Problem 1. (10 points)

A. Constraints

What are holonomic constrains? Under what circumstances can we use the Lagrangian formalism for solving a mechanical problem with non-holonomic contraints? What implications does this have in terms of the number of generalized coordinates? Provide a proper written explanation where you may use some equations also. [6 p]

B. Total time-derivative of kinetic energy for a falling particle

Consider the following Hamiltonian of a falling particle: $H = T + V = \frac{p^2}{2m} + mgy$
Derive the total time-derivative of kinetic energy T by using the Poisson bracket formalism. The vertical coordinate of the particle (mass m) is denoted with y and the related momentum is p.

Confirm your result by considering the time-derivative of total energy. [4p]

Problem 2. Sliding particle with air resistance (8 points)

Let us consider a body (mass m) sliding down an inclined plane without any friction from the support. The body experiences a drag force that is directly proportional to velocity and has the form $F = -kmv$, where k is a constant. The inclination angle of the plane is θ.

(a) Write out the Lagrangian and solve the equation of motion. [3p]

(b) The body starts from rest ($v_0 = 0$). Solve the equation for velocity as a function of time. What is the upper limit (called terminal velocity)? [3p]

Hint: Express $\ddot{x} = \frac{dv}{dt}$ and carry out the integrations.

(c) Let us assume that the body has achieved 90% of its maximum velocity. When does this occur (t) and how far (x) has the body slid along the plane? [2p]

Problem 3. Heavy spinning top (8 points)

Let us consider a heavy spinning top as described in the lectures which rotates frictionless around a fixed point on the ground and experiences gravity. The kinetic energy is of the form

$$T = \frac{1}{2}I(\omega_1^2 + \omega_2^2) + \frac{1}{2}I_3\omega_3^2$$

Let us choose the Euler angles (φ, θ, ψ) as the generalized coordinates. What physical quantities do they represent? The distance of the center of mass from the fixed tip is denoted with h. Write out the explicit Lagrangian in terms of the generalized coordinates. Which component of $\overline{\omega}$ is a constant and what does it represent? [4 p]

Derive the Lagrangian equations of motion and the canonical momenta. What are the conserved quantities? [4p]

Hint: There is no need to expand the final time-derivative explicitly for a constant quantity. Instead you may use $\frac{d}{dt}(\dots) = 0$.

Problem 4. Special theory of relativity – creation of a kaon particle (8 points)

(a) Starting from the 4-vector for position (event) x_μ, derive the momentum 4-vector P_μ where its 4th component (according to the covariant 3+1 formulation) is expressed in terms of total energy. Calculate also the Lorentz invariant property $P_\mu P_\mu$ by assuming rest frame. These will be useful in the following. [3p]

(b) Let us consider a relativistic collision (reaction): $\pi^+ + n \rightarrow K^+ + \Lambda_0$.

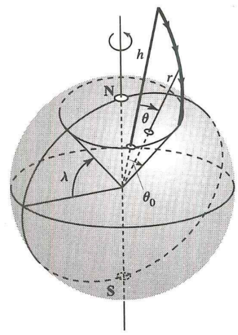
What is the threshold kinetic energy of the pion (π^+) to create kaon (K^+) at an angle of 90 degrees in the rest frame of the neutron (n)? Express your solution in terms of particle masses. The inherent nature of the particles is not relevant, just consider them as particles with different masses. [5p]

Hint: We have conservation of 4-momentum. Correspondingly, modify the momentum equation such that you will have $\left(P_\mu^\Lambda\right)^2 = P_\mu^\Lambda P_\mu^\Lambda$ alone on the left-hand side. Next, consider the individual terms taking into account the above-mentioned conditions.

(Final answer: $\frac{T_\pi}{c^2} \geq \frac{m_\Lambda^2 - m_\pi^2 - m_n^2 - m_K^2 + 2m_n m_K}{2(m_n - m_K)} - m_\pi$)

Problem 5. Coriolis effect from another perspective (12 points)

Let us revisit the problem of dropping a pellet down from a tower as seen from outer space (inertial frame), see the figure. Let us denote that latitude on Earth is λ and angular velocity of rotation is ω. The radius of Earth is R and the height of the tower is h. The pellet starts falling down from rest (tower) in the rotating non-inertial frame of Earth.



Note: Points (a)-(c) are straightforward. Come back to (d)-(f) later if you are short of time.

(a) Consider for a short moment that Earth is represented by a single mass point in its center. What shape does the trajectory of the falling particle have? In reality, the particle is able to travel only a small sequence of this trajectory before hitting the ground ($r = R$). What location does the particle have on this orbit in the beginning (tower)? Write out the horizontal velocity v_{hor} of the particle and the corresponding angular momentum ℓ in the beginning ($r = R + h$), as seen from outside (inertial frame). [2p]

(b) Going back to the solution of the Kepler problem, choose that the orbit angle is $\theta = 0$ in the beginning and use

$$\frac{p}{r} = 1 - \varepsilon \cos\theta,$$

Show that the distance r has the general solution [2p]

$$r = \frac{(1-\varepsilon)(R+h)}{1 - \varepsilon\cos\theta}$$

(c) Use the constant areal velocity (Kepler II) in order to derive the following expression for time [2p]

$$t = \frac{1}{\omega\cos\lambda}\int_0^\theta \frac{(1-\varepsilon)^2}{(1-\varepsilon\cos\theta)^2} d\theta$$

(d) Denote the ground landing point as $\theta = \theta_0$ ($r = R$). Show that [2p]

$$\frac{R+h}{R} = 1 + \frac{2\varepsilon}{1-\varepsilon}\sin^2\frac{\theta_0}{2} \;\Longrightarrow\; \frac{h}{R} \cong \frac{\varepsilon\theta_0^2}{2(1-\varepsilon)}$$

Hint: Use the trigonometric relation: $\cos\theta = 1 - 2\sin^2(\theta/2)$ and a Taylor expansion in the last step (θ_0 small).

(e) Modify the equation for time in (c) for the actual falling time $t(\theta = \theta_0) = T$ such that it becomes [2p]

$$T \cong \frac{1}{\omega\cos\lambda}\int_0^{\theta_0} \frac{d\theta}{[1 + (h\theta^2/R\theta_0^2)]^2}$$

(f) We can now take a Taylor expansion, integrate, and solve θ_0 (+ Taylor once more) such that it becomes (you do not need to show this!)

$$\theta_0 \cong \omega T\cos\lambda\left(1 + \frac{2h}{3R}\right)$$

By using this result, calculate the deflection d of the landing point on the ground as a function of tower height h. What is the direction of deflection? [2p]

Hint: Free fall without air resistance in a non-rotating frame gives you an expression (approximation) for time T that you can use here for the final step.

Fun fact: The result will be identically the same as what we obtained earlier in the exercises by using the expression of Coriolis force in a rotating frame!

GOOD LUCK!

MERRY CHRISTMAS AND HAPPY NEW YEAR!