TFY4345 – Classical Mechanics, home exam 18.12.2020

1. Hybrid essay + calculation: Heavy spinning top

Let us consider a heavy spinning top (mass m) with principal moments of inertia $I_1 = I_2 \equiv I \neq I_3$. The distance of the center-of-mass from the tip of the top is denoted as l. The tip is fixed on the ground and the top rotates without friction.

(a) (5 p.) Explain briefly:

- What are the Euler angles (ϕ, θ, ψ) in general, what is their purpose, and what physical quantities do they correspond to in this particular example? You may use illustrative drawings.
- How can one express the angular velocity $\overline{\omega}$ components along the body axes using the Euler angles? How is this transformation derived? You may use some equations on the side but do not make it too extensive!

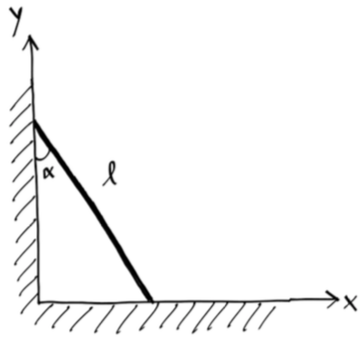
(b) (5 p.) The Lagrangian function for the spinning top is now

$$L = \frac{1}{2}I({\omega_1}^2 + {\omega_2}^2) + \frac{1}{2}I_3{\omega_3}^2 - mglcos\theta = \frac{1}{2}I(\dot{\phi}^2 sin^2\theta + \dot{\theta}^2) + \frac{1}{2}I_3(\dot{\phi}cos\theta + \dot{\psi})^2 - mglcos\theta$$

Which component of $\overline{\omega}$ is a constant and what does it represent? Derive the corresponding Lagrangian equations of motion. What are the conserved quantities?

2. Tilted plane sliding against a wall

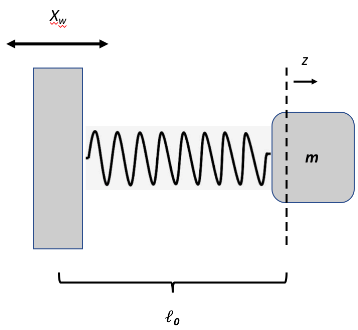
Let us consider a plate leaning against a wall. The original inclination angle is denoted as α. Suddenly, friction disappears both in the lower and upper ends of the plate and it starts falling down. The length of the plate is ℓ and its mass is m.



(a) (4 p.) Let us assume that the upper edge of the plate does not lose contact with the wall in the first place. Write out the Lagrangian function and the corresponding equation of motion for the plate. Choose your generalized coordinate such that it corresponds to the instantaneous angle $\theta(t)$ between the plate and the wall with the initial value $\theta(0) = \alpha$. The initial angular velocity is $\dot{\theta}(0) = 0$. The plate itself is homogeneous and its principal moment of inertia around the perpendicular principal z-axis and the center-of-mass is $I_z = \frac{m\ell^2}{12}$.
[**Answer:** The equation of motion will be $\ddot{\theta} = 3g/(2\ell)sin\theta$.]

(b) (4 p.) At which value of $\theta(t)$ will the plate lose its contact with the wall? In other words, the force of support of the wall will vanish at this point and $\ddot{x} = 0$ from here on. Derive this result **without using** the Lagrange's undetermined multipliers. You will need the derivatives of $\ddot{\theta}$ and $\dot{\theta}$ as a function of θ. The latter you can derive, for example, by considering the conservation of energy.

3. Hamiltonian for a system with moving wall



(5 p.) A system comprises a mass m and a wall and they are connected by a horizontal spring with a spring constant k and equilibrium length l_0. The wall moves back and forth with a position $X_w = A\,sin\,(\omega t)$. Let us denote the displacement (stretch) of the spring by z. Derive the Hamiltonian in terms of z and its conjugate momentum, and then derive Hamilton's equations. Finally, combine these two equations to a single second order differential equation that corresponds to the Lagrange equation of motion. Is H the energy? Is H conserved?

You can focus on horizontal motion completely, *i.e.* there is no need to consider gravity.

4. Central forces and scattering

Consider a particle with mass m moving under a repulsive central force $\bar{F}(r) = f(r)\hat{e}_r$, where $f(r) = -\frac{dV}{dr}$. The particle location is described by distance $r(t)$ and orbit angle $\theta(t)$.

(a) (3 p.) Write out the Lagrangian and derive the corresponding (general) equation of motion using the implicit function $f(r)$. Further, using this result, derive the equation that relates the orbit to the force

$$\frac{\ell^2 u^2}{m}\left(\frac{d^2u}{d\theta^2} + u\right) = -f(u),$$

where $u = \frac{1}{r}$ and ℓ is the angular momentum. You will need to use here a conservation law. In case you get stuck, move on to (b) and come back later.

(b) (6 p.) The repulsive central force is now of the explicit form $f(r) = \frac{km}{r^3}$. The particle has a speed v_0 at the periapsis distance r_0. Let us also denote the velocity in the beginning ($r \to \infty$) as v_∞.

(i) Derive the orbital equation $r(\theta)$ for the particle motion, expressing it in terms of the constants r_0 and v_0. Use the main result from part (a) for this purpose and you will encounter a familiar differential equation.

[**Answer:** The orbital equation will be of the form $1/r = 1/r_0 \times \left(sin\left(\sqrt{1 + k/({r_0}^2{v_0}^2)}\theta\right)\right)$.]

(ii) Derive the equations for the impact parameter s and the total angular deflection Θ in terms of the constants r_0 and v_0 and by assuming that the particle approaches from large distance $r \to \infty$.

(iii) Sketch the particle trajectory, showing the impact parameter and the total deflection calculated for the case $s = r_0/2$ in part (ii).

5. Special theory of relativity – 007 space odyssey

(4 p.) Let us consider a moon rocket by Drax Industries (fictitious company) which moves away from Earth towards Moon with speed v. You may choose this direction as the x-axis. Next, the moon rocket fires a landing module in the forward direction at a speed v relative to itself. Further, the pilot of the landing module (007 in undercover) launches a probe vessel, including unconscious and tied Mr. Jaws in its hull, in the forward direction at speed v relative to the landing module.

(a) What is the speed of the landing module (and 007) relative to the Earth, as observed by Ms. Moneypenny?

(b) What is the speed of the probe (and Mr. Jaws) relative to the Earth?

Explain your answers properly.

Hint: Check the feasibility of your answers by considering the lower and upper limits in both cases.

GOOD LUCK AND HAPPY HOLIDAYS!