PROBLEM 1 - Euler angles and a heavy spinning top

a. Short descriptions for the questions of the hybrid essay. For more, see the lecture notes. One is expected to provide a **thorough description** of the Euler angles and their general context in terms of orthogonal transformations and linear algebra. Drawings and equations can be used for supporting this description.

Euler angles are a convention to achieve three independent angles (generalized coordinates) and corresponding orthogonal transformations (rotations) to describe the orientation/rotation of a rigid body. They enable transformation between the laboratory and body coordinate systems, and describe fully the orientation of a rigid body and its rotational degrees of freedom. There are twelve conventions for defining Euler's angles (here: ZXZ), and further, there are also other conventions that can be used for defining orthogonal transformations, such as the one used in aircrafts (pitch roll, yaw).

Euler angles are the natural generalized coordinates for a heavy spinning top as ϕ describes precession around the z-axis, θ marks the inclination with respect to the vertical and ψ denotes the rotation (spinning) around the body axis z'.

The transformation of the ω-vector between fixed and rotating coordinate systems is achieved by decomposing the angular velocity vector in the components of Euler angles, where $\omega_\phi = \dot{\phi}$, $\omega_\theta = \dot{\theta}$ and $\omega_\psi = \dot{\psi}$.

$$\vec{\omega} = \vec{\omega}_\phi + \vec{\omega}_\theta + \vec{\omega}_\psi \tag{1}$$

Since ϕ is a rotation around the original z-axis, we shall need to transform $\vec{\omega}_\phi$ in the body coordinate system by applying the full transformation matrix $\mathbf{A} = \mathbf{BCD}$, such that $\vec{\omega}_\phi{}' = \mathbf{A}\ \vec{\omega}_\phi$. Further, θ is a rotation with respect to the line of nodes (intermediate x-axis), and $\vec{\omega}_\theta$ only requires a rotation with respect to ψ, i.e. matrix $\mathbf{B}$. The last rotation with respect to ψ occurs in the body coordinate system with respect to the z'-axis, and therefore, no rotation is required for $\vec{\omega}_\psi$=$\vec{\omega}'_\psi$.

Finally, the components of $\vec{\omega}'_\phi$, $\vec{\omega}'_\theta$ and $\vec{\omega}'_\psi$ need to be collected to obtain the transformed angular velocity $\vec{\omega}'$. See the compendium, page 63, for a full description and the final result. (Just typing the final result is not a valid answer here.)

b. Lagrangian reveals immediately that ϕ and ψ are cyclic (do not appear explicitly), therefore the associated **canonical momenta** are conserved. These correspond to the angular momenta that involve precession and spinning of the top. The system is conservative meaning that **total energy** is conserved as well.

Lagrange equations are:

$$\frac{d}{dt}(I_1\dot{\phi}\sin^2\theta + I_3\omega_3\cos\theta) = 0 \quad (2)$$

$$I_1\ddot{\theta} - I_1\dot{\phi}^2\cos\theta\sin\theta + I_3\omega_3\dot{\phi}\sin\theta - mgh\sin\theta = 0 \quad (3)$$

$$\frac{d}{dt}(I_3\omega_3) = 0 \quad (4)$$

These forms for ϕ and ψ are more illustrative than carrying out the time derivative explicitly, but both choices are valid, of course. Further, the explicit forms of p_ϕ and p_ψ can be seen inside the parentheses, respectively. The angular velocity component $\omega_3 = \dot{\phi}\cos\theta + \dot{\psi}$ that corresponds to the spin is the constant component of angular velocity.

PROBLEM 2 - Plate sliding against a wall

a. Using the notation given in the problem statement, we choose as the generalized coordinate the angle the plate makes with the vertical. Denote this by θ. For a homogeneous plate, the distance form the center of mass to either end is $\ell/2$, and by symmetry also the distance from the centre of mass to the corner of the wall and floor is $\ell/2$. The key idea is the following: as the plate starts sliding, its center of mass moves along a circle of radius $\ell/2$ until the plate looses contact with the wall as $\ddot{x} = 0$. In terms of θ, the x and y coordinates are

$$x = \frac{\ell}{2}\sin\theta, \qquad y = \frac{\ell}{2}\cos\theta,$$

so that the velocity of the center of mass is $\dot{x}^2 + \dot{y}^2 = \frac{\ell^2}{4}\dot{\theta}^2$ and the angular velocity is $\omega = \dot{\theta}$.

Let us assume a general moment of inertia with respect to the rotation around the z axis through the center of mass, I_z, and fix its precise value later. The Lagrangian is

$$L = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 - V = \frac{m}{8}\ell^2\dot{\theta}^2(1 + \frac{4}{m\ell^2}I_z) - mg\frac{\ell}{2}\cos\theta. \tag{5}$$

For part (a), the Lagrangian equation of motion is

$$\frac{m\ell^2}{4}(1 + \frac{4}{m\ell^2}I_z)\ddot{\theta} = mg\frac{\ell}{2}\sin\theta, \tag{6}$$

which simplifies to $\ddot{\theta} = 2g/(\ell(1 + 4I_z/(m\ell^2))\sin\theta$. For $I_z = m\ell^2/12$ this becomes $\ddot{\theta} = 3g/(2\ell)\sin\theta$.

b. Let us first note that total energy of the plate is

$$E = \frac{m}{8}\ell^2\dot{\theta}^2(1 + \frac{4}{m\ell^2}I_z) + mg\frac{\ell}{2}\cos\theta = \frac{m\ell^2}{6}\dot{\theta}^2 + \frac{mg\ell}{2}\cos\theta, \tag{7}$$

where the latter equality is again for $I_z = m\ell^2/12$. By energy conservation this is equal to $\frac{mg\ell}{2}\cos\alpha$, so we can solve for $\dot{\theta}^2$ as

$$\dot{\theta}^2 = 4(1 + 4I_z/(m\ell^2))^{-1}\frac{g}{\ell}(\cos\alpha - \cos\theta) = \frac{3g}{\ell}(\cos\alpha - \cos\theta), \tag{8}$$

where we again show the result for general I_z and the specific $I_z = m\ell^2/12$. Then we can find out when $\ddot{x}$ vanishes. Taking the derivatives we first obtain

$$\ddot{x} = -\frac{\ell}{2}\sin\theta\dot{\theta}^2 + \frac{\ell}{2}\cos\theta\ddot{\theta}, \tag{9}$$

and substituting $\ddot{\theta}$ and $\dot{\theta}^2$ from above,

$$\begin{aligned}
\ddot{x} &= -\frac{\ell}{2}\sin\theta \cdot 4(1 + 4I_z/(m\ell^2))^{-1}\frac{g}{\ell}(\cos\alpha - \cos\theta) \\
&+ \frac{\ell}{2}\cos\theta(1 + 4I_z(m\ell^2))^{-1}\frac{2g}{\ell}\sin\theta \\
&= -\frac{3g}{2}\sin\theta(\cos\alpha - \cos\theta) + \frac{3g}{4}\sin\theta\cos\theta
\end{aligned} \tag{10}$$

Setting $\ddot{x} = 0$ and looking for the solution $\sin\theta \neq 0$, we find that $\cos\theta = \frac{2}{3}\cos\alpha$. Note that this result holds independently of I_z (as is easily seen from the first line in Eq. (10)).

PROBLEM 3 - System with a moving wall

The starting point of this problem is that the wall imposes an external force to the body. Correspondingly, the wall position oscillates as $X_w = A\sin(\omega t)$. The position of the object is then

$$\begin{aligned} x &= \ell_0 + z + X_w & (11) \\ \dot{x} &= \dot{z} + \dot{X}_w = \dot{z} + A\omega\cos(\omega t), & (12) \end{aligned}$$

where ℓ_0 is the equilibrium length and z is the stretch of the spring. Correspondingly

$$L = \frac{1}{2}m(\dot{z} + A\omega\cos(\omega t))^2 - \frac{1}{2}kz^2 \quad (13)$$

The canonical momentum is $p_z = \frac{\partial L}{\partial \dot{z}} = m(\dot{z} + A\omega\cos(\omega t))$ which can be modified as $\dot{z} = p_z/m - A\omega\cos(\omega t)$.

Next, the Hamiltonian will be constructed via Legendre transformation

$$H = p_z \dot{z} - L = \dots = \frac{p_z^2}{2m} - p_z A\omega \cos(\omega t) + \frac{1}{2} k z^2 \tag{14}$$

Hamilton's equations are

$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} - A\omega \cos(\omega t) \tag{15}$$

$$\dot{p}_z = -\frac{\partial H}{\partial z} = -kz, \tag{16}$$

Take a time derivative of $\dot{z}$ and combine the two equations to get the final Lagrange equation of motion

$$\ddot{z} + \frac{k}{m} = A\omega^2 \sin(\omega t) \tag{17}$$

The Lagrangian contains explicit time-depence, i.e. $\frac{\partial L}{\partial t} = \frac{dH}{dt} \neq 0$. Hamiltonian is not conserved and it is not the total energy.

PROBLEM 4 - Scattering problem

a. As typical for central forces, this is a 2D problem with two generalized coordinates r and θ. The Lagrangian in polar coordinates is of the form:

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r) \tag{18}$$

We can see that θ is a cyclic coordinate, and therefore, its canonical momentum is conserved.

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \longrightarrow \dot{p}_\theta = \frac{d}{dt}(mr^2\dot{\theta}) = 0 \tag{19}$$

The latter is the equation of motion for θ, and it leads to the conservation of angular momentum

$$mr^2\dot{\theta} = \ell^2 \tag{20}$$

The resulting Lagrange equation is then

$$\frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0 \tag{21}$$

Modify this further

$$m\ddot{r} - mr\dot{\theta}^2 = f(r) \longrightarrow m\ddot{r} - \frac{\ell^2}{mr^3} = f(r) \tag{22}$$

This is now the equation that we must convert to the final form by making the substitution $u = 1/r$. Let us first use the conservation of angular momentum

$$\ell dt = mr^2 d\theta \longrightarrow \frac{d}{dt} = \frac{\ell}{mr^2}\frac{d}{d\theta} \tag{23}$$

Substitute this in the Lagrange equation

$$\frac{1}{r^2}\frac{d}{d\theta}\left(\frac{\ell}{mr^2}\frac{dr}{d\theta}\right) - \frac{\ell^2}{mr^3} = f(r) \tag{24}$$

Introduce u in the equation by taking into account that $du = -dr/r^2$

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{\ell^2}\frac{d}{du}V\left(\frac{1}{u}\right) \tag{25}$$

The rest follows easily

$$\frac{\ell^2 u^2}{m}\left(\frac{d^2u}{d\theta^2} + u\right) = -f(u) \tag{26}$$

b.i We shall use the previous result as a starting point for solving the orbit equation. Note that the force can be expressed now as

$$f(r) = \frac{km}{r^3} = kmu^3 = f(u) \tag{27}$$

The differential equation becomes

$$\frac{d^2u}{d\theta^2} + u = -\frac{km^2}{\ell^2}u \tag{28}$$

$$\frac{d^2u}{d\theta^2} + \left(1 + \frac{km^2}{\ell^2}\right)u = 0 \tag{29}$$

Behold! This is a differential equation of the same form as that for the harmonic oscillator with the general solution

$$u = A\sin(B\theta + C), \qquad B^2 = 1 + \frac{km^2}{\ell^2} \tag{30}$$

Note that angular momentum is conserved, i.e. $\ell = msv_\infty = mr_0v_0$, and this leads to $B^2 = 1 + k/r_0^2v_0^2$. Further, consider the boundary conditions: (i) Very far away $\theta = 0$ and $u = 0$ leading to the conclusion that $C = 0$. (ii) At the periapsis $\theta = \theta_0$ and $r = r_0 = r_{min}$ meaning that $u = u_{max}$.

$$u_{max} = \frac{1}{r_0} = A\sin(B\theta_0) = A\sin(\pi/2) \longrightarrow B\theta_0 = \pi/2, \qquad A = 1/r_0 \tag{31}$$

The orbit equation becomes then

$$\frac{1}{r} = \frac{1}{r_0}\sin\left(\sqrt{1 + k/(r_0^2v_0^2)}\,\theta\right) \tag{32}$$

b.ii This will not require any complicated mathematics and integrals. Based on the conservation of angular momentum, the impact parameter is $s = r_0 v_0 / v_\infty$. Let us consider here the conservation of energy

$$E = \frac{1}{2} m v_\infty^2 = \frac{1}{2} m v_0^2 + \frac{km}{2 r_0^2} \longrightarrow v_\infty^2 = v_0^2 + \frac{k}{r_0^2} \tag{33}$$

meaning that

$$s^2 = \frac{r_0 v_0}{v_0^2 + k/r_0^2} = \frac{r_0^2}{B^2} \longrightarrow s = \frac{r_0}{B} = \frac{r_0}{\sqrt{1 + k/r_0^2 v_0^2}} \tag{34}$$

The orbit angle at the periapsis is $\theta_0 = \pi/(2B)$ as solved previously, and since $B = r_0/s$, this can be written as $\theta_0 = \pi s/(2r_0)$. This is the same as the angle Ψ in the scattering theory. The total scattering deflection angle becomes then

$$\Theta = \pi - 2\Psi = \pi - 2\theta_0 = \pi\left(1 - \frac{s}{r_0}\right) = \pi\left(1 - \frac{1}{\sqrt{1 + k/r_0^2 v_0^2}}\right) \tag{35}$$

b.iii Based on the previous results, we can calculate directly that if $s = r_0/2$ then $\theta_0 = \pi/4$. This means that the total deflection angle will be $\Theta = \pi/2$, i.e. 90 degrees. Graphically, this sets the deflection asymptote perpendicular with respect to the original particle direction. The corresponding plot should show a trajectory where an incoming particle ($s = r_0/2$) gets deflected near the origin (periapsis distance r_0 apart from the centre) such that the outgoing particle will approach the deflection asymptote up to (positive) infinity.

PROBLEM 5 - 007 space odyssey

The problem is as simple as it looks. We can consider three inertial frames, Earth (Ms. Moneypenny) as S, moon rocket as S', landing module (007) as S'' and probe vessel (Mr. Jaws) moving in the latter by a velocity v. The velocity difference between subsequent inertial frames is always v as well.

Since this is collinear motion, we can apply directly Einstein's velocity addition formula (kudos for those who derive it from scratch). It ensures that the new velocity does not exceed the speed of light.

a. We consider the velocity difference between S and S''. The solution is

$$v' = \frac{v + v}{\sqrt{1 + v^2/c^2}} = \frac{2v}{\sqrt{1 + \beta^2}} \tag{36}$$

b. We consider the total velocity change between Earth (S) and the probe vessel (velocity v in S'') by making a subsequent velocity addition by using Einstein's velocity addition formula, again.

$$v'' = \frac{v + v'}{\sqrt{1 + vv'/c^2}} = ... = \left[\frac{3 + \beta^2}{\sqrt{1 + 3\beta^2}}\right]v \tag{37}$$

Checking the limits show that for $v << c$, we get $2v$ and $3v$, respectively, as expected for the classical Galilei transformation. For $v = c$, both cases reduce to c, as required.