where the buck moves in a circular orbit. Show that the radius of this orbit is [2p]:

$$r_0^3 = \frac{\mathcal{L}^2}{Mmg}.$$

(d) Investigate small deviations in the equation of motion. Use the trial function $r(t) = r_0 + \delta(t)$ and show that small deviations lead to the solution [2p]:

$$\ddot{\delta} = -\omega_\delta^2 \delta$$

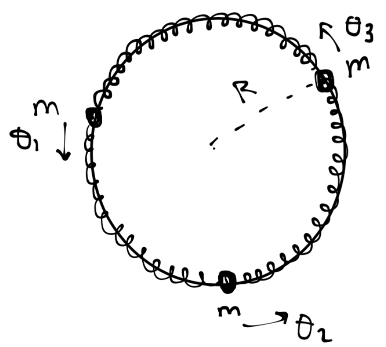
What is ω_δ^2? Express it as a function of r_0.

Hint: You may find here the following Taylor expansion useful:

$$(1+x)^p \approx 1 + px + \frac{p(p+1)}{2}x^2 + \cdots, \; x \ll 1$$

Problem 3. Three coupled oscillating rings [6p]. Three rings of mass *m* are coupled with three massless springs (spring constant *k*) as shown in the figure.

The rings slide without friction around the circular hoop of radius *R*. Assuming small oscillations, determine the eigenfrequencies and eigenvectors for the motion of the system. Parametrize the deviations from the equilibrium positions by using the angles θ_1, θ_2 and θ_3 as shown in the figure.



What kind of motions do the solutions of the eigenvectors correspond to?

Problem 4. Fast moving particle in two inertial frames [6p]. We shall consider a particle with rest mass *m* seen from two different inertial reference systems. In the reference system *S* the particle has a velocity $\mathbf{u} = (u_x, u_y, u_z)$. The reference system *S'* is moving along the *z*-axis with a constant velocity *v* relative to the reference system *S*. The velocity of the particle is $\mathbf{u'} = (u'_x, u'_y, u'_z)$ in the reference system *S'*. Find the explicit interrelationship between $\mathbf{u'}$ and $\mathbf{u}$, i.e. derive the transformation that gives $\mathbf{u'}$ from the components of $\mathbf{u}$ based on the Special Theory of Relativity.

Problem 5. Hamiltonian mechanics [6p].

(a) For a one-dimensional system with the Hamiltonian

$$H = \frac{p^2}{2} - \frac{1}{2q^2}$$

show that there is a constant of the motion [3p]

$$D = \frac{pq}{2} - Ht.$$

(b) Show that the time-independent transformation

$$P = p + q^2 + pq^2, \qquad Q = tan^{-1}q$$

is canonical by obtaining a suitable generating function of type $F_3(p, Q)$ [3p].

Hint: You may find the trigonometric relation $1 + tan^2x = sec^2x$ useful here.

Some useful formula and equations on a separate document:

Cylindrical coordinates:

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2$$

Displacement in Minkowski-space:

$$ds^2 = dx_\mu dx_\mu = dx_1^2 + dx_2^2 + dx_3^2 - c^2 dt^2$$

Coupled oscillations:

$$V = \frac{1}{2}\sum_{j,k} A_{jk} q_j q_k; \quad T = \frac{1}{2}\sum_{j,k} m_{ij} \dot{q}_j \dot{q}_k$$

Lorentz transformation:

$$\begin{aligned} x' &= x \\ y' &= y \\ z' &= \gamma(z - vt) \\ t' &= \gamma(t - \frac{vz}{c^2}; \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \end{aligned}$$

Canonical transformations, generating function F_3

$$q = -\frac{\partial F_3(p,Q)}{\partial p}; \quad P = -\frac{\partial F_3(p,Q)}{\partial Q}; \quad K = H + \frac{\partial F_3(p,Q)}{\partial t}$$

Poisson brackets:

$$[u,v]_{q,p} = \sum_{i=1}^{n} \left(\frac{\partial u}{\partial q_i}\frac{\partial v}{\partial p_i} - \frac{\partial v}{\partial q_i}\frac{\partial u}{\partial p_i} \right)$$

Hamilton-Jacobi theory:

$$H + \frac{\partial S}{\partial t} = 0; \quad S = S(q_1, ..., q_n, \alpha_1, ..., \alpha_n, t)$$

$$p_i = \frac{\partial}{\partial q_i} S(q,\alpha,t); \quad Q_i = \frac{\partial}{\partial \alpha_i} S(q,\alpha,t) = \beta_i; \quad q_i = q_i(\alpha,\beta,t)$$