Solutions to examination in TFY4345 Classical mechanics
Saturday May 16, 2015

1a) The kinetic energy is

$$T = \frac{1}{2}\,m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\ .$$

Two dimensional polar coordinates recommend themselves, so we write

$$x = \rho\cos\varphi\ , \qquad y = \rho\sin\varphi\ ,$$

which gives that

$$\dot{x} = \dot{\rho}\cos\varphi - \rho\dot{\varphi}\sin\varphi\ , \qquad \dot{y} = \dot{\rho}\sin\varphi + \rho\dot{\varphi}\cos\varphi\ ,$$

and

$$\dot{x}^2 + \dot{y}^2 = \dot{\rho}^2 + \rho^2\dot{\varphi}^2\ .$$

The constraint gives that

$$z = b\rho\ , \qquad \dot{z} = b\dot{\rho}\ ,$$

and we get the Lagrangian

$$L = T - V = \frac{1}{2}\,m((1+b^2)\dot{\rho}^2 + \rho^2\dot{\varphi}^2) - mgb\rho\ .$$

The Euler–Lagrange equations are:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\rho}}\right) - \frac{\partial L}{\partial\rho} &= \frac{\mathrm{d}}{\mathrm{d}t}\big(m(1+b^2)\dot{\rho}\big) - m\rho\dot{\varphi}^2 + mgb = 0\ ,\\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\varphi}}\right) - \frac{\partial L}{\partial\varphi} &= \frac{\mathrm{d}}{\mathrm{d}t}\big(m\rho^2\dot{\varphi}\big) = 0\ .
\end{aligned}$$

1b) Since φ is a cyclic coordinate, its Euler–Lagrange equation is a conservation law for its conjugate momentum, which is the angular momentum ℓ,

$$m\rho^2\dot{\varphi} = \ell = \text{constant}\ .$$

Since the Lagrangian is not explicitly time dependent, the Hamiltonian

$$H = \dot{\rho}\,\frac{\partial L}{\partial\dot{\rho}} + \dot{\varphi}\,\frac{\partial L}{\partial\dot{\varphi}} - L = T + V = \frac{1}{2}\,m((1+b^2)\dot{\rho}^2 + \rho^2\dot{\varphi}^2) + mgb\rho$$

is a constant of motion. This is the energy E. When we eliminate $\dot{\varphi}$ the energy conservation equation becomes an equation of motion for ρ alone,

$$\frac{1}{2}\,m(1+b^2)\dot{\rho}^2 + \frac{\ell^2}{2m\rho^2} + mgb\rho = E\ .$$

Another way to derive this equation is to eliminate $\dot{\varphi}$ between the two Euler–Lagrange equations, this gives the equation

$$m(1+b^2)\ddot{\rho} - \frac{\ell^2}{m\rho^3} + mgb = 0\ ,$$

with ℓ constant. Then we multiply by $\dot{\rho}$ and integrate.

1c) Assuming ρ to be constant we get from the last equation that

$$-\frac{\ell^2}{m\rho^3} + mgb = 0 \;, \qquad \ell^2 = m^2 g b \rho^3 \;.$$

The period of the circular orbit is then

$$T = \frac{2\pi}{\dot{\varphi}} = \frac{2\pi m \rho^2}{\ell} = 2\pi \sqrt{\frac{\rho}{gb}} = 2\pi \left(\frac{\ell}{m(gb)^2} \right)^{\frac{1}{3}} .$$

This is a good enough answer, but of course the distance to the origin, measured along the conical surface, is not ρ but $\rho\sqrt{1+b^2}$.

An alternative approach to problem 1 is to introduce the quadratic constraint equation

$$f(x,y,z) = z^2 - b^2(x^2+y^2) = 0 \;,$$

a corresponding Lagrange multiplier λ, and then write the Lagrangian

$$L = T - V - \lambda f(x,y,z) = \frac{1}{2}\, m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz - \lambda(z^2 - b^2(x^2+y^2)) \;.$$

The Euler–Lagrange equation for λ is the constraint equation,

$$-\frac{\partial L}{\partial \lambda} = z^2 - b^2(x^2+y^2) = 0 \;.$$

Since this equation must hold at any time, its time derivative must also hold:

$$z\dot{z} - b^2(x\dot{x} + y\dot{y}) = 0 \;.$$

The Euler–Lagrange equations for x, y, z are

$$\begin{aligned} m\ddot{x} - 2\lambda b^2 x &= 0 \;, \\ m\ddot{y} - 2\lambda b^2 y &= 0 \;, \\ m\ddot{z} + mg + 2\lambda z &= 0 \;. \end{aligned}$$

There are several ways to eliminate the Lagrange multiplier λ from these equations. Multiplying the first equation by y, the second by x, and subtracting, we get that

$$m(x\ddot{y} - y\ddot{x}) = \frac{\mathrm{d}}{\mathrm{d}t}\,(m(x\dot{y} - y\dot{x})) = 0 \;.$$

This equation says that the angular momentum is conserved:

$$m(x\dot{y} - y\dot{x}) = \ell = \text{constant} \;.$$

Multiplying the first equation by $\dot{x}$, the second by $\dot{y}$, the third by $\dot{z}$, adding and using the time derivative of the constraint equation, we get that

$$m(\ddot{x}\dot{x} + \ddot{y}\dot{y} + \ddot{z}\dot{z}) + mg\dot{z} = 0 \;.$$

Integrating this equation we get the energy conservation equation

$$\frac{1}{2}\, m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + mgz = E = \text{constant} \;.$$

It seems that the Lagrange multiplier method is not very useful in this particular example, although we easily get these conservation laws.

2a) The formula for $r \leq R$ and the formula for $r \geq R$ both give that

$$\phi(R) = -\frac{4\pi}{3}\, G\rho R^2 \, .$$

For $r \leq R$ we have that

$$\phi'(r) = \frac{4\pi}{3}\, G\rho r \to \frac{4\pi}{3}\, G\rho R \qquad \text{as} \qquad r \to R^- \, .$$

For $r \geq R$ we have that

$$\phi'(r) = \frac{4\pi}{3r^2}\, G\rho R^3 \to \frac{4\pi}{3}\, G\rho R \qquad \text{as} \qquad r \to R^+ \, .$$

2b) The mass inside the radius r is

$$M(r) = \frac{4\pi r^3 \rho}{3} \quad \text{for} \quad r \leq R \, , \qquad M(r) = M(R) = \frac{4\pi R^3 \rho}{3} \quad \text{for} \quad r \geq R \, .$$

The force on the mass m is radial. For $r \leq R$ it is

$$F = -m\phi'(r) = -\frac{4\pi}{3}\, G\rho r m = -\frac{GM(r)m}{r^2} \, .$$

For $r \geq R$ it is

$$F = -m\phi'(r) = -\frac{4\pi}{3r^2}\, G\rho R^3 m = -\frac{GM(R)m}{r^2} = -\frac{GM(r)m}{r^2} \, .$$

2c) A circle of radius a is the same as an ellipse with semi-major axis a and eccentricity zero. The period of the circular orbit is

$$T_c = \frac{2\pi a^{3/2}}{\sqrt{G(M+m)}} \, .$$

When the two particles fall radially towards each other, the orbit is a degenerate ellipse with semi-major axis $a/2$ and eccentricity one. This degenerate ellipse has a period of $T_c/2^{3/2}$. The fall until the particles collide is only half of the full elliptical orbit. Hence the free fall time is

$$T_f = \frac{T_c}{4\sqrt{2}} = \frac{\pi a^{3/2}}{2\sqrt{2G(M+m)}} \, .$$

The alternative method of solution is to solve the equations of motion. Let the positions of the two particles be x_1 and x_2, the distance between them $x = x_1 - x_2 > 0$, and the masses $m_1 = m$ and $m_2 = M$. Then

$$\ddot{x}_1 = -\frac{GM}{x^2} \, , \qquad \ddot{x}_2 = \frac{Gm}{x^2} \, , \qquad \ddot{x} = \ddot{x}_1 - \ddot{x}_2 = -\frac{G(M+m)}{x^2} \, .$$

Multiplying by $\dot{x}$ and integrating, and using that $\dot{x} = 0$ when $x = a$, we get that

$$\frac{1}{2}\,\dot{x}^2 = G(M+m)\left(\frac{1}{x} - \frac{1}{a}\right) = \frac{G(M+m)(a-x)}{ax} \, .$$

The free fall time is

$$T_f = \int_0^{T_f} \mathrm{d}t = \int_a^0 \mathrm{d}x \, \frac{\mathrm{d}t}{\mathrm{d}x} = \int_0^a \frac{\mathrm{d}x}{|\dot{x}|} = \sqrt{\frac{a}{2G(M+m)}} \int_0^a \mathrm{d}x \, \sqrt{\frac{x}{a-x}} \; .$$

We substitute for example $x = a \sin^2 u$ and get that

$$\int_0^a \mathrm{d}x \, \sqrt{\frac{x}{a-x}} = 2a \int_0^{\pi/2} \mathrm{d}u \, \sin^2 u = a \int_0^{\pi/2} \mathrm{d}u \, (1 - \cos(2u)) = \frac{\pi a}{2} \; .$$

Thus we arrive at the same answer for T_f.

2d) We use the above formula for T_f. The mass M in the formula is the mass inside the radius a,

$$M = \frac{4\pi}{3} \, a^3 \rho_0 \; .$$

Here ρ_0 is the mass density before the collapse, which is assumed constant in space. Of course the density is not constant in time, it increases with time when the core collapses! The mass m is the mass of the falling particle, which is negligible compared to M. Thus

$$T_f = \frac{\pi a^{3/2}}{2\sqrt{2GM}} = \frac{1}{4} \sqrt{\frac{3\pi}{2G\rho_0}} \; ,$$

independent of a. One solar mass within a radius of 1000 km gives a density of

$$\rho_0 = \frac{3 \times 2 \times 10^{30} \text{ kg}}{4\pi \times 10^{18} \text{ m}^3} = 4.77 \times 10^{11} \text{ kg/m}^3 \; ,$$

and a free fall time of

$$T_f = 0.096 \text{ s} \; .$$

The whole core collapses within one tenth of a second.

Since all parts of the core take the same time to collapse, the density at one instant of time is always constant in space, inside the core.

3a)

$$\begin{aligned} S_k &= \int_{t_k}^{t_k+\tau} \mathrm{d}t \left(\frac{1}{2} \, m v_k^2 - \frac{1}{2} \, m\omega^2 (x_k + v_k(t - t_k))^2 \right) \\ &= \frac{1}{2} \, m \left(v_k^2 \tau - \omega^2 \left(x_k^2 \tau + x_k v_k \tau^2 + \frac{1}{3} \, v_k^2 \tau^3 \right) \right) \; . \end{aligned}$$

We get the given formula when we insert

$$v_k = \frac{x_{k+1} - x_k}{\tau} \; .$$

3b) Only S_{k-1} and S_k depend on x_k. The equations to be satisfied are, for $k = 1, 2, \ldots, N-1$:

$$\frac{\partial S}{\partial x_k} = \frac{\partial S_{k-1}}{\partial x_k} + \frac{\partial S_k}{\partial x_k} = m\left(\frac{2x_k - x_{k-1} - x_{k+1}}{\tau} - \frac{\omega^2\tau}{6}\,(x_{k-1} + 4x_k + x_{k+1})\right) = 0\ .$$

3c) If $x(t) = \mathrm{e}^{\epsilon \mathrm{i}\lambda t}$ with $\epsilon = \pm 1$ then $x_{k\pm 1} = x_k\,\mathrm{e}^{\pm\epsilon \mathrm{i}\lambda\tau}$, and we get the single equation

$$\frac{2 - 2\cos(\lambda\tau)}{\tau} - \frac{\omega^2\tau}{6}\,(4 + 2\cos(\lambda\tau)) = 0\ ,$$

which we rewrite as

$$\cos(\lambda\tau) = \frac{6 - 2(\omega\tau)^2}{6 + (\omega\tau)^2}\ .$$

3d) We may write the same equation as

$$(\omega\tau)^2 = \frac{6(1 - \cos(\lambda\tau))}{2 + \cos(\lambda\tau)}\ .$$

When τ is small we may use the series expansion

$$\cos(\lambda\tau) = 1 - \frac{(\lambda\tau)^2}{2} + \frac{(\lambda\tau)^4}{24} + \cdots$$

which gives that

$$\begin{aligned}\frac{6(1 - \cos(\lambda\tau))}{2 + \cos(\lambda\tau)} &= \frac{3(\lambda\tau)^2 - \frac{(\lambda\tau)^4}{4} + \cdots}{3 - \frac{(\lambda\tau)^2}{2} + \cdots} = \left((\lambda\tau)^2 - \frac{(\lambda\tau)^4}{12} + \cdots\right)\left(1 + \frac{(\lambda\tau)^2}{6} + \cdots\right)\\ &= (\lambda\tau)^2 + \frac{(\lambda\tau)^4}{12} + \cdots\ .\end{aligned}$$

We see that we get $\lambda = \omega$ up to correction terms of order τ^2. Thus our method is of second order.

Not surprising, since the whole method is based on the linear approximation in equation (1) in the examination paper, and this has an error of second order. To get the next term in the approximation we would assume constant acceleration a_k and write

$$x(t) = x_k + v_k(t - t_k) + \frac{1}{2}\,a_k(t - t_k)^2\ .$$

4a) Given two events A and B with space and time coordinates x_A, t_A and x_B, t_B. We use the Lorentz transformation for the intervals $\Delta x = x_B - x_A$ and $\Delta t = t_B - t_A$,

$$\Delta x' = \gamma(\Delta x - V\Delta t)\ , \qquad \Delta t' = \gamma\left(\Delta t - \frac{V}{c^2}\,\Delta x\right),$$

with $V = 0.6\,c$ and

$$\gamma = \frac{1}{\sqrt{1 - 0.6^2}} = \frac{1}{0.8} = 1.25\ .$$

The inverse transformation is

$$\Delta x = \gamma(\Delta x' + V\Delta t')\ , \qquad \Delta t = \gamma\left(\Delta t' + \frac{V}{c^2}\,\Delta x'\right) .$$

Let event A be the start of the space ship from the Earth, and event B be the arrival at Sirius. Then obviously the observer on the Earth measures the distance to be

$$\Delta x = 9 \text{ light years}\ ,$$

and the travel time to be

$$\Delta t = \frac{\Delta x}{V} = \frac{9 \text{ light years}}{0.6\,c} = 15 \text{ years}\ .$$

It follows from the Lorentz transformation that $\Delta x' = 0$, this means that the observer on the space ship does not move relative to his reference frame. Of course.

The travel time measured by the traveller is, by the Lorentz transformation,

$$\Delta t' = \gamma\left(\Delta t - \frac{V}{c^2}\,\Delta x\right) = \gamma\left(\Delta t - \frac{V^2}{c^2}\,\Delta t\right) = \Delta t\sqrt{1-\frac{V^2}{c^2}} = 0.8\,\Delta t = 12 \text{ years}\ .$$

This is the time dilatation effect that the moving clock measures a shorter time than the properly synchronized stationary clocks.

One student gives an interesting argument to derive the time dilatation. Let us describe the process first in the inertial system where the Earth is at rest (or the solar system is at rest, we neglect the relative velocities of the Earth and the Sun). Imagine a light signal emitted at an angle θ from the direction between Earth and Sirius such that $\cos\theta = 0.6$. This light signal has a velocity component along the direction from Earth to Sirius which is $c\cos\theta = 0.6c$. While it travels 9 light years in this direction it travels 9 light years times $\tan\theta = 0.8/0.6 = 4/3$ in the orthogonal direction, that is 12 light years. The distance it travels, as seen from this inertial system, is $\sqrt{9^2+12^2} = 15$ light years. Obviously, since the time is 15 years.

Now see the process from the inertial system of the space ship. In this system the light travels a perpendicular distance which is the same, 12 light years. The direction of the light signal in this system is perpendicular to the direction from Earth to Sirius, and the speed of light in the direction it travels is c. Obviously, it takes the light signal 12 years to travel 12 light years. Hence 12 years is the travel time of the space ship from Earth to Sirius, as seen from the space ship.

4b) So what is the travel distance as measured by the traveller? From the point of view of the space ship the rest of the Universe moves with a speed of $V = 0.6\,c$ in the negative x' direction. During the 12 years of travel time the rest of the Universe, including the Earth, moves a distance of

$$0.6\,c \times 12 \text{ years} = 7.2 \text{ light years}\ .$$

This is the distance from the Earth to Sirius as measured by the observer on the space ship. It is 9 light years Lorentz contracted by the factor 0.8.

A more formal deduction using the Lorentz transformation may go as follows. We consider two events A on the Earth and B at Sirius, thus $\Delta x = x_B - x_A = 9$ light years, as before. But now we require that A and B are simultaneous as seen from the space ship, that is, $\Delta t' = 0$. Then by the inverse Lorentz transformation we get $\Delta x = \gamma \Delta x'$, or

$$\Delta x' = \frac{\Delta x}{\gamma} = 0.8 \times 9 \text{ light years} .$$

To find the length of the meter stick on the space ship as measured by observers at rest relative to the Earth, we reason in a similar way. Now we require that $\Delta x' = 1$ meter and $\Delta t = 0$. Then by the Lorentz transformation we get $\Delta x' = \gamma \Delta x$, or

$$\Delta x = \frac{\Delta x'}{\gamma} = 0.8 \text{ meter} .$$

These two results together look very much like a paradox. The earthly observer sees the space ship observer measuring a distance of 9 light years with a meter stick which is only 0.8 meter. The result of the measurement is 9 light years times 0.8, not 9 light years divided by 0.8, as one would expect.

4c) Now we take event A to be the emission of the light from Sirius and B to be the observation of the light on the Earth. Then $\Delta x = x_B - x_A = -9$ light years, and the travel time for the light, according to the observer on the Earth, is

$$\Delta t = \frac{-9 \text{ light years}}{-c} = 9 \text{ years} .$$

By the Lorentz transformation, the travel time for the light, according to the observer on the space ship, is

$$\begin{aligned} \Delta t' &= \gamma \left(\Delta t - \frac{V}{c^2}\,\Delta x \right) = \gamma \left(\Delta t - \frac{V}{c^2}\,(-c\,\Delta t) \right) = \gamma \left(1 + \frac{V}{c} \right) \Delta t \\ &= \sqrt{\frac{1+\frac{V}{c}}{1-\frac{V}{c}}}\ \Delta t = \sqrt{\frac{1+0.6}{1-0.6}}\ \Delta t = 18 \text{ years} . \end{aligned}$$

Another way to calculate the same answer is to note that to the observer on the space ship the distance between the light signal and the Earth decreases with a speed of $c - 0.6c = 0.4c$. According to the observer on the space ship, when the light starts from Sirius the distance to the Earth is 7.2 light years. Hence the travel time of the light signal is

$$\Delta t' = \frac{7.2 \text{ light years}}{0.4c} = 18 \text{ years} .$$

We may check the result once more as follows. The light takes 18 years to catch up with the Earth, because during that time the Earth moves another $0.6\,c \times 18$ years $=$ 10.8 light years away.

Finally, we take event A to be the emission of the light from the Sun and B to be the arrival of the light at Sirius. Then $\Delta x = x_B - x_A = 9$ light years, and the travel time for the light, according to the observer on the Earth, is

$$\Delta t = \frac{9 \text{ light years}}{c} = 9 \text{ years} .$$

By the Lorentz transformation, the travel time for the light, according to the observer on the space ship, is

$$\begin{aligned}\Delta t' &= \gamma\left(\Delta t - \frac{V}{c^2}\,\Delta x\right) = \gamma\left(\Delta t - \frac{V}{c^2}\,(c\,\Delta t)\right) = \gamma\left(1 - \frac{V}{c}\right)\Delta t \\ &= \sqrt{\frac{1-\frac{V}{c}}{1+\frac{V}{c}}}\,\Delta t = \sqrt{\frac{1-0.6}{1+0.6}}\,\Delta t = 4.5 \text{ years}\ .\end{aligned}$$

Another way to calculate the same answer again is that to the observer on the space ship the distance between the light signal and the Earth decreases with a speed of $c + 0.6c = 1.6c$. Hence the travel time of the light signal is

$$\Delta t' = \frac{7.2 \text{ light years}}{1.6c} = 4.5 \text{ years}\ .$$

We check this result also. According to the observer on the space ship, Sirius rushes to meet the light. During the 4.5 years the light is on its way, Sirius moves $0.6\,c \times 4.5$ years $=$ 2.7 light years closer, so that the light has to travel only 4.5 light years and not 7.2 light years.