Classical Mechanics TFY4345 - Exam 2016

1. Motion of a particle confined to a surface

1a)

Position of the particle on the cylinder:

$$\vec{r} = R\cos(\phi)\vec{e}_x + R\sin(\phi)\vec{e}_y + z\vec{e}_z \tag{1}$$

Velocity:

$$\begin{aligned} \vec{v}^2 &= \left(\frac{d\vec{r}}{dt}\right)^2 && (2) \\ &= R^2\dot{\phi}^2 + \dot{z}^2 && (3) \\ & && (4) \end{aligned}$$

Potential energy:

$$V = mgz \tag{5}$$

Lagrangian:

$$L = T - V = \frac{1}{2}mv^2 - mgz = \frac{1}{2}m\left(R^2\dot{\phi}^2 + \dot{z}^2\right) - mgz \tag{6}$$

1b)

Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \phi} \tag{7}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{z}} = \frac{\partial L}{\partial z} \tag{8}$$

which implies the conservation law :

$$\frac{d}{dt}\left(mR^2\dot{\phi}\right) = mR^2\ddot{\phi} = 0 \tag{9}$$

This implies that the angular velocity is a constant $\dot{\phi} = \omega_0 =$ constant. Equation of motion for the z-coordinate:

$$m\ddot{z} = -mg \tag{10}$$

1c) At time $= 0$ the velocity in the z-direction is zero, and the total velocity is then $v = v_0 = \omega_0 R$, where ω_0 is constant. The solution for the ϕ and z coordinates:

$$\phi = \omega_0 t \tag{11}$$

$$z = z_0 - \frac{1}{2}gt^2 \tag{12}$$

This is a helice where the pitch increases with time.

$$x = R\cos(\omega_0 t) \tag{13}$$

$$y = R\sin(\omega_0 t) \tag{14}$$

$$z = z_0 - \frac{1}{2}gt^2 \tag{15}$$

1d) Spherical coordinates for a particle confined to the surface of a cone:

$$x = r\sin(\theta)\cos(\phi) \tag{16}$$
$$y = r\sin(\theta)\sin(\phi) \tag{17}$$
$$z = r\cos(\theta) \tag{18}$$

The motion of the particle on the cone can be described by these spherical coordinates setting $\theta = \alpha =$ constant.
Lagrangian for the particle on the cone :

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V \tag{19}$$
$$= \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\phi}^2\sin^2(\alpha)\right] - mgr\cos(\alpha) \tag{20}$$

1e)
Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \phi} \tag{21}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r} \tag{22}$$

Inserting the Lagrangian in equation (20) into equations (21) and (22) gives :

$$mr^2\dot{\phi} = \text{constant} = \ell \tag{23}$$

$$m\ddot{r} = mr\dot{\phi}^2\sin^2(\alpha) - mg\cos(\alpha) \tag{24}$$

1f) The angular frequency of the particle is $\omega = \dot{\phi}$, stable motion implies $\ddot{r} = 0$, and equation (24) implies:

$$\omega = \dot{\phi} = \sqrt{\frac{g\cos(\alpha)}{r\sin^2(\alpha)}} = \sqrt{\frac{g\cos(\alpha)}{R\sin(\alpha)}} \tag{25}$$

1g) Combining the conservation law and equation of motion gives:

$$\ddot{r} = -g\cos(\alpha) + \frac{\ell^2}{r^3\sin^2(\alpha)} \tag{26}$$

We now look at small perturbation around the stable motion: $r = r_0 + \delta(t)$, where r_0 is the radius of the stable motion in question 1f), i.e. $r_0 = R/\sin(\alpha)$ Expanding to linear order in δ gives

$$\ddot{\delta} = -\frac{3\ell^2}{r_0^4\sin^2(\alpha)}\delta = -3\sin^2(\alpha)\omega^2\delta \tag{27}$$

Which means that the frequency of the oscillatory motion is:

$$\Omega = \sqrt{3}\sin(\alpha)\omega \tag{28}$$

We therefore get that $\omega = \Omega$ when $\sin\alpha = \frac{1}{\sqrt{3}}$, i.e. $\alpha \approx 35,2^\circ$

2. Particle in a central force potential
2a)

$$\frac{d}{dt}\vec{L} = \vec{N} \tag{29}$$

Central force implies zero torque:

$$\vec{N} = \vec{F} \times \vec{r} = -\nabla V(r) \times \vec{r} = -V'(r)\vec{e}_r \times \vec{r} = 0 \tag{30}$$

This implies $\frac{d}{dt}\vec{L} =$, i.e. conservation of angular momentum. Which means that the particle stays in the same plane.
2b) Velocity:

$$v^2 = \dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\phi}^2 \tag{31}$$

Lagrangian:

$$L = T - V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + \frac{k}{r^4} \tag{32}$$

L does not depend on ϕ, which gives the conservation law

$$\frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} = constant = \ell \tag{33}$$

This is the angular momentum. In addition the Lagrangian does not depend on time, which implies conservation of energy.
2c)
Lagrange equation for r coordinate:

$$m\ddot{r} = mr\dot{\phi}^2 - 4\frac{k}{r^5} \tag{34}$$

Using the conservation of angular momentum:

$$m\ddot{r} = mr\dot{\phi}^2 - 4\frac{k}{r^5} = \frac{\ell^2}{mr^3} - 4\frac{k}{r^5} \tag{35}$$

This is equivalent to a 1D problem:

$$m\ddot{r} = -\frac{dV_{\text{eff}}}{dr} \tag{36}$$

$$V_{\text{eff}} = \frac{\ell^2}{2mr^2} - \frac{k}{r^4} \tag{37}$$

2d)

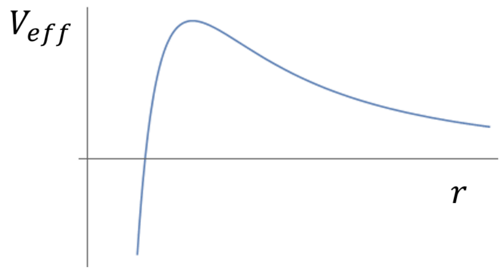

Figure 1: Sketch of potential $V_{\rm eff} = \frac{\ell^2}{2mr^2} - \frac{k}{r^4}$ as a function of r. For small r the negative term $-\frac{k}{r^4}$ dominates, for large r the postive term $\frac{\ell^2}{2mr^2}$ dominates.

2e) r = constant implies $\ddot{r} = 0$. Equation (36) hence implies $\frac{dV_{\rm eff}}{dr}(r) = 0$, which has the solution $r_0 = \frac{2\sqrt{km}}{\ell}$.

2f) The orbits $r = r_0$ = constant are unstable, since $V_{\rm eff}$ is maximum at r_0. Any small displacement from r_0 will result in a gain potential energy, and the particle will move away from the orbit.

3. Cone rolling on a plane

3a) Velocity of centre of mass:

$$V_{\rm cm} = \ell \cos(\alpha)\dot{\phi} \tag{38}$$

3b) Cone rotational motion is effectively a rotation around the instantaneous axis OA. The angular velocity around OA is

$$\omega = \frac{V_{\rm cm}}{\ell \sin \phi} = \frac{\cos(\alpha)}{\sin(\alpha)}\dot{\phi} \tag{39}$$

3c) Let (x_1, x_2, x_3) be the coordinate system aligned with the principal axes of the cone. The projection of $\vec{\omega}$ on these axes is:

$$\vec{\omega} = \omega_1 \vec{e}_{x_1} + \omega_2 \vec{e}_{x_2} + \omega_3 \vec{e}_{x_3} = \omega \sin(\alpha)\vec{e}_{x_1} + \omega \cos(\alpha)\vec{e}_{x_3} \tag{40}$$

Component of ω along the x_3 axis:

$$\omega_3 = \omega \cos\alpha = \frac{\cos^2(\alpha)}{\sin(\alpha)}\dot{\phi} \tag{41}$$

3d)

$$\begin{aligned} T &= \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_3 \omega_3^2 && (42)\\ &= \frac{1}{2} I_1 \left[\omega \sin(\alpha)\right]^2 + \frac{1}{2} I_3 \left[\omega \cos(\alpha)\right]^2 && (43)\\ &= \frac{3}{40} H^2 m \dot{\phi}^2 \left[1 + 5\cos^2(\alpha)\right] && (44) \end{aligned}$$

4. Light from a fluorescent tube

4a)

In the S frame the tube lights up at the point z at time t. Seen from coordinate system S' :

$$\begin{aligned} z' &= \gamma(z - vt) && (45)\\ t' &= \gamma(t - \frac{vz}{c^2}) && (46) \end{aligned}$$

In the S frame the tube lights up at the point $z + \Delta z$ at time t. Seen from coordinate system S' :

$$\begin{aligned} z' + \Delta z' &= \gamma(z + \Delta z - vt) && (47) \\ t' + \Delta t' &= \gamma(t - \frac{v(z + \Delta z)}{c^2}) && (48) \end{aligned}$$

4b) From 4a) we get that:

$$\begin{aligned} \Delta z' &= \gamma \Delta z && (49) \\ \Delta t' &= -\frac{\gamma v \Delta z}{c^2} && (50) \end{aligned}$$

Seen from S' the fluroscent tube does not light up instantaneously everwhere (like in S), the lighting up propagates with the velocity:

$$u = \frac{\Delta z'}{\Delta t'} = \frac{\gamma \Delta z}{-\frac{\gamma v \Delta z}{c^2}} = -\frac{c^2}{v} \quad (51)$$