Classical Mechanics TFY4345 - Exam 2016

1.

(1a)

Position of the center of the wheel:

$$\begin{aligned} X &= b\theta && (1) \\ Y &= b && (2) \end{aligned}$$

Position of point mass m:

$$\begin{aligned} x &= b\theta - b\sin\theta && (3) \\ y &= b - b\cos\theta && (4) \end{aligned}$$

Velocity of mass M:

$$V^2 = \dot{X}^2 + \dot{Y}^2 = b^2\dot{\theta}^2 \tag{5}$$

Velocity of mass m:

$$v^2 = \dot{x}^2 + \dot{y}^2 = b^2\dot{\theta}^2(1-\cos\theta)^2 + b^2\dot{\theta}^2\sin^2\theta = 2b^2\dot{\theta}^2(1-\cos\theta) \tag{6}$$

(1b)

Lagrangian:

$$\begin{aligned} L &= \frac{1}{2}mv^2 + \frac{1}{2}MV^2 - mgy && (7) \\ &= mb^2\dot{\theta}^2(1-\cos\theta) + \frac{1}{2}Mb^2\dot{\theta}^2 - mgb(1-\cos\theta) && (8) \end{aligned}$$

(1c)

Lagrange equation:

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = \frac{\partial L}{\partial\theta} \tag{9}$$

Inserting equation (8) into (9) gives the equation of motion:

$$2mb\ddot{\theta}(1-\cos\theta) + Mb\ddot{\theta} + mb\dot{\theta}^2\sin\theta + mg\sin\theta = 0 \tag{10}$$

(1d)

Linearising equation (10) gives:

$$\ddot{\theta} + \frac{mg}{Mb}\theta = 0 \tag{11}$$

which gives harmonic oscillations with frequency:

$$\omega = \sqrt{\frac{mg}{Mb}} \tag{12}$$

2a)

$$L = T - V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{K}{r^6} \tag{13}$$

2b) The Lagrangian does not depend on θ:

$$\frac{\partial L}{\partial\theta} = 0 \tag{14}$$

This implies the conservation law:

$$\frac{\partial L}{\partial \dot{\theta}} = \text{constant} \tag{15}$$

$$mr^2\dot{\theta} = \text{constant} = \ell \tag{16}$$

The conserved quantity is the angular momentum of the particle.
2c) We may derive the equation of motion from the conservation of angular momentum and from the Lagrange equation:

$$\frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0 \tag{17}$$

It is however easier to derive the equation of motion from conservation of total energy (as in the Kepler problem). The total energy is:

$$E = T + V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{K}{r^6} \tag{18}$$

We can eliminate $\dot{\theta}$ by using Eq. (16) on conservation angular momentum conservation:

$$E = \frac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2} - \frac{K}{r^6} \tag{19}$$

Solving for $\dot{r}$ and using the angular momentum law once more gives:

$$\frac{dr}{d\theta} = \frac{mr^2}{\ell}\sqrt{\frac{2E}{m} - \frac{\ell^2}{m^2r^2} + \frac{2K}{mr^6}} \tag{20}$$

2d) We may express the left term in Eq. (20) as $\frac{dr}{d\theta} = \frac{1}{2r}\frac{d}{d\theta}(r^2)$. Inserting $r^2 = c^2\cos 2\theta$ in Eq. (20) we find after some algebra that:

$$(c^4\ell^2 - 2Km) - 2c^6mE\cos^3(2\theta) = 0 \tag{21}$$

This implies that $r = c\sqrt{\cos 2\theta}$ is a solution provided that

$$E = 0 \tag{22}$$

$$c^4 = \frac{2mK}{\ell^2} \tag{23}$$

i.e. the total energy is $E = 0$.
2e) Combining the solution $r = c\sqrt{\cos 2\theta}$ with the equation of conservation of angular momentum gives:

$$\ell = mr^2\frac{\mathrm{d}\theta}{\mathrm{d}t} \Rightarrow \ell\mathrm{d}t = mc^2\cos(2\theta)\mathrm{d}\theta \tag{24}$$

Integrating this equation gives

$$\ell t = mc^2\sin(2\theta) \tag{25}$$

Which gives:

$$\theta(t) = \frac{1}{2}\arcsin\left(\frac{2\ell}{mc^2}t\right) \tag{26}$$

2f) At time $t = 0$ the particle starts in the position $r = c$ and $\theta = 0$. The particle moves towards the origin, corresponding to the position $r = 0$, which implies $\theta = \frac{\pi}{4}$. Therefore $0 < \theta < \frac{\pi}{4}$ corresponds to a quarter of a orbit (orbital period τ) This implies

$$\frac{2\ell}{mc^2}\frac{\tau}{4} = 1 \tag{27}$$

according to Eq. (26). The total orbital period is:

$$\tau = \frac{2mc^2}{\ell} = \frac{2m}{\ell^2}\sqrt{2mK} \tag{28}$$

3.

3a)

The Euler equation free body (no torque):

$$\left(\frac{d\vec{L}}{dt}\right)_{body} + \vec{\omega} \times \vec{L} = 0 \tag{29}$$

From Eq. (29) we find the Euler equation on component form:

$$I_1\dot{\omega}_{x'} + \omega_{y'}\omega_{y'}(I_3 - I_2) = 0 \tag{30}$$
$$I_2\dot{\omega}_{y'} + \omega_{x'}\omega_{z'}(I_1 - I_3) = 0 \tag{31}$$
$$I_2\dot{\omega}_{z'} + \omega_{x'}\omega_{y'}(I_2 - I_1) = 0 \tag{32}$$

Angular velocities in body frame:

$$\omega_{x'} = \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \tag{33}$$
$$\omega_{y'} = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \tag{34}$$
$$\omega_{z'} = \dot{\varphi}\cos\theta + \dot{\psi} \tag{35}$$

3b) When $I_1 = I_2$ Eq. (32) implies that $\omega_{z'}$ = constant. This again implies that the angular momentum around z' axis body frame is constant, i.e. $L'_z = L\cos\theta$ = constant, which implies

$$\theta = \text{constant} \tag{36}$$

, i.e. $\dot{\theta} = 0$. Eq. (30) and (31) can now be expressed as:

$$\dot{\omega}_{x'} = -\Omega\omega_{y'} \tag{37}$$
$$\dot{\omega}_{y'} = \Omega\omega_{x'} \tag{38}$$

with $\Omega = \frac{I_3 - I_1}{I_1}\omega_{z'}$ = constant. This implies [1] that $\omega_{x'}^2 + \omega_{y'}^2$ = constant. Which in turn implies:

$$\omega_{x'}^2 + \omega_{y'}^2 = (\dot{\varphi}\sin\theta\sin\psi)^2 + (\dot{\varphi}\sin\theta\cos\psi)^2 = \dot{\varphi}^2\sin\theta = \text{constant} \tag{39}$$

We thus find that:

$$\dot{\varphi} = c_1 \tag{40}$$

where c_1 is a constant. Inserting Eq. (33) and (34) into Eq. (37) and (38) gives

$$\dot{\varphi}\dot{\psi}\sin\theta\cos\psi = -\Omega\dot{\varphi}\sin\theta\cos\psi \tag{41}$$
$$-\dot{\varphi}\dot{\psi}\sin\theta\sin\psi = \Omega\dot{\varphi}\sin\theta\sin\psi \tag{42}$$

This implies that

$$\dot{\psi} = -\Omega = -\frac{(I_3 - I_1)}{I_1}\omega'_z = \left(\frac{1}{I_3} - \frac{1}{I_1}\right)L\cos\theta \tag{43}$$

[1] Using Eq. (37) and (38) we find: $\frac{\mathrm{d}}{\mathrm{dt}}\left[\omega_{x'}^2 + \omega_{y'}^2\right] = 2\omega_{x'}\dot{\omega}_{x'} + 2\omega_{y'}\dot{\omega}_{y'} = -2\Omega\omega_{x'}\omega_{y'} + 2\Omega\omega_{y'}\omega_{x'} = 0$ which shows that $\omega_{x'}^2 + \omega_{y'}^2$ does not change with time.

Inserting Eq. (43) into Eq. (35) gives

$$\dot{\phi} = \frac{L}{I_1} \tag{44}$$

3c)

$$\frac{\dot{\varphi}}{\omega'_z} = \frac{\dot{\varphi}}{L\cos(\theta)/I_3} = \frac{L/I_1}{L\cos(\theta)/I_3} = \frac{I_3}{I_1\cos\theta} \approx \frac{I_3}{I_1} = 2 \tag{45}$$

4a)

Using the Lorentz transformation for the endpoints of the rod gives:

$$z'_2 - z'_1 = \gamma(z_2 - vt) - \gamma(z_1 - vt) = \gamma(z_2 - z_1) \tag{46}$$

Which implies

$$L = \frac{L'}{\gamma} = L'\sqrt{1 - \frac{v^2}{c^2}} \tag{47}$$

4b)

$$L_z = \frac{L'_z}{\gamma} = \frac{L'}{\gamma}\cos\theta_0 \tag{48}$$

$$L_y = L'_y = L'\sin\theta_0 \tag{49}$$

$$L^2 = L_y^2 + L_z^2 = L'^2\left(1 - \frac{v^2}{c^2}\cos^2\theta_0\right) \tag{50}$$

4c)

$$\frac{L'_y}{L'_z} = \frac{L_y}{\gamma L_z} = \tan\theta_0 \tag{51}$$

which implies:

$$\tan\theta = \frac{L_y}{L_z} = \gamma\tan\theta_0 = \frac{\tan\theta_0}{\sqrt{1 - v^2/c^2}} \tag{52}$$