Exam 11.8.2018 – Solutions

1. a) $z' = \gamma(z - vt)$

$$L' = z_2' - z_1' = \gamma(z_2 - vt) - \gamma(z_1 - vt) = \gamma(z_1 - z_2)$$

$$\underline{L' = \gamma L}$$

Distance 494 km, $v = 0.8c$, $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$

$$\Longrightarrow \quad L = \frac{L'}{\gamma} = L' \sqrt{1 - v^2/c^2} = L' \cdot \frac{3}{5} = \underline{296 \text{ km}}$$

b) $\bar{x} = (x_1, x_2, x_3, ict)$; $\quad \bar{v}^2 = \left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2 + \left(\frac{dx_3}{dt}\right)^2$

$$dx_\mu = (dx_1, dx_2, dx_3, ic\,dt)$$

$$ds^2 = dx_\mu dx_\mu = dx_1^2 + dx_2^2 + dx_3^2 - c^2 dt^2 \quad \text{(invariant)}$$

$$-c^2 d\tau^2 = dx_1^2 + dx_2^2 + dx_3^2 - c^2 dt^2 = (v^2 - c^2) dt^2$$

rest frame

$$\Longrightarrow \quad dt = \frac{d\tau}{\sqrt{1 - \beta^2}} = \gamma d\tau$$

Velocity 4-vector:

$$u_\mu = \frac{dx_\mu}{d\tau} = \left(\frac{dx_i}{d\tau}, ic\frac{dt}{d\tau}\right) = \gamma(\bar{v}, ic)$$

proper time

Momentum 4-vector:

$$p_\mu = m u_\mu = (m\gamma\bar{v}, ip_4)$$

$$p_4 = \gamma mc$$

Recall $T = \underbrace{\gamma mc^2}_{E} - mc^2 \iff E = T + mc^2 = \gamma mc^2$

rest energy

$$\Longrightarrow \quad \underline{p_\mu = \left(m\gamma\bar{v}, i\frac{E}{c}\right)}$$

2. Sliding pendulum

Holomic constraints
(x-axis, rod)

a) 1° $U_1 = 0$, $T_1 = \frac{1}{2} m \dot{x}^2$

2° $U_2 = -m_2 g l \cos\theta$, $T_2 = \frac{1}{2} m_2 [\dot{x}\hat{\imath} + l\dot{\theta}(\hat{\imath}\cos\theta + \hat{\jmath}\sin\theta)]^2$

$$= \ldots = \frac{1}{2} m_2 (\dot{x}^2 + l^2\dot{\theta}^2 + 2\dot{x} l \dot{\theta} \cos\theta)$$

$$L = T - U = \ldots = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + \frac{1}{2} m_2 l^2 \dot{\theta}^2 + m_2 l \cos\theta \, (\dot{x}\dot{\theta} + g)$$

b) x: $\frac{\partial L}{\partial x} = 0 \Rightarrow \frac{d}{dt}\left[(m_1 + m_2)\dot{x} + m_2 l \cos\theta \cdot \dot{\theta}\right] = 0$

$$\Rightarrow \underline{(m_1 + m_2)\ddot{x} + m_2 l \cos\theta \cdot \ddot{\theta} - m_2 l \sin\theta \cdot \dot{\theta}^2 = 0}$$

θ: $-m_2 l \sin\theta\,(\dot{x}\dot{\theta} + g) - \frac{d}{dt}(m_2 l^2 \dot{\theta} + m_2 l \cos\theta \cdot \dot{x}) = 0$

$$\Rightarrow \ldots \Rightarrow \underline{m_2 g l \sin\theta + m_2 l^2 \ddot{\theta} + m_2 l \cos\theta \cdot \ddot{x} = 0}$$

c) $\dot{x} \to 0 \Rightarrow \ddot{x} \to 0$

The equation of motion for x becomes trivial.

θ: $m_2 g l \sin\theta + m_2 l^2 \ddot{\theta} = 0$

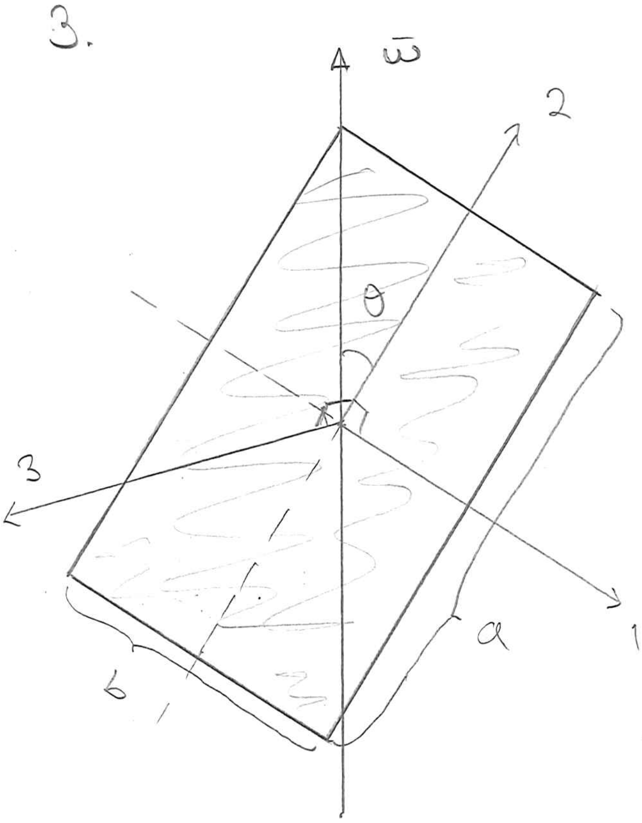
$$\Rightarrow \ddot{\theta} + \frac{g}{l}\sin\theta = 0 \qquad \text{(ordinary pendulum)}$$

If θ small, $\sin\theta \approx \theta$

$$\Rightarrow \ddot{\theta} + \omega^2 \theta = 0, \qquad \omega = \sqrt{\frac{g}{l}}$$

$\theta(t) = \theta_0 \cos(\omega t)$ ($\theta(0) = \theta_0$ and $\dot{\theta}(0) = 0$)
init. conditions

3.



$\bar\omega$ in 1-2 plane

$I_1 = \frac{1}{12} m a^2$

$I_2 = \frac{1}{12} m b^2$

$I_3 = \frac{1}{12} m (a^2+b^2)$

tensor $\bar{\bar{I}}$, diagonal

$$\underset{\sim}{\bar L} = \bar{\bar{I}} \cdot \bar\omega = \omega_1 I_1 \hat e_1 + \omega_2 I_2 \hat e_2 + \omega_3 I_3 \hat e_3$$

$$= \underset{\sim}{\frac{1}{12} m\omega \left[-\sin\theta \cdot a^2 \hat e_1 + \cos\theta \cdot b^2 \hat e_2 \right]}$$

$$L^2 = \left(\frac{m\omega}{12}\right)^2 \left[\sin^2\theta \cdot a^4 + \cos^2\theta \cdot b^4 \right]$$

$$\sin\theta = \frac{b}{\sqrt{a^2+b^2}}, \quad \cos\theta = \frac{a}{\sqrt{a^2+b^2}}$$

$$\Rightarrow \quad L^2 = \left(\frac{m\omega}{12}\right)^2 \left[\frac{b^2 a^4}{a^2+b^2} + \frac{a^2 b^4}{a^2+b^2} \right]$$

$$= \left(\frac{m\omega}{12}\right)^2 a^2 b^2$$

$$\underset{\sim}{L = \frac{1}{12} m\omega ab} \qquad (*)$$

b) $\bar\omega \cdot \bar L = \omega L \cos\alpha$ (α unknown)

$$= \omega(-\sin\theta, \cos\theta) \cdot \frac{1}{12} m\omega (-\sin\theta \cdot a^2, \cos\theta \cdot b^2)$$

$$= \frac{1}{12} m\omega^2 (a^2 \sin^2\theta + b^2 \cos^2\theta)$$

$$= \frac{1}{12} m\omega^2 \left[\frac{b^2 a^2 + a^2 b^2}{a^2+b^2} \right] = \frac{1}{12} m\omega ab \cdot \cos\alpha \quad (*)$$

$$\Rightarrow \quad \cos\alpha = \frac{2ab}{a^2+b^2} \quad \Rightarrow \quad \underline{\alpha = \arccos \frac{2ab}{a^2+b^2}}$$

E.g. $b = 1$, $a = 2$ $\rightarrow$ $\alpha = 36.7^\circ$, $\theta = 26.6^\circ$

c) $\underline{T_{rot}} = \frac{1}{2} \bar\omega \cdot \bar L = \underline{\frac{1}{12} m\omega^2 \cdot \frac{a^2 b^2}{a^2+b^2}}$

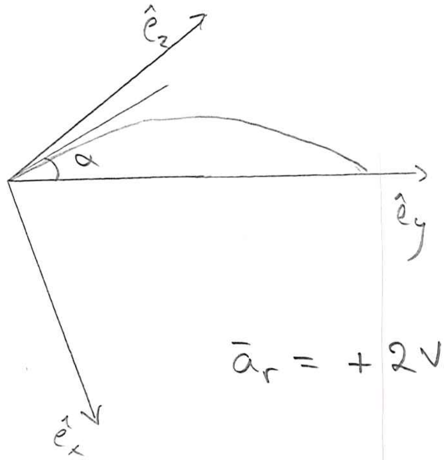
4. Coriolis effect

$$\bar{a}_r = \bar{g} - 2\bar{\omega} \times \bar{v}_r$$

$$\begin{cases} \omega_x = -\omega \cos\lambda \quad \leftarrow \text{latitude} \\ \omega_y = 0 \\ \omega_z = \omega \sin\theta \end{cases}$$

$$\begin{cases} \dot{x} \approx 0 \\ \dot{y} = v_0 \cos\alpha \quad \leftarrow \text{shooting angle} \\ \dot{z} = v_0 \sin\alpha - g t \end{cases}$$

$$\omega \times \bar{v}_r = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ -\omega\cos\lambda & 0 & \omega\sin\lambda \\ 0 & v_0\cos\alpha & v_0\sin\alpha - gt \end{vmatrix}$$



Gravity: $\bar{g} = g\,\hat{e}_z$

$$\bar{a}_r = +2v_0\cos\alpha\cdot\omega\sin\lambda\,\hat{e}_x - 2(v_0\sin\alpha - gt)\cdot\omega\cos\lambda\,\hat{e}_y + (g + 2v_0\cos\alpha\cdot\omega\cos\lambda)\,\hat{e}_z$$

b) Neglect Coriolis in $\hat{e}_z$-direction (vertical)

$$a_{r,x} = \frac{d^2x}{dt^2} = +2\omega\sin\lambda\cdot v_0\cos\alpha \quad \leftarrow \text{displacement}$$

init. cond. $v_x = 0$ $\Rightarrow$ $\dfrac{dx}{dy} = +2\omega\sin\lambda\cdot v_0 t\cos\alpha$ solve time

init. cond. $x=0$ $\Rightarrow$ $x(t) = +2\omega\sin\lambda\cdot\frac{1}{2}v_0 t^2\cos\alpha = \omega\sin\lambda\cdot v_0 t^2\cos\alpha$

$$a_{r,z} = \frac{d^2z}{dt^2} = g + \underbrace{2v_0\cos\alpha\cdot\omega\cos\lambda}_{\text{neglect this}}$$

$$\Rightarrow \ldots \Rightarrow z(t) = \underset{\uparrow 0}{z(0)} + \underbrace{v_0\sin\alpha\cdot t}_{\text{init. cond.}} + \frac{1}{2}gt^2$$

$$\Rightarrow t = \frac{2v_0\sin\alpha}{g}$$

$$\Rightarrow \underline{x(v_0,\alpha)} = +\omega\sin\lambda\cdot v_0\cdot\left(\frac{2v_0\sin\alpha}{g}\right)^2\cos\alpha$$

$$= +\frac{4v_0^3}{g^2}\,\omega\sin\lambda\,\sin^2\alpha\,\cos\alpha \qquad (+ \leftrightarrow \text{South})$$

5. Hamiltonian dynamics

a) $x^2+y^2=R^2$

$$\bar{F} = -\frac{k}{r^2} = -\nabla U(r) \implies U = -\frac{k}{r} = -\frac{k}{\sqrt{R^2+z^2}}$$

$$\bar{v}^2 = \underset{0}{\dot{R}^2} + R^2\dot{\theta}^2 + \dot{z}^2$$

$$T = \frac{1}{2}m\left(R^2\dot{\theta}^2+\dot{z}^2\right)$$

$$L = T - U = \frac{1}{2}m\left(R^2\dot{\theta}^2+\dot{z}^2\right) + \frac{k}{\sqrt{R^2+z^2}} \qquad (\text{no } \theta\,!)$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta}, \qquad p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

$$\underline{H(z,p_z,p_\theta)} = T+U = \underline{\frac{p_\theta^2}{2mR^2} + \frac{p_z^2}{2m} - \frac{k}{\sqrt{R^2+z^2}}}$$

b) θ cyclic coordinate $\rightarrow$ p_θ conserved (angular momentum)

Total energy conserved

a) (continues...)

$$\begin{cases} \dot{p}_\theta = -\dfrac{\partial H}{\partial \theta} = 0 \\ \dot{p}_z = -\dfrac{\partial H}{\partial z} = +2zk\cdot\left(-\frac{1}{2}\right)\cdot\left(R^2+z^2\right)^{-\frac{3}{2}} = -\dfrac{kz}{\sqrt{R^2+z^2}^{\,3}} \\ \dot{\theta} = \dfrac{\partial H}{\partial p_\theta} = \dfrac{p_\theta}{mR^2} \\ \dot{z} = \dfrac{\partial H}{\partial p_z} = \dfrac{p_z}{m} \end{cases}$$

Hamilton's equation of motion

c) $\dfrac{dT}{dt} = [T,H]_{p,q} + \dfrac{\partial T}{dt}$; T time-independent $\implies \dfrac{\partial T}{\partial t} = 0$

$$[T,H]_{p,q} = \sum_i \left(\frac{\partial T}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial T}{\partial p_i}\frac{\partial H}{\partial q_i}\right)$$

$$= \underbrace{\frac{\partial T}{\partial \theta}\frac{\partial H}{\partial p_\theta}}_{0} - \underbrace{\frac{\partial T}{\partial p_\theta}\frac{\partial H}{\partial \theta}}_{0} + \underbrace{\frac{\partial T}{\partial z}\frac{\partial H}{\partial p_z}}_{0} - \frac{\partial T}{\partial p_z}\frac{\partial H}{\partial z}$$

$$= +\frac{p_z}{m}\cdot\frac{kz}{\sqrt{R^2+z^2}^{\,3}} \implies \underline{\frac{dT}{dt} = \frac{p_z}{m}\cdot\frac{kz}{\sqrt{R^2+z^2}}}$$

$[T,H]_{p,q} = 0$ if z constant $\longrightarrow p_z = 0$

$\implies$ circular orbit