PROBLEM 1. True or false (where explanations matter)

i. TRUE. Only such orthogonal transformations which approach continuously the identity operation qualify. In terms of the corresponding orthogonal transformation matrices, the determinant has to equal to +1 (not -1). This is related to Noether's theorem which states that if there is a continuous symmetry in the system, then there is a related conserved quantity.

ii. TRUE. According to the intermediate axis theorem, the rotations around the I_1 and I_2 principal axes are unstable, but rotation around I_3 is stable (the case of two degenerate eigenvalues). We derived the intermediate axis theorem in the lectures from Euler's equations for rigid bodies.

iii. FALSE. There are two perspectives for "full" solution. Firstly, in terms of the equations of motion, we can use the holonomic constraints for reducing the number generalized coordinates down to $3N - k$ and solve those equations of motion. The remaining k generalized coordinates are already defined by the constraint equations. Therefore, one does not have to solve $3N$ Lagrangian equations of motion. Secondly, if one expects to calculate the forces of constrains, one needs solve a set of $3N + k$ equations by using the method of Lagrange's undetermined multipliers.

iv. Both TRUE/FALSE can be accepted. TRUE: The lateral Coriolis correction on the Northern hemisphere is systematically on the left although its magnitude varies with the shooting direction. However, if we consider shooting East (+) or West (-) in our local coordinate system, the magnitude of the lateral correction (due to the cross-product $\vec{\omega} \times \vec{v}$) remains the same. The deflection is towards South and North, respectively. However, also the target has been rotated by 180 degrees such that in terms of rifle adjustment there is no change! FALSE: There is also a vertical deflection (down / up) due to the Coriolis effect which **depends on the direction** and will require a rifle adjustment. FALSE is also accepted, if the student correctly points that the lateral displacement (not correction) is in opposite directions taking into account a possible confusion with the problem assignment.

v. FALSE. The angular momentum does not have to point in the same direction as $\vec{\omega}$. One example is the precessing free body where $\vec{\omega}$ changes it direction while $\vec{L}$ is a constant (no external torque). The second example is a case where the body has been made to rotate around some arbitrary direction (fixed $\vec{\omega}$) which does not coincide with the principal access. Correspondingly, one has to apply torque which causes that angular momentum changes.

vi. FALSE. There are $3N$ degrees of freedom for a system of N particles in the first place. A molecule has $3N - 6$ non-trivial vibrational modes. The 6 trivial modes correspond to translation (3) and rotation (3) of the whole molecule and their associated frequencies are naturally 0 cm^{-1}. Note: Also planar molecules have out-of-the-plane vibrational modes (e.g. benzene C_6H_6).

PROBLEM 2. Condition for circular orbits

This is a problem that is supposed to be easy but may have caused some trouble for students because of its general (implicit) nature. Our starting point is the 2D central field motion of a particle, as for the Kepler problem.

(a) Let us start by identifying the generalized coordinates. The central force problem can be treated in two dimensions and the convenient coordinates are polar coordinates. This means that the coordinates in question are the distance r and the orbit angle θ. The corresponding Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r) \quad (1)$$

By inspecting the Lagrangian, we acknowledge immediately that θ is a cyclic coordinate. Also total energy is concerved due to $V(r)$ (conservative).

This means that the corresponding canonical momentum, $p_\theta = mr^2\dot{\theta} = \ell$ is a constant. In other words, angular momentum ℓ is conserved and its derivative $\dot{p}_\theta = \frac{d}{dt}(mr^2\dot{\theta}) = 0$. Actually, this is the second Lagrange equation of motion (for θ).

The first Lagrangian equation of motion (for r) follows easily by taking the derivatives

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = m\ddot{r} - mr\dot{\theta}^2 - f(r) = 0 \tag{2}$$

This can be modified further

$$m\ddot{r} = f(r) + \frac{\ell^2}{mr^3} = -\frac{\partial V_{eff}(r)}{\partial r} \tag{3}$$

(**b**) To solve the circular orbit $r = r_0$, we must first write out the total energy

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = \frac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2} + V(r) \tag{4}$$

The orbit is circular $\dot{r} = 0$ and $r = r_0$, and therefore the energy becomes

$$E_0 = \frac{\ell^2}{2mr_0^2} + V(r_0) = V_{eff}(r_0) \tag{5}$$

We cannot solve this further without knowing the explicit form of $V(r)$. The condition for the force follows from the requirement that we are in the minimum of the effective potential

$$\left.\frac{\partial V_{eff}(r)}{\partial r}\right|_{r=r_0} = 0 \quad \Longrightarrow \quad f(r_0) = -\frac{\ell^2}{mr_0^3}; \quad r_0 = \left[-\frac{\ell^2}{mf(r_0)}\right]^{1/3} \tag{6}$$

(**c**) Inspection of the derivative of the effective potential yields information of local extrema (minima and maxima). To distinguish between those two cases, one has to look at the curvature of the effective potential (the shape of V_{eff}), *i.e.* its second derivative must be positive around a local minimum.

$$\left.\frac{\partial^2 V_{eff}(r)}{\partial r^2}\right|_{r=r_0} = -\left.\frac{\partial f(r)}{\partial r}\right|_{r=r_0} + \frac{3\ell^2}{mr_0^4} > 0 \tag{7}$$

Taking into account the previous result for circular force, we can modify this

$$\left.\frac{\partial f(r)}{\partial r}\right|_{r=r_0} < -\frac{3f(r_0)}{r_0} \tag{8}$$

Let us consider the potential $V(r) = kr^{n+1}$ which leads to a force $f(r) = -kr^n$.

Direct implemention of Eq. (8) leads to

$$-knr_0^{n-1} < 3kr_0^{n-1} \quad \Longrightarrow \quad n > -3 \tag{9}$$

This means that $f(r) > -kr^{-3}$ and $|V(r)| > kr^{-2}$ is the condition for being able to have a stable a circular orbit (note $k < 0$, attractive). This makes sense if you consider the shape of the effective potential V_{eff}. If you have have a potential $V(r) = kr^{n+1}$ with $n < -3$, the repulsive centrifugal effect $\sim r^{-2}$ will not predominate at small distances and V_{eff} diverges to minus infinity as $r \rightarrow 0$. Correspondingly, the point where the first derivative of V_{eff} disappears (cyclic orbit) becomes a local maximum, and it is therefore not stable. This is the first result for the derivation of the famous Bertrand's theorem for closed orbits.

PROBLEM 3. Hoop rolling down an inclined plane

In this problem, a hoop is rolling down an inclined plane (wedge) which itself can slide frictionless on a horisontal surface.

(a) The convenient generalized coordinates are X (inclined plane), s (hoop) and θ (hoop), where the two latter ones are connected via the slipping constraint $\dot{s} = R\dot{\theta}$ (note $R = a$ in the figure, my bad). We can solve the problem using X and s by eliminating $\dot{\theta}$ using the constraint.

Let us start from the position coordinates of the center of the hoop:

$$x = X + s\cos\alpha - R\sin\alpha \quad \Longrightarrow \quad \dot{x} = \dot{X} + \dot{s}\cos\alpha \tag{10}$$

$$y = s\sin\alpha + R\cos\alpha \quad \Longrightarrow \quad \dot{y} = \dot{s}\sin\alpha \tag{11}$$

Note that the center is tilted with respect to the contact due to inclination. The Lagrangian becomes (here: $\dot{s} = R\dot{\theta}$)

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m\dot{X}^2 - mgy \tag{12}$$

$$= \frac{1}{2}\left(m + \frac{I}{R^2}\right)\dot{s}^2 + \frac{1}{2}(M+m)\dot{X}^2 + m\dot{X}\dot{s}\cos\alpha - mgs\sin\alpha - mgR\cos\alpha \tag{13}$$

where the last term is a constant (can be neglected, equal to shifting potential).

(b) The Lagrangian reveals that X is cyclic, therefore the associated canonical momentum

$$p_X = \frac{\partial L}{\partial \dot{X}} = (M+m)\dot{X} + m\cos\alpha\dot{s} \tag{14}$$

is a constant and the associated Lagrangian equation reduces to $\dot{p}_X = 0$ and can be written out as

$$(M+m)\ddot{X} + m\cos\alpha\ddot{s} = 0 \quad \Longrightarrow \quad \ddot{s} = \frac{M+m}{m\cos\alpha}\ddot{X} \tag{15}$$

The equation of motion is coupled meaning that we have to solve the Lagrange equation for s before we can solve both of them in finalized forms. After taking the derivatives, the Lagrange equation for s becomes

$$\Big(m + \frac{I}{R^2}\Big)\ddot{s} + m\cos\alpha\ddot{X} + mg\sin\alpha = 0 \tag{16}$$

The mass m cancels out and the equation can be written as

$$\Big(1 + \frac{I}{mR^2}\Big)\ddot{s} + \cos\alpha\ddot{X} = -g\sin\alpha \tag{17}$$

This is the second coupled Lagrange equation. We can use now the previous result of Eq. (15) for p_X:

$$\Big(1 + \frac{I}{mR^2}\Big)\Big(-\frac{M+m}{m\cos\alpha}\Big) + \cos\alpha\ddot{X} = -g\sin\alpha \tag{18}$$

The equations of motion become

$$\ddot{X} = \frac{g\sin\alpha\cos\alpha}{\Big(1 + \frac{I}{mR^2}\Big)\Big(1 + \frac{M}{m}\Big) - \cos^2\alpha} = a_X \tag{19}$$

$$\ddot{s} = -\frac{g\Big(1 + \frac{M}{m}\Big)\sin\alpha\cos\alpha}{\Big(1 + \frac{I}{mR^2}\Big)\Big(1 + \frac{M}{m}\Big) - \cos^2\alpha} = a_s \tag{20}$$

One can now write the solutions for $X(t)$ and s(t) taking into account initial conditions (not defined herein)

$$X(t) = X(0) + \dot{X}(0)t + \frac{1}{2}a_X t^2 \tag{21}$$

$$s(t) = s(0) + \dot{s}(0)t + \frac{1}{2}a_s t^2 \tag{22}$$

Note that $a_s < 0$ while $a_X > 0$ and both are constants. Further, as many will remember for a homogeneous ring $I = mR^2$, which implemented in Eqs. (19) and (20) will make them somewhat simpler.

PROBLEM 4. Hamitonian mechanics and canonical transformations

This is a one-dimensional problem with a dissipative damping force proportional to instantaneous velocity. In terms of the resulting physics, the problem is simple and reduces in the end to the underdamped harmonic oscillator, but we shall focus on the Hamiltonian formalism.

(a) Lagrangian is given and we shall solve it in a straightforward fashion

$$2\gamma e^{2\gamma t} m\dot{q} + e^{2\gamma t} m\ddot{q} + \frac{dV}{dq} e^{2\gamma t} = 0 \tag{23}$$

This reduces to

$$m\ddot{q} = -\frac{dV}{dq} - 2m\gamma\dot{q} \tag{24}$$

which is the requested equation of motion demonstrating a force that results in from the conservative potential $V(q)$ and another (external) force contribution from the dissipative term.

(b) The transformation to Hamiltonian involves the equation

$$H = p\dot{q} - L = e^{2\gamma t} m\dot{q}^2 - e^{2\gamma t} \frac{1}{2} m\dot{q}^2 + e^{2\gamma t} V(q) = e^{2\gamma t} \Big(\frac{1}{2} m\dot{q}^2 + V(q) \Big) \tag{25}$$

The canonical momentum is $p = \frac{\partial L}{\partial \dot{q}} = e^{2\gamma t} m\dot{q}$, which results in the final Hamiltonian

$$H(q, p, t) = e^{-2\gamma t} \frac{p^2}{2m} + e^{2\gamma t} V(q) \tag{26}$$

(c) The generating function of the canonical transformation is

$$F(q,Q,P,t) = e^{\gamma t}qP - QP = F_2(q,P,t) - QP \tag{27}$$

By inspection of F we can already distinguish that we are dealing with a canonical transformation of the second kind. However, we are supposed to derive the corresponding transformation equations from scratch. We will need to start from the basic equation:

$$p\dot{q} - H = P\dot{Q} - K + \frac{dF}{dt} \tag{28}$$

Let us calculate the time-derivative first:

$$\frac{dF(q,Q,P,t)}{dt} = \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial Q}\dot{Q} + \frac{\partial F}{\partial P}\dot{P} + \frac{\partial F}{\partial t} \tag{29}$$

$$= e^{2\gamma t}P\dot{q} - P\dot{Q} - Q\dot{P} + e^{2\gamma t}q\dot{P} + \gamma qP \tag{30}$$

Insert this into Eq. (28)

$$p\dot{q} - H = -K + e^{2\gamma t}P\dot{q} - Q\dot{P} + e^{2\gamma t}q\dot{P} + \gamma e^{2\gamma t}qP \tag{31}$$

$$\Longrightarrow (p - e^{2\gamma t}P)\dot{q} + (Q - e^{2\gamma t}q)\dot{P} = H - K + \gamma e^{2\gamma t}qP \tag{32}$$

Since q and P are independent variables the coefficients before $\dot{q}$ and $\dot{P}$ must vanish independently

$$
\begin{aligned}
p &= e^{\gamma t} P && (33)\\
Q &= e^{\gamma t} q && (34)\\
K &= H + \gamma e^{\gamma t} q P && (35)
\end{aligned}
$$

The transformed Hamiltonian becomes

$$K = \frac{p^2 e^{-2\gamma t}}{2m} + e^{2\gamma t} V(q) + \gamma e^{2\gamma t} q P \tag{36}$$

but this is still in the mixed form because we do not know $V(q)$.

(d) For $V(q) = \frac{1}{2} m\omega^2 q^2$, the transformed Hamiltonian becomes

$$
\begin{aligned}
K &= \frac{e^{2\gamma t} P^2 e^{-2\gamma t}}{2m} + e^{2\gamma t} \frac{1}{2} m\omega^2 Q^2 e^{-2\gamma t} + \gamma e^{2\gamma t} e^{-2\gamma t} Q P && (37)\\
&= \frac{P^2}{2m} + \frac{1}{2} m\omega^2 Q^2 + \gamma Q P = K(Q, P, t) && (38)
\end{aligned}
$$

Note that transformed Hamiltonian $K(Q, P, t)$ does not explicitly depend on time so that it is a constant of motion (note: $[K, K] = 0$, trivial).

(e) Using Hamilton's equations for the canonical transformation, we get

$$\dot{Q} = \frac{\partial K}{\partial P} = \frac{P}{m} + \gamma Q \quad (39)$$

$$\dot{P} = -\frac{\partial K}{\partial Q} = -m\omega^2 Q - \gamma P \quad (40)$$

Taking derivative on both sides of Eq. (39) we get

$$\ddot{Q} = = \frac{\dot{P}}{m} + \gamma \dot{Q} = -\frac{1}{m}(m\omega^2 Q + \gamma P) + \gamma(P/m + \gamma Q) \quad (41)$$

$$= -\omega^2 Q + \gamma^2 Q \quad (42)$$

which results in the simple differential equation familiar from harmonic oscillator

$$\ddot{Q} + (\omega^2 - \gamma^2) Q = 0 \quad (43)$$

For the case $\gamma < \omega$, we have $\omega^2 - \gamma^2 = \Omega^2 > 0$, and $Q(t)$ has the oscillating solution

$$Q(t) = Ae^{i\Omega t} + Be^{-i\Omega t} \quad (44)$$

where A and B are integration constants set by initial conditions. Use the transformation between q and Q for the final underdamped solution of $q(t)$

$$q(t) = e^{-\gamma t}\left(Ae^{i\Omega t} + Be^{-i\Omega t}\right) \quad (45)$$

PROBLEM 5. Compton scattering

(a) The conserved quantities are total energy and momentum 4-vector (alternatively, one can say that relativistic linear momentum conserved). The 4-momentum can be derived from the event 4-vector by taking a time-derivative with respect to eigentime

$$P_\mu = m\frac{d}{d\tau} = m\left[\frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}, \frac{d(ict)}{d\tau}\right] = [\gamma\vec{p}_0, i\gamma mc] = [\gamma\vec{p}_0, iE/c] \tag{46}$$

where we have used $E = \gamma mc^2$. For a photon, the relativistic momentum is $p = h/\lambda$, and we shall denote the relativistic linear momentum of the recoiled electron as p_e.

(b) Let us write down the 4-momentum before the collision in along z-direction

$$P_{\mu,i} = P_{\mu,\gamma} + P_{\mu,e} = \left[0, 0, \frac{h}{\lambda}, \frac{ih}{\lambda}\right] + \left[0, 0, 0, imc\right] = \left[0, 0, \frac{h}{\lambda}, i\left(mc + \frac{h}{\lambda}\right)\right] \tag{47}$$

The photon hits the electron changing its wavelength to λ', scatters in angle θ, and the electron recoils in angle ϕ in the xz-plane. After the collision, the 4-momentum becomes

$$
\begin{aligned}
P_{\mu,f} &= P_{\mu,\gamma'} + P_{\mu,e'} = \left[\frac{h}{\lambda'}\sin\theta, 0, \frac{h}{\lambda'}\cos\theta, \frac{ih}{\lambda'}\right] && (48)\\
&+ \left[p_e\sin\phi, 0, p_e\cos\phi, i\sqrt{m^2c^2 + p_e^2}\right] && (49)\\
&= \left[\frac{h}{\lambda'}\sin\theta + p_e\sin\phi, 0, \frac{h}{\lambda'}\cos\theta + p_e\cos\phi, \left(\frac{ih}{\lambda'}\right) + i\sqrt{m^2c^2 + p_e^2}\right] && (50)
\end{aligned}
$$

We have used here the relativistic dispersion relation for electron energy $E^2 = m^2c^4 + p_e^2c^2$. By setting the two equations equal (conservation of 4-momentum) we achieve three equations

$$
\begin{aligned}
(h/\lambda')\sin\theta + p_e\sin\phi &= 0 && (51)\\
(h/\lambda')\cos\theta + p_e\cos\phi &= h/\lambda && (52)\\
h/\lambda' + \sqrt{m^2c^2 + p_e^2} &= mc + h/\lambda && (53)
\end{aligned}
$$

Next, solve from Eq. (51) the relationship between θ and ϕ

$$
\sin\phi = -\frac{h}{\lambda' p_e}\sin\theta \quad \Longrightarrow \quad \cos\phi = \left[1 - \left(\frac{h}{\lambda' p_e}\right)^2 \sin^2\theta\right]^{1/2} \qquad (54)
$$

Insert this in Eq. (52) and after some elaboration the result looks like

$$p_e^2 = \frac{h^2}{\lambda^2} + \frac{h^2}{\lambda'^2} - 2\frac{h^2}{\lambda\lambda'}\cos\theta \tag{55}$$

On the other hand, we can modify Eq. (53) such that it becomes

$$p_e^2 = h^2\Big(\frac{1}{\lambda} - \frac{1}{\lambda'}\Big)^2 + 2mch\Big(\frac{1}{\lambda} - \frac{1}{\lambda'}\Big) \tag{56}$$

Comparing the two previous equations (set them equal, few intermediate steps) leads to the conclusion

$$\cos\theta = 1 - \frac{mc}{h}(\lambda' - \lambda) \tag{57}$$

which results in the final result

$$\sin^2\frac{\theta}{2} = \frac{mc}{2h}(\lambda' - \lambda) = \frac{1}{2\lambda_c}(\lambda' - \lambda) \tag{58}$$

where $\lambda_c = h/(mc)$ is the predefined Compton wavelength.

(c) The kinetic energy of the electron after collision can be solved by considering the conservation of energy and the result of the previous section. The initial photon energy is hc/λ and the electron is at rest (mc^2). Let us write the total energy before and after collision:

$$mc^2 + hc/\lambda = \gamma mc^2 + hc/\lambda' \tag{59}$$

Note that the total energy for a moving electron is γmc^2. By rearranging terms, we achieve

$$(\gamma - 1)mc^2 = hc\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) = K \tag{60}$$

where K stands for kinetic energy.

$$K = hc\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right) = h\left(\frac{\lambda' - \lambda}{\lambda\lambda'}\right) \tag{61}$$

$$= hc\left(\frac{2\lambda_c \sin^2 \frac{\theta}{2}}{\lambda[\lambda + 2\lambda_c \sin^2 \frac{\theta}{2}]}\right) \tag{62}$$

$$= h\nu\left(\frac{2\chi \sin^2 \frac{\theta}{2}}{1 + 2\chi \sin^2 \frac{\theta}{2}}\right) \tag{63}$$

where we have used $\chi = \lambda_c/\lambda$ for shortening the final notation.