PROBLEM 1

Observer in a rotating coordinate system experiences an effective force:

$$\vec{F}_{eff} = \vec{F} - 2m(\vec{\omega} \times \vec{v_r}) - m\vec{\omega} \times \vec{\omega} \times \vec{r} \tag{1}$$

Let's start from the general rule of thumb that Coriolis force causes deflection on the right on the Northern hemisphere. However, we have to consider also the vertical deflection due to Coriolis and remember the existence of centripetal accelaration (latter term in the equation).

To make things easier, note that the distance in centripetal acceleration is measured from Earth's center and it does not change! This means that there is no effect from this source.

One needs to consider the lateral ($\mathbf{e}_y$) and vertical deflection ($\mathbf{e}_z$) due to Coriolis by writing out the cross-product $-2m(\vec{\omega} \times \vec{v_r})$. It is not necessary to include gravitation (marginal effect).

The components of ω are $(-\omega cos\lambda, 0, \omega sin\lambda)$, velocity is $\vec{v_r} = v_r\mathbf{e}_y$, and the determinant of the cross-product gives us the Coriolis force components while shooting towards East

$$
\begin{aligned}
F_x &= +2m\omega v_r sin\lambda && (2)\\
F_y &= 0 && (3)\\
F_z &= +2m\omega v_r cos\lambda && (4)
\end{aligned}
$$

The lateral deflection is towards right (South) and the shooter corrects it correspondingly (adjustment to the left). Similarly, the vertical deflection is up and the correction is down.

Consider next the reversed shooting direction with $\vec{v_r} = -v_r\mathbf{e}_y$:

$$
\begin{aligned}
F_x &= -2m\omega v_r sin\lambda && (5)\\
F_y &= 0 && (6)\\
F_z &= -2m\omega v_r cos\lambda && (7)
\end{aligned}
$$

The lateral deflection is now towards right as well (North) with the same magnitude and the previous correction is valid! Note here that the target is in the reversed direction, and the situation of the lateral trajectory component is identical with the previous case.

The only thing that remains is the vertical deflection. It is negative (down) while the previous correction deflects the trajectory even further downwards. The vertical deflection doubles!

Time of flight: $t = 100/600$ s = 1/6 s, $\lambda = 60°$, $\omega = 7.29 \cdot 10^{-5}$ 1/s

$$\Delta s = 2 \times \frac{1}{2} a_z t^2 = -2\omega v_r cos\lambda t^2 = -1.215 \cdot 10^{-3}\mathrm{m} \tag{8}$$

As a summary, the systematic deflection is 1.2 mm downwards due to the Coriolis effect. Surely, the magnitude is negligible in practice.

PROBLEM 2

The Lagrangian is:

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}ma^2\dot{\phi}^2 - mgrcos\theta \tag{9}$$

Constraints are: (1) $r = R + a$ and (2) $(R + a)\dot{\theta} = a\dot{\phi}$. The first one is a holonomic constraint $f_1 = r - (R + a) = 0$ whereas the second one is non-holonomic but integrable constraint $f_2 = -(R + a)\dot{\theta} + a\dot{\phi} = 0$. The former we describe with a multiplier λ (normal force) and the latter with μ (tangential force required for rolling motion).

Generalized coordinates are now (r, θ, ϕ) and the Lagrange equations are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \left(\frac{\partial L}{\partial q_j}\right) = \lambda\left(\frac{\partial f_1}{\partial q_j}\right) + \mu\left(\frac{\partial f_2}{\partial \dot{q}_j}\right) \tag{10}$$

The equations of motion are:

$$\begin{aligned} m\ddot{r} - mr\dot{\theta}^2 + mgcos\theta &= \lambda && (11) \\ mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} - mgrsin\theta &= -\mu(R+a) && (12) \\ ma^2\ddot{\phi} &= \mu a && (13) \\ & && (14) \end{aligned}$$

The last equation results in that $\mu = ma\ddot{\phi} = m(R+a)\ddot{\theta}$ (from constraint 2).

Assume next that the hoop stays on the surface, i.e. $\dot{r} = 0$. The second equation reduces to:

$$\ddot{\theta} = \frac{gsin\theta}{2(R+a)} \tag{15}$$

Now multiply the equation with $\dot{\theta}$ and use the hint $\ddot{\theta}d\theta = \dot{\theta}d\dot{\theta}$. Integrate both sides.

$$\dot{\theta}^2 = \frac{g}{(R+a)}(1 - cos\theta) \tag{16}$$

Use this now for the first equation, remember to keep $\dot{r} = 0$.

$$\begin{aligned}
\lambda &= -mr\dot{\theta}^2 + mgcos\theta && (17)\\
&= -m(R+a)\frac{g}{(R+a)}(1 - cos\theta) + mgcos\theta && (18)\\
&= mg(2cos\theta - 1) && (19)\\
& && (20)
\end{aligned}$$

In the beginning, while $\theta = 0$, the constraint for normal force is $\lambda = mg$ (as it should!). The constraint will change its sign once the hoop leaves the surface, i.e. $\lambda \leq 0$. This occurs once $\lambda \geq 60°$.

PROBLEM 3

The Lorentz transformations can be achieved directly from:

$$L_{jk} \quad = \delta_{jk} + (\gamma - 1)\beta_j\beta_k/\beta^2; \;\; L_{j4} = i\gamma\beta_j; \;\; L_{4k} = -i\gamma\beta_k; \;\; L_{44} = \gamma \tag{21}$$

This results in the requested matrices

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\beta\gamma \\ 0 & 0 & -i\beta\gamma & \gamma \end{bmatrix};$$

$$L' = \begin{bmatrix} \gamma' & 0 & 0 & i\beta'\gamma' \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta'\gamma' & 0 & 0 & \gamma' \end{bmatrix}$$

Now, it it is more efficient to consider each orthogonal transformation L and L' separately rather than forming a new matrix L'' for the overall transition. Firstly,

$$\bar{x'} = \bar{L} \cdot \bar{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & i\beta\gamma \\ 0 & 0 & -i\beta\gamma & \gamma \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ ict \end{bmatrix} = \begin{bmatrix} x \\ y \\ \gamma(z - vt) \\ i\gamma(ct - \beta z) \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \\ ict' \end{bmatrix}$$

This will just result in the familiar Lorentz transformation (along z-axis) that we have seen in the lectures. The second transformation follows

$$\bar{x''} = \bar{L'} \cdot \bar{x'} = \begin{bmatrix} \gamma' & 0 & 0 & i\beta'\gamma' \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta'\gamma' & 0 & 0 & \gamma' \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \\ ict' \end{bmatrix} = \begin{bmatrix} \gamma'(x' - vt') \\ y' \\ z' \\ i\gamma'(ct' - \beta' x') \end{bmatrix} = \begin{bmatrix} x'' \\ y'' \\ z'' \\ ict'' \end{bmatrix}$$

Let us now collect the results to get the new coordinates in terms of the oldest ones, i.e. corresponding to $\bar{x''} = \bar{L''} \cdot \bar{x}$

$$x'' = \gamma'\left[x - v'\gamma\left(t - \frac{vz}{c^2}\right)\right] \tag{22}$$

$$y'' = y \tag{23}$$

$$z'' = \gamma(z - vt) \tag{24}$$

$$t'' = \gamma'\left[\gamma\left(t - \frac{vz}{c^2}\right) - \frac{v'x}{c^2}\right] \tag{25}$$

The coordinates are now expressed with respect to the first coordinate system and the individual velocities of the inertial moving frames.

PROBLEM 4

This problem for coupled oscillations follows the lecture example closely, only the masses and spring constants are being varied.

The Lagrangian of the system is

$$\begin{aligned} L &= \frac{1}{2}m_1\dot{x_1}^2 + \frac{1}{2}m_2\dot{x_2}^2 - \frac{1}{2}k_1x_1^2 - \frac{1}{2}k_2(x_2 - x_1)^2 - \frac{1}{2}k_3x_2^2 && (26) \\ &= m\dot{x_1}^2 + \frac{1}{2}m\dot{x_2}^2 - 2kx_1^2 - \frac{1}{2}k(x_2 - x_1)^2 - kx_2^2 && (27) \end{aligned}$$

The components of the secular determinant follow from the second derivatives

$$\begin{aligned} m_{ij} &= \frac{\partial^2 T}{\partial \dot{x}_i \partial \dot{x}_j} && (28) \\ A_{ij} &= \frac{\partial^2 V}{\partial x_i \partial x_j} && (29) \end{aligned}$$

The secular determinant can be next formed as $\det(A_{ij} - \omega^2 m_{ij})$. These are the eigenfrequencies.

This leads to

$$\begin{vmatrix} 5k - 2m\omega^2 & -k \\ -k & 3k - m\omega^2 \end{vmatrix} = 0$$

and the solutions are $\omega^2 = 7k/2m$ and $\omega^2 = 2k/m$.

Solve the eigenfrequencies by using the equation:

$$\sum_{jk} (A_{jk} - \omega_r^2 m_{jk}) a_{jr} = 0 \tag{30}$$

Implement here $\omega_1^2 = 7k/2m$ and it results in the relation $a_{1(1)} = -2a_{2(1)}$. This is the interrelationship between the components in the eigenvector. After normalization, the (out-of-phase) result is

$$\vec{a}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Similarly, $\omega_2^2 = 2k/m$ results in that $a_{1(2)} = a_{2(2)}$ and the normalised (in-phase) eigenvector is

$$\vec{a}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

PROBLEM 5

We consider the simple case of free fall in uniform gravitational field. The motion occurs in y-direction and the corresponding Hamiltonian is

$$H = \frac{p^2}{2m} + mgy \tag{31}$$

The basic equation of the Hamilton-Jacobi theory is:

$$H + \frac{\partial S}{\partial t} = 0; \quad S = S(q_1, ..., q_n, \alpha_1, ..., \alpha_n, t) \tag{32}$$

We replace now momentum with the partial derivative the Hamilton's principal function S and solve it. First, however, we note that

$$K = H + \frac{\partial S}{\partial t} = \frac{p^2}{2m} + mgy - \alpha = 0 \tag{33}$$

where α is now the total energy E, as suggested. Note also that the momentum is

$$p = \sqrt{2m(\alpha - mgy)} \tag{34}$$

The partial differential equation becomes

$$\left(\frac{\partial S}{\partial y}\right)^2 = 2m(\alpha - mgy) \tag{35}$$

Note that the Hamiltonian does not contain time explicitly. We can therefore write (in general) $S = W - \alpha t$ where W is the Hamilton's characteristic function. We can replace S in the partial differential equation and integrate the solution

$$\left(\frac{\partial W}{\partial y}\right)^2 = 2m(\alpha - mgy) \tag{36}$$

in this case leads to the solution

$$W(y) = -\frac{1}{3m^2 g}\Big[2m(\alpha - mgy)\Big]^{3/2} + C \tag{37}$$

Correspondingly,

$$S(y) = -\frac{1}{3m^2 g}\Big[2m(\alpha - mgy)\Big]^{3/2} + C - \alpha t \tag{38}$$

Next, we introduce the new coordinates via the relation

$$Q_i = \frac{\partial}{\partial \alpha_i} S(q, \alpha, t) = \beta_i \tag{39}$$

where α of concern is now related to time. Take a partial derivative with respect to α

$$\frac{\partial S(y, \alpha, t)}{\partial \alpha} = -\frac{1}{mg}\Big[2m(\alpha - mgy)\Big]^{1/2} = t + \beta \tag{40}$$

By applying the inside-out strategy, solve this with respect to y

$$y = -\frac{1}{2} g (t + \beta)^2 + \frac{\alpha}{mg} \tag{41}$$

Consider here the initial conditions y_0, p_0, and $t = 0$. The total energy and position become as

$$E = \alpha = \frac{p_0^2}{2m} + mgy_0 \tag{42}$$

$$y_0 = -\frac{1}{2} g \beta^2 + \frac{p_0^2}{2m^2 g} + y_0 \tag{43}$$

From here one can solve β with respect to initial conditions

$$\beta = \frac{p_0}{mg} \tag{44}$$

Once this is inserted in the original equation for position (Eq. (41)), one achieves the final result

$$y = -\frac{1}{2}gt^2 - \frac{p_0}{m}t + y_0 \tag{45}$$

which is the familiar basic equation for constant acceleration. Note that once β is introduced it becomes implicitly clear from Eq. (41) that the solution is of the form

$$y = -\frac{1}{2}gt^2 + c_1 t + c_2 \tag{46}$$

The constants can be solved from the initial conditions, as shown above.