PROBLEM 1

A. Short answers to the questions. For more, see the lecture notes.

i. Holonomic constraints can be expressed as an implicit equation

$$f(q_1, ..., q_n, t) = 0. \tag{1}$$

Note that there is no dependence on generalized velocities, only generalized coordinates and time are considered. Example: A pendulum bob which attached to a wire of length l, the distance r is constrained by $r - l = 0$.

ii. Monogenic forces can be derived from a general potential function that may depend on the generalized velocities as well. This makes the formulation of classical mechanics more general than the derivation via D'Alembert's principle (assuming conservative force field) since Hamilton's principle is valid for monogenic systems.

iii. Euler angles are a convention to achieve three independent angles (generalized coordinates) and corresponding orthogonal transformations (rotations) to describe the orientation/rotation of a rigid body. They enable transformation between the laboratory and body coordinate systems. There are twelve conventions for defining Euler's angles.

iv. The basic equation of the Hamilton-Jacobi theory is

$$H + \frac{\partial S}{\partial t} = 0; \quad S = S(q_1, ..., q_n, \alpha_1, ..., \alpha_n, t) \tag{2}$$

where S is the Hamilton's principal function. It is a generating function of the second type (F_2) and provides a canonical transformation to cyclic coordinates that are related to the initial constants of the physical problem (integration constants). Not to be confused with Hamiltonian H or Hamilton's characteristic function W.

v. Coriolis effect is a fictitious force that a moving observer in a rotating coordinate system experiences. It is the middle term of the total effective force

$$\vec{F}_{eff} = \vec{F} - 2m(\vec{\omega} \times \vec{v}_r) - m\vec{\omega} \times \vec{\omega} \times \vec{r} \tag{3}$$

and it depends on velocity. One can observe it easily for fast moving objects, such as projectiles, but it also shows up in the winds and ocean currents. Because of the cross-product with $\vec{\omega}$, the effect has opposite signs on different hemispheres.

B. Two cylinders:

We have **two rigid bodies**. In the first place, their location is characterized by 3 translational and 3 rotational coordinates. Since we can project the situation in 2D, we are left with 2+1 coordinates (translations + rotation) for each cylinder, thereby we have 3+3=6 generalized coordinates to start with. Adding **4 constraints**, we shall have **6-4=2 degrees of freedom** left. Correspondingly, we can choose ϕ_1 and θ as the final generalized coordinates (or x_1 and θ).

Constraints:

$$y_1 = R_1 \tag{4}$$

$$r = r_1 - r_2 \tag{5}$$

$$x_1 = R_1\phi_1 \quad \text{or} \quad \dot{x}_i = R_1\dot{\phi}_1 \tag{6}$$

$$r_2\phi_2 = r_1(\phi_1 + \theta) \quad \text{or} \quad r_2\dot{\phi}_2 = r_1(\dot{\phi}_1 + \dot{\theta}) \tag{7}$$

The slipping constrains (two latter equations) can be expressed either as holonomic or semiholonomic. Further treatment via Lagrange's undetermined multipliers would result in the same equations of motion.

PROBLEM 2

a) The puck may rotate in plane and the z-coordinate of the mass M experiences a holonomic constraint $z = r - l_0$. The Lagrangian is:

$$L = \frac{1}{2}(m+M)\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - mg(r - l_0) \tag{8}$$

By applying Lagrange's equation, the equations of motion become (r, θ):

$$(m+M)\ddot{r} - mr\dot{\theta}^2 + Mg = 0 \tag{9}$$

$$2mr\dot{r}\dot{\theta} + mr\ddot{\theta} = 0 \tag{10}$$

b) Note based on Eq. (8) that θ is a cyclic coordinate. The associated generalized momentum is conserved. Eq. (10) can be expressed also as

$$\frac{d}{dt}\left(mr\dot{\theta}\right) = 0; \qquad \mathcal{L} = mr\dot{\theta} \Rightarrow \qquad \dot{\theta} = \frac{\mathcal{L}}{mr} \tag{11}$$

One possible approach is to substitute this in Eq. (9):

$$(m+M)\ddot{r} - \frac{\mathcal{L}^2}{mr^3} + Mg = 0 \tag{12}$$

Modify further to get an expression that matches with Newton II.

$$(m+M)\ddot{r} = -\frac{\partial}{\partial r}\left(\frac{\mathcal{L}^2}{2mr^2} + Mgr\right) = 0 \tag{13}$$

We can see now based on $\vec{F} = -\nabla V_{eff}$ that the effective potential is of the form

$$\underline{V_{eff} = \frac{\mathcal{L}^2}{2mr^2} + Mgr} \tag{14}$$

c) When there is a finite $\mathcal{L}$, there exists some radius r_0 where $V'_{eff} = 0$.

$$V'_{eff} = -\frac{\mathcal{L}^2}{mr_0^3} + Mg = 0 \Rightarrow \qquad \underline{r_0^3 = \frac{\mathcal{L}^2}{mMg}} \tag{15}$$

d) Use $r = r_0 + \delta(t)$ in Eq. (12) for describing a small perturbation around the potential minimum. Note that $\ddot{r} = \ddot{\delta}$.

$$(m + M)\ddot{\delta}(t) = -Mg + \frac{\mathcal{L}^2}{m}(r_0 + \delta(t))^{-3} \tag{16}$$

$$\frac{\mathcal{L}^2}{m}(r_0 + \delta(t))^{-3} = \frac{\mathcal{L}^2}{mr_0^3}\left(1 + \frac{\delta(t)}{r_0}\right)^{-3} = Mg\left(1 + \frac{\delta(t)}{r_0}\right)^{-3} \approx Mg\left(1 - 3\frac{\delta(t)}{r_0}\right) \tag{17}$$

$$\underline{\ddot{\delta} = -\frac{3Mg}{(M + m)r_0}\delta \equiv -\omega_\delta^2\delta} \tag{18}$$

Since ω_δ is positive, we are dealing with an oscillatory motion around the equilibrium distance r_0. The circular orbit at r_0 is stable based on this analysis.

PROBLEM 3

The displacement along the ring is given by $r\theta$ for each particle. The Lagrangian of the system becomes

$$T = \frac{1}{2}mR^2(\dot{\theta}_1{}^2 + \dot{\theta}_2{}^2 + \dot{\theta}_1{}^2) \quad (19)$$

$$V = \frac{1}{2}kR^2[(\theta_1 - \theta_2)^2 + (\theta_2 - \theta_3)^2 + (\theta_3 - \theta_1)^2] \quad (20)$$

$$L = \frac{1}{2}mR^2(\dot{\theta}_1{}^2 + \dot{\theta}_2{}^2 + \dot{\theta}_1{}^2) - kR^2(\theta_1^2 + \theta_2^2 + \theta_3^2 - \theta_1\theta_2 - \theta_2\theta_3 - \theta_3\theta_1) \quad (21)$$

The components of the secular determinant follow from the second derivatives

$$m_{ij} = \frac{\partial^2 T}{\partial\dot{\theta}_i \partial\dot{\theta}_j} \quad (22)$$

$$A_{ij} = \frac{\partial^2 V}{\partial\theta_i \partial\theta_j} \quad (23)$$

The secular determinant can be next formed as $\det(A_{ij} - \omega^2 m_{ij})$.

This leads to

$$\begin{vmatrix} R^2(2k-m\omega^2) & -kR^2 & -kR^2 \\ -kR^2 & R^2(2k-m\omega^2) & -kR^2 \\ -kR^2 & -kR^2 & R^2(2k-m\omega^2) \end{vmatrix} = 0$$

and the solutions of the corresponding polynomial equation

$$R^6[-m^3\omega^6 + 6km^2\omega^4 - 9k^2m\omega^2] = 0 \tag{24}$$

are $\underline{\omega^2 = 0}$ and $\underline{\omega^2 = 3k/m}$ (double degenerate).

Solve the eigenfrequencies by using the equation:

$$\sum_{jk}(A_{jk} - \omega_r^2 m_{jk})a_{jr} = 0 \tag{25}$$

Implement here $\omega_1^2 = 0$ and it results in the relation $a_{1(1)} = a_{2(1)} = a_{3(1)}$. After normalization, the result is

$$\vec{a}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

This corresponds to a trivial solution where the whole ring rotates (no vibration $\rightarrow \omega_1 = 0$).

Next, $\omega_2^2 = 3k/m$ eigenvalue is double degenerate and leads to a solution

$$-\theta_1 - \theta_2 - \theta_3 = 0 \tag{26}$$

where the two eigenvectors $\vec{a}_2$ and $\vec{a}_3$ are in plane but not uniquely determined. Any given pair of eigenvectors whose components satisfy this while being orthogonal with respect to each other (and $\vec{a}_1$) are valid solutions.

PROBLEM 4

From the Lorentz transformations we can see immediately:

$$x' = x \tag{27}$$

$$y' = y \tag{28}$$

$$z' = \gamma(z - vt) \tag{29}$$

$$t' = \gamma(t - \frac{vz}{c^2}); \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{30}$$

Let us differentiate:

$$u'_x = \frac{dx'}{dt'} = \frac{dx}{dt'} = \frac{dx}{\gamma(dt - \frac{vdz}{c^2})} = \frac{dx/dt}{\gamma(dt/dt - \frac{v(dz/dt)}{c^2})} = \underline{\frac{u_x}{\gamma(1 - \frac{vu_z}{c^2})}} \tag{31}$$

$$u'_y = \frac{dy'}{dt'} = \frac{dy}{dt'} = \dots = \underline{\frac{u_y}{\gamma(1 - \frac{vu_z}{c^2})}} \tag{32}$$

As one can see, also the x and y velocity components change.

What remains is the z-component

$$u'_z = \frac{dz'}{dt'} = \frac{\gamma(dz - vdt)}{\gamma(dt - \frac{vz}{c^2})} = \frac{dz/dt - v(dt/dt)}{dt/dt - \frac{v(dz/dt)}{c^2}} = \underline{\frac{u_z - v}{1 - \frac{vu_z}{c^2}}} \quad (33)$$

This completes the velocity transformation. The results is know as the Einstein's velocity addition formula (general case). It ensures that the new velocity does not exceed the speed of light. In this case, the u_x and u_y components are affected although there is movement between S and S' along the z-axis only.

PROBLEM 5

a) D is a constant of motion. This means that its total time-derivative is zero. Mathematically speaking, this relation is expressed as

$$\frac{dD}{dt} = [D, H]_{q,p} + \frac{\partial D}{\partial t} = 0 \tag{34}$$

where

$$[D, H]_{q,p} = \sum_{i=1}^{n} \left(\frac{\partial D}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial q_i} \frac{\partial D}{\partial p_i} \right) \tag{35}$$

Collect the partial derivatives

$$\frac{\partial D}{\partial q} = \frac{p}{2}, \qquad \frac{\partial D}{\partial p} = \frac{q}{2}, \qquad \frac{\partial D}{\partial t} = -H \tag{36}$$

$$\frac{\partial H}{\partial q} = \frac{1}{q^3}, \qquad \frac{\partial H}{\partial p} = p \tag{37}$$

$$\frac{dD}{dt} = \frac{\partial D}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial D}{\partial p} \frac{\partial H}{\partial q} + \frac{\partial D}{\partial t} = \frac{p^2}{2} - \frac{1}{2q^2} - H = \underline{H - H = 0} \quad \Rightarrow \text{OK!} \tag{38}$$

b) We have the partial derivatives with respect to the generating function

$$q = -\frac{\partial F_3(p,Q)}{\partial p}, \qquad P = -\frac{\partial F_3(p,Q)}{\partial Q} \tag{39}$$

$$\text{Now :} \quad \frac{\partial F_3(p,Q)}{\partial p} = -q = \tan Q \quad \Rightarrow \quad F_3(p,Q) = -p\tan Q + f(Q) \tag{40}$$

$$P = p(1+q^2) + q^2 = p\sec^2 Q + \tan^2 Q = -\frac{\partial F_3(p,Q)}{\partial Q} \tag{41}$$

$$\begin{aligned} F_3(p,Q) &= -\int (p\sec^2 Q + \tan^2 Q)dQ && (42)\\ &= -p\int \sec^2 Q dQ - \int \tan^2 Q dQ && (43)\\ &= -p\tan Q - \tan Q + Q + C' = -(p+1)\tan Q + Q + g(p) && (44) \end{aligned}$$

Comparing Eqs. (40) and (44) $\Rightarrow \underline{F_3(p,Q) = -(p+1)\tan Q + Q + C}$