Exam: 12.12.2017 — Solutions

Problem 1.

a) Relativistic Doppler effect: $\nu = \frac{\sqrt{1 \pm \beta}}{\sqrt{1 \mp \beta}} \nu_0$

Mary: $v = 0.8c$, $\nu_0 = 1$ signal / year

Away: $\beta = -0.8 \implies \nu = \frac{\sqrt{1-0.8}}{\sqrt{1+0.8}} \nu_0 = \frac{\nu_0}{3} \longrightarrow$ 3 years

Return: $\beta = 0.8 \implies \nu_0 = \frac{\sqrt{1+0.8}}{\sqrt{1-0.8}} \nu_0 = 3\nu_0 \longrightarrow$ 4 months

[Frank receives message signals every 3 years until 18 years have passed. Then he receives 6 more signals every 4 months. Mary is already on her way back once Frank observes the change.]

b)

$\bar{x} = (x_1, x_2, x_3, ict)$ $\qquad \bar{v} = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$

$dx_\mu = (dx, dy, dz, icdt)$

$ds^2 = dx_\mu dx_\mu = dx^2 + dy^2 + dz^2 - c^2dt^2 \leftarrow$ invariant

$-cd\tau = dx^2 + dy^2 + dz^2 - c^2dt^2 = (v^2 - c^2)dt^2$

rest frame

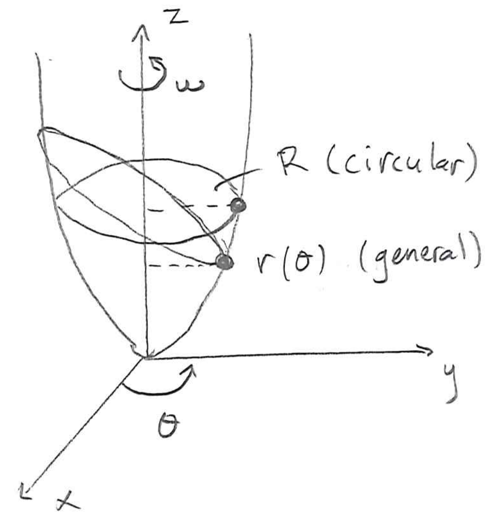
$\implies dt = \frac{d\tau}{\sqrt{1-\beta^2}} = \gamma d\tau$

Now: $u_\mu = \frac{dx_\mu}{d\tau} = \left(\frac{dx_i}{d\tau}, ic\frac{dt}{d\tau}\right) = \gamma(\bar{v}, ic)$

proper time

Problem 2.

a) $z = cr^2 \longleftrightarrow V = 0$ at $z = 0$



Coordinates: r, θ, z ← not independent

$$T = \frac{m}{2}\left(\dot{r}^2 + \dot{z}^2 + (r\dot{\theta})^2\right)$$

$$V = mgz$$

$$z = cr^2 \longrightarrow \dot{z} = 2c\dot{r}r$$

$$\theta = \omega t \longrightarrow \dot{\theta} = \omega$$

$$L = T - V = \frac{m}{2}\left(\dot{r}^2 + 4c^2\dot{r}^2 r^2 + r^2\omega^2\right) - mgcr^2$$

System is holonomic because the constraint $z = cr^2$ depends only on generalized coordinates

b) $$\frac{\partial L}{\partial \dot{r}} = \frac{m}{2}\left(2\dot{r} + 8c^2 r^2 \dot{r}\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{m}{2}\left(2\ddot{r} + 16c^2 r\dot{r}^2 + 8c^2 r^2 \ddot{r}\right)$$

$$\frac{\partial L}{\partial r} = m\left(4c^2 r\dot{r}^2 + r\omega^2 - 2gcr\right)$$

$$\Longrightarrow \ddot{r}\left(1 + 4c^2 r^2\right) + \dot{r}^2\left(4c^2 r\right) + r\left(2gc - \omega^2\right) = 0 \quad (*)$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \omega} = mr^2\omega =: l; \qquad \frac{\partial L}{\partial \theta} = 0 \Longrightarrow \underline{\theta \text{ cyclic!}}$$

$$\Longrightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt} l = 0$$ ← trivial

c) $$r = R \longrightarrow \dot{r} = 0, \ \ddot{r} = 0$$

$$\overset{(*)}{\Longrightarrow} R\left(2gc - \omega^2\right) = 0 \Longrightarrow c = \frac{\omega^2}{2g}$$

Problem 3.

a) $T = \frac{1}{2} m b^2 \dot{\theta}^2 + \frac{1}{2} m b^2 \sin^2\theta \, \dot{\phi}^2$ (coordinates θ, ϕ)

$V = -mgb\cos\theta$ (zero-level at the attachment point)

$$p_\theta = \frac{\partial L}{\partial \dot\theta} = mb^2 \dot\theta \longrightarrow \dot\theta = \frac{p_\theta}{mb^2}$$

$$p_\phi = \frac{\partial L}{\partial \dot\phi} = mb^2 \sin^2\theta \cdot \dot\phi \longrightarrow \dot\phi = \frac{p_\phi}{mb^2\sin^2\theta}$$

$$H = T + V = \frac{p_\theta^2}{2mb^2} + \frac{p_\phi^2}{2mb^2\sin^2\theta} - mgb\cos\theta$$

$$\begin{cases} \dot\theta = \dfrac{\partial H}{\partial p_\theta} = \dfrac{p_\theta}{mb^2} \\ \dot\phi = \dfrac{\partial H}{\partial p_\phi} = \dfrac{p_\phi}{mb^2\sin^2\phi} \end{cases} + \begin{cases} \dot p_\theta = -\dfrac{\partial H}{\partial \theta} = \dfrac{p_\phi^2\cos\theta}{mb^2\sin^3\theta} - mgb\sin\theta \\ \dot p_\phi = -\dfrac{\partial H}{\partial \phi} = 0 \end{cases}$$

b) Because ϕ is cyclic $\Rightarrow$ p_ϕ is constant

$H = T + V = E$ conserved

c) General relation: $\dfrac{dF}{dt} = \dfrac{\partial F}{\partial t} + [F, H]_{q,p}$

Now: $\dfrac{\partial T}{\partial t} = 0$ (no explicit time-dependence)

$$[T,H]_{p,q} = \sum_i \left(\frac{\partial T}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial T}{\partial p_i}\frac{\partial H}{\partial q_i} \right) \qquad \left(T = \frac{p_\theta^2}{2mb^2} + \frac{p_\phi^2}{2mb^2\sin^2\theta} \right)$$

$$= \frac{\partial T}{\partial \theta}\frac{\partial H}{\partial p_\theta} - \frac{\partial T}{\partial p_\theta}\frac{\partial H}{\partial \theta} + \underbrace{\frac{\partial T}{\partial \phi}}_{\to 0}\frac{\partial H}{\partial p_\phi} - \frac{\partial T}{\partial p_\phi}\underbrace{\frac{\partial H}{\partial \phi}}_{\to 0}$$

$$= -\frac{p_\phi^2\cos\theta}{mb^2\sin^3\theta}\cdot\frac{p_\theta}{mb^2} - \frac{p_\theta}{mb^2}\left(-\frac{p_\phi^2\cos\theta}{mb^2\sin^3\theta} + mgb\sin\theta \right)$$

$$= -\frac{p_\theta}{mb^2}\cdot mgb\sin\theta = -\dot\theta \cdot mgb\sin\theta$$

T constant of motion if $[T,H]_{q,p} = 0$ ←

If $\theta = \text{constant} \longrightarrow \dot\theta = 0 \longrightarrow p_\theta = 0 \longrightarrow$ circular orbit

Problem 4.

up & down

a) Particle thrown upwards vertically $\rightarrow h$ $(\theta > 0)$

$"\theta" = "\lambda"$ latitude

$$\bar{F}_{eff} = \bar{F} + 2m\omega \bar{v}_r \times \bar{\omega} - m\bar{\omega} \times (\bar{\omega} \times \bar{r})$$

centripetal force

$$\hookrightarrow \bar{a}_r = \bar{g} - 2\bar{\omega} \times \bar{v}_r$$

coriolis

$\bar{g} = -g\hat{e}_z$

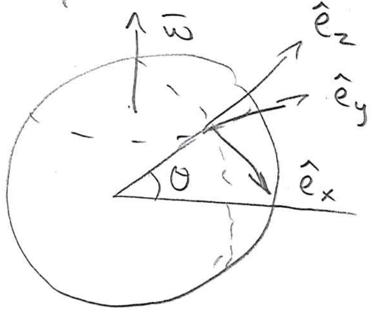
$$\begin{cases} \dot{x} \approx 0 \\ \dot{y} \approx 0 \\ \dot{z} \approx v_0 - gt \end{cases}$$

$$\bar{\omega} \times \bar{v}_r = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ -\omega\cos\theta & 0 & \omega\sin\theta \\ 0 & 0 & v_0 - gt \end{vmatrix}$$

$\omega_x' = -\omega\cos\theta$

$\omega_y' = 0$

$\omega_z' = \omega\sin\theta$



$$\omega \times \bar{v}_r = +\omega\cos\theta\,(v_0 - gt)\,\hat{e}_y$$

$$\bar{a}_r = -2\omega\,(v_0 - gt)\cos\theta\,\hat{e}_y - g\hat{e}_z$$

b) Displacement: $\frac{dy^2}{dt^2} = -2\omega\,(v_0 - gt)\cos\theta \Rightarrow \frac{dy}{dt} = -2\omega\,(v_0 t - \frac{1}{2}gt^2)\cos\theta$

$$\Rightarrow y(t) = -2\omega\left(\tfrac{1}{2}v_0 t^2 - \tfrac{1}{6}gt^3\right)\cos\theta = \left(\tfrac{1}{3}gt^3 - v_0 t^2\right)\omega\cos\theta$$

$$z(t) = \underbrace{z(0)}_{0} + v_0 t - \tfrac{1}{2}gt^2 = 0 \Rightarrow v_0 = \tfrac{1}{2}gt$$

$$\Rightarrow y(t) = -\tfrac{1}{6}gt^3 \cdot \omega\cos\theta$$

$$z(\tfrac{1}{2}t) = h = \tfrac{1}{2}g(\tfrac{1}{2}t)^2 \Rightarrow t = 2\sqrt{\tfrac{2h}{g}}$$

$$\Rightarrow \underline{y(t)} = -\tfrac{1}{6}g \cdot 8\left(\tfrac{2h}{g}\right)^{3/2}\cos\theta \cdot \omega = \underline{-\tfrac{4}{3}\omega\cos\theta\sqrt{\tfrac{8h^3}{g}}}$$

$\therefore$ The particle lands in the west from the launching point $\leftarrow$ $\bar{\omega} \times \bar{v}_r(t)$ is the cause

Problem 5.

a) $T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} \cdot nm \dot{x}_2^2$

$V = \frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2 + \frac{1}{2} k_{12} x_1 x_2 \rightarrow A_{11} = \left.\frac{\partial^2 V}{\partial x_1^2}\right|_0 = k + k_{12}$ etc.

$$\bar{\bar{A}} = \begin{bmatrix} k+k_{12} & -k_{12} \\ -k_{12} & k+k_{12} \end{bmatrix} \leftarrow V = \frac{1}{2} \sum_{j,k} A_{jk} x_j x_k$$

$$T = \frac{1}{2} \sum_{j,k} m_{jk} \dot{x}_j \dot{x}_k \longrightarrow \bar{\bar{m}} = \begin{bmatrix} m & 0 \\ 0 & n \cdot m \end{bmatrix}$$

$$\det(A_{jk} - \omega^2 m_{jk}) = \begin{vmatrix} k + k_{12} - m\omega^2 & -k_{12} \\ -k_{12} & k + k_{12} - n \cdot m \omega^2 \end{vmatrix}$$

$$\Rightarrow (k + k_{12} - m\omega^2)(k + k_{12} - nm\omega^2) - k_{12}^2 = 0$$

b)

$$nm^2\omega^4 - (n+1) m (k+k_{12}) \omega^2 + (k+k_{12})^2 - k_{12}^2 = 0$$

$$\omega^2 = \frac{(n+1)m(k+k_{12}) \pm \sqrt{[(n+1)m(k+k_{12})]^2 - 4nm^2[(k+k_{12})^2 - k_{12}^2]}}{2nm^2}$$

$$= \frac{(n+1)(k+k_{12})}{2nm} \pm \frac{1}{2} \sqrt{\left[\frac{(n+1)(k+k_{12})}{nm}\right]^2 - \frac{4[(k+k_{12})^2 - k_{12}^2]}{nm^2}}$$

(this does not become much prettier from here)

$n \rightarrow \infty$: $\omega^2 = \frac{k+k_{12}}{2m} \pm \frac{1}{2} \sqrt{\left(\frac{k+k_{12}}{m}\right)^2 - 0}$

$\omega^2 = \frac{k+k_{12}}{2m} \cdot 2 = \frac{k+k_{12}}{m} \longrightarrow \omega = \sqrt{\frac{k+k_{12}}{m}}$

(or $\omega^2 = 0$)

$n = 1$: $\omega^2 = \frac{k + k_{12} \pm k_{12}}{m} \longrightarrow \omega_S^2 = \frac{k}{m}, \; \omega_A^2 = \frac{k + 2k_{12}}{m}$

$\therefore$ Body 1 oscillates with $\omega = \sqrt{\frac{k+k_{12}}{m}}$ while 2 is at rest $\longrightarrow$ "wall"; ω^2 is in between ω_S^2 and ω_A^2 $(n=1)$