

# Classical Mechanics TFY4345 - Exam 2015

1.

1a)

Particle position:

$$\vec{r} = \ell \cos \theta \vec{e}_x + \ell \sin \theta \vec{e}_y \quad (1)$$

Velocity:

$$\vec{v}^2 = \left( \frac{d\vec{r}}{dt} \right)^2 \quad (2)$$

$$= \dot{\ell}^2 + \ell^2 \dot{\theta}^2 \quad (3)$$

$$= \dot{\ell}^2 + \ell^2 \omega^2 \quad (4)$$

Lagrangian:

$$L = \frac{1}{2} m v^2 = \frac{1}{2} m (\dot{\ell}^2 + \ell^2 \omega^2) \quad (5)$$

1b)

Lagrange equation:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\ell}} = \frac{\partial L}{\partial \ell} \quad (6)$$

which implies:

$$\ddot{\ell} = \omega^2 \ell \quad (7)$$

The solution is:

$$\ell = A e^{\omega t} + B e^{-\omega t} \quad (8)$$

With initial conditions  $\ell(0) = \ell_0$  and  $\dot{\ell}(0) = 0$  the solution is:

$$\ell(t) = \frac{1}{2} \ell_0 (e^{\omega t} + e^{-\omega t}) = \ell_0 \cosh \omega t \quad (9)$$

1c) Lagrangian:

$$L = T - V = \frac{1}{2} m (\dot{\ell}^2 + \ell^2 \omega^2) - mg\ell \sin \omega t \quad (10)$$

Lagrange equation:

$$\ddot{\ell} = \omega^2 \ell - g \sin \omega t \quad (11)$$

1d)

$$\ell(t) = \frac{1}{2} \left( \ell_0 - \frac{g}{2\omega^2} \right) e^{\omega t} + \frac{1}{2} \left( \ell_0 + \frac{g}{2\omega^2} \right) e^{-\omega t} + \frac{g}{2\omega^2} \sin \omega t \quad (12)$$

1e)

If  $(\ell_0 - \frac{g}{2\omega^2}) < 0$  we must have that  $\ell(t) \rightarrow -\infty$  when  $t \rightarrow \infty$ . I.e. when  $\omega^2 < \frac{g}{2\ell_0}$   $\ell(t)$  must become equal to  $l = 0$  at some time  $t$ .

2.

2a)

Velocity:

$$v^2 = \dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2 \dot{\theta}^2 \quad (13)$$

Lagrangian:

$$L = T - V = \frac{1}{2} (\dot{r}^2 + r^2 \dot{\theta}^2) - \frac{1}{2} K r^2 \quad (14)$$

2b) The Lagrangian does not depend on  $\theta$ , which gives the conservation law:

$$\frac{\partial L}{\partial \dot{\theta}} = \text{const.} \quad (15)$$

Setting in  $L$  gives:

$$mr^2\dot{\theta} = \ell \quad (16)$$

where we call the constant  $\ell$  (not to be confused with the length in Problem 1).  $\ell$  is the angular momentum, similarly to the Kepler problem treated in the lectures and Brevik compendium.

2c)

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}Kr^2 \quad (17)$$

Inserting the equation for angular momentum conservation gives:

$$e = \frac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2} + \frac{K}{2}r^2 \quad (18)$$

Solving for  $\dot{r}$  and using the angular momentum law once more gives:

$$\frac{d\theta}{dr} = \frac{\ell}{mr^2\sqrt{\frac{2E}{m} - \frac{\ell^2}{m^2r^2} - \frac{K}{m}r^2}} \quad (19)$$

2d)

Substituting  $u = 1/r^2$  gives:

$$d\theta = -\frac{\ell}{2}\frac{du}{\sqrt{2Emu - \ell^2u^2 - Km}} \quad (20)$$

Using the integral given in the problem we find:

$$(\theta - \theta_0) = -\frac{1}{2}\arccos\left[-\frac{Em - \ell^2u}{\sqrt{E^2m^2 - \ell^2mK}}\right] + \frac{1}{2}\arccos\left[-\frac{Em - \ell^2u_0}{\sqrt{E^2m^2 - \ell^2mK}}\right] \quad (21)$$

This is equivalent to

$$\cos(2\theta + c) = \frac{Em - \ell^2u}{\sqrt{E^2m^2 - \ell^2mK}} \quad (22)$$

where  $c$  is a constant. which gives

$$r^2 = \frac{\ell^2/(Em)}{1 + \sqrt{1 - \frac{\ell^2K}{E^2m}}\cos(2\theta + c)} \quad (23)$$

We may chose  $\theta = 0$  as we like, setting  $c = 0$  correspond to taking  $r(\theta = 0) = r_{min}$ .

$$r^2 = \frac{\ell^2/(Em)}{1 + \sqrt{1 - \frac{\ell^2K}{E^2m}}\cos(2\theta)} \quad (24)$$

i.e.

$$A = \ell^2/(Em) \quad (25)$$

$$B = \sqrt{1 - \frac{\ell^2K}{E^2m}} \quad (26)$$

2e) Inserting Eq (24) into  $x^2/a^2 + y^2/b^2 = 1$  gives :

$$a^2 = \frac{A^2}{1+B} \quad (27)$$

$$b^2 = \frac{A^2}{1-B} \quad (28)$$

### 3.

3a)

$$\begin{bmatrix} L'_x \\ L'_y \\ L'_z \end{bmatrix} = BCD \begin{bmatrix} 0 \\ 0 \\ L \end{bmatrix} = BC \begin{bmatrix} 0 \\ 0 \\ L \end{bmatrix} = B \begin{bmatrix} 0 \\ L \sin \theta \\ L \cos \theta \end{bmatrix} = \begin{bmatrix} L \sin(\psi) \sin(\theta) \\ L \cos(\psi) \sin(\theta) \\ L \cos(\theta) \end{bmatrix} \quad (29)$$

3b)

Inserting the expressions for the angular velocity in the body frame (rotating body) into the equation of motion gives (components):

$$L \sin \psi \sin \theta = I_1 \left[ \dot{\varphi} \sin \theta \sin \psi + \dot{\theta} \cos \psi \right] \quad (30)$$

$$L \cos \psi \sin \theta = I_2 \left[ \dot{\varphi} \sin \theta \cos \psi - \dot{\theta} \sin \psi \right] \quad (31)$$

$$L \cos \theta = I_3 \left[ \dot{\varphi} \cos \theta + \dot{\psi} \right] \quad (32)$$

We may rewrite these equations on the following form:

$$L = I_1 \left[ \dot{\varphi} + \dot{\theta} \frac{\cos \psi}{\sin \psi} \right] \quad (33)$$

$$L = I_2 \left[ \dot{\varphi} - \dot{\theta} \frac{\sin \psi}{\cos \psi} \right] \quad (34)$$

$$L \cos \theta = I_3 \left[ \dot{\varphi} \cos \theta + \dot{\psi} \right] \quad (35)$$

When  $I_1 = I_2$  we may subtract the two first equations above, and obtain:

$$\dot{\theta} \left[ \frac{\cos \psi}{\sin \psi} + \frac{\sin \psi}{\cos \psi} \right] = 0 \quad (36)$$

This implies

$$\dot{\theta} = 0 \quad (37)$$

The remaining equations now decouple, and the solution is simple:

$$\dot{\varphi} = \frac{L}{I_1} \quad (38)$$

$$\dot{\psi} = L \left( \frac{1}{I_3} - \frac{1}{I_1} \right) \cos \theta \quad (39)$$

An alternative (and accepted) solution is to plug the suggested solution (problem text) into the equations of motion, and find the constants  $c_1, c_2, c_3$ .

3c)

$$(\omega_{x'}, \omega_{y'}) = (\dot{\varphi} \sin \theta \cos \psi, \dot{\varphi} \sin \theta \cos \psi) = \frac{L}{I_1} \sin \theta \left( \sin(\dot{\psi}t), \cos(\dot{\psi}t) \right) \quad (40)$$

which is a rotating vector with (constant) angular velocity  $\Omega = \dot{\psi} = L \left( \frac{1}{I_3} - \frac{1}{I_1} \right) \cos \theta$

3d) The Euler equation free body (no torque):

$$\left( \frac{d\vec{L}}{dt} \right)_{body} + \vec{\omega} \times \vec{L} = 0 \quad (41)$$

In the solution above (Eq. 37)  $\theta = const.$ , which implies that the time derivative in Eq. (41) is:

$$\left( \frac{d\vec{L}}{dt} \right)_{body} = \frac{d}{dt} \begin{bmatrix} L \sin(\psi) \sin(\theta) \\ L \cos(\psi) \sin(\theta) \\ L \cos(\theta) \end{bmatrix} = \begin{bmatrix} L \dot{\psi} \cos(\psi) \sin(\theta) \\ -L \dot{\psi} \sin(\psi) \sin(\theta) \\ 0 \end{bmatrix} \quad (42)$$

The second term in the Euler equation Eq. (41) is:

$$\vec{\omega} \times \vec{L} = \begin{bmatrix} \omega_{y'} L_{z'} - \omega_{z'} L_{y'} \\ \omega_{z'} L_{x'} - \omega_{x'} L_{z'} \\ \omega_{x'} L_{y'} - \omega_{y'} L_{x'} \end{bmatrix} \quad (43)$$

The x' component of Eq. (43) is

$$\omega_{y'} L_{z'} - \omega_{z'} L_{y'} = \dot{\varphi} \sin(\theta) \sin(\psi) L \cos(\theta) - (\dot{\varphi} \cos(\theta) + \dot{\psi}) L \sin(\theta) \cos(\psi) \quad (44)$$

$$= -L \dot{\psi} \cos(\psi) \sin(\theta) \quad (45)$$

The y' component:

$$\omega_{z'} L_{x'} - \omega_{x'} L_{z'} = L \dot{\psi} \sin(\psi) \sin(\theta) \quad (46)$$

The z' component is zero:

$$\omega_{x'} L_{y'} - \omega_{y'} L_{x'} = \dot{\varphi} \sin(\theta) \sin(\psi) L \sin(\theta) \cos(\psi) - \dot{\varphi} \sin(\theta) \cos(\psi) L \sin(\theta) \sin(\psi) \quad (47)$$

$$= 0 \quad (48)$$

This shows that the solution found in 3b) satisfies the Euler equation for a free body (as it should).

**4.**

4a)

We calculate  $\frac{du_z}{dt}$  using the Lorentz transform, ie.

$$\frac{du_z}{dt} = \frac{dz' + v dt'}{dt' + \frac{v}{c^2} dz'} \quad (49)$$

which gives

$$u_x = \frac{u'_x}{\gamma \left( 1 + \frac{v u'_z}{c^2} \right)} \quad (50)$$

$$u_y = \frac{u'_y}{\gamma \left( 1 + \frac{v u'_z}{c^2} \right)} \quad (51)$$

$$u_z = \frac{u'_z + v}{1 + \frac{u'_z v}{c^2}} \quad (52)$$

$$(53)$$

4b)

Let  $u = u_z$ . We have  $u' = c/n$

$$u = \frac{c}{n} \frac{1 + n\beta}{1 + \beta/n} \quad (54)$$

which gives

$$u \approx \frac{c}{n} + v\left(1 - \frac{1}{n^2}\right) \quad (55)$$