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INVERSE FRACTAL STATISTICS IN TURBULENCE AND FINANCE

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We consider inverse statistics in turbulence and financial data. By inverse statistics, also sometimes called exit time statistics, we "turn" the variables around such that the fluctuating variable becomes the fixed variable, while the fixed variable becomes fluctuating. In that sense we can probe distinct regimes of the data sets. In the case of turbulence, we obtain a new set of (multi)-scaling exponents which monitor the dissipation regime. In the case of economics, we obtain a distribution of waiting times needed to achieve a predefined level of return. Such a distribution typically goes through a maximum at a time called the *optimal investment horizon* τ^*_ρ, since this defines the most likely waiting time for obtaining a given return ρ. By considering equal positive and negative levels of return, we report on a quantitative gain-loss asymmetry most pronounced for short horizons.

1. Inverse Statistics of Turbulence Data

The understanding of intermittency effects in fully developed turbulence and the associated multiscaling spectrum of exponents, is probably the most fundamental open problem in turbulence research.[1] The traditional way of describing this is, as already suggested by Kolmogorov,[2] to consider the velocity difference between two points of the turbulent state, raise this difference to the moment q, and then study the variation with respect to the distance between the two points, also called structure functions where the corresponding scaling exponents are called structure function exponents.[1] It has become clear both from many experimental,[3–5] numerical[6] and theoretical considerations,[7] that this set of exponents is very non-trivial, defining an infinity of independent exponents leading to at "curved" variation of the scaling exponent with the moment. We propose to "invert" the structure function equation, and consider instead averaged moments of the distance between two points, given a velocity difference between those points. This leads to an alternative

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way of describing and analyzing a turbulent velocity field and one obtains a new set of exponents.

Let us introduce the well known structure functions for the velocity field $\mathbf{u}(\mathbf{x}, t)$ of a fully developed turbulent state, obtained either from the Navier–Stokes equations or from measurements

$$\langle \Delta u_{\mathbf{x}}(\ell)^q \rangle \sim \ell^{\zeta_q} \tag{1}$$

where the difference is defined as

$$\Delta u_{\mathbf{x}}(\ell) = \mathbf{u}(\mathbf{x} + \mathbf{r}) - \mathbf{u}(\mathbf{x}) \,, \qquad \ell = |\mathbf{r}| \,. \tag{2}$$

The average in Eq. (1) is over space (and maybe time). We have assumed full isotropy of the velocity field. The set of exponents ζ_q forms a multiscaling spectrum.[7]

Alternatively, we now consider the following quantities, which is denoted the *inverted structure functions*[8,9]

$$\langle \ell(\Delta u_{\mathbf{x}})^q \rangle \sim |\Delta u_{\mathbf{x}}|^{\delta_q} \tag{3}$$

where the difference $\Delta u_{\mathbf{x}}$ is again defined as in Eq. (2) and $\ell(\Delta u_{\mathbf{x}})$ is understood as the *minimal* distance in $\mathbf{r}$, measured from $\mathbf{x}$, for which the velocity difference exceeds the value $\Delta u_{\mathbf{x}}$. In other words, we fix a certain set of values of the velocity difference $\Delta u_{\mathbf{x}}$. Starting out from the point $\mathbf{x}$, we monitor the distances $\ell(\Delta u_{\mathbf{x}})$ where the velocity differences are equal to the prescribed values. Performing an average over space (and maybe time) the inverted structure functions Eq. (3) are obtained. By assuming self-similarity of the small scale velocity differences, one expects a trivial set of exponents δ_q where the variation with the moment q is determined by one exponent. Say, in the standard Kolmogorov theory we know that the velocity differences behave as $\Delta u \sim \ell^{1/3}$, forgetting for a moment the averaging brackets. Inverting this equations, we of course obtain $\ell \sim \Delta u^3$ and would expect a trivial relation $\delta_q = 3q$. In case of an intermittent and singular velocity field without self-similarity of the small scale velocity differences, this would be completely different and the averaging brackets will be crucial, relating to the statistics of the varying quantity that is averaged. We will show, based on shell model calculations, that in turbulence there exists a new spectrum δ_q. Let us for a moment reflect on the case $q = 1$. Using the standard value $\zeta_1 \sim 0.38 - 0.40$, the simple inversion gives $\delta_1 \sim 2.5$. Our calculations indicate that this value is not obtained in a turbulent model field. Instead we find a value $\delta_1 \sim 2.0 - 2.1$.

To apply this scheme we employ the Gledzer–Ohkitani–Yamada, GOY, shell model[10,11] which has be intensively studied over the last years.[12,13] This model is a rough approximation to the Navier–Stokes equations and is formulated on a discrete set of k-values, $k_n = r^n$. We use the standard value $r = 2$. In term of a complex Fourier mode, u_n, of the velocity field the model reads

$$\left(\frac{d}{dt} + \nu k_n^2\right) u_n = i k_n \left(a_n u_{n+1}^* u_{n+2}^* + \frac{b_n}{2} u_{n-1}^* u_{n+1}^* + \frac{c_n}{4} u_{n-1}^* u_{n-2}^*\right) + f \delta_{n,4} \,, \tag{4}$$

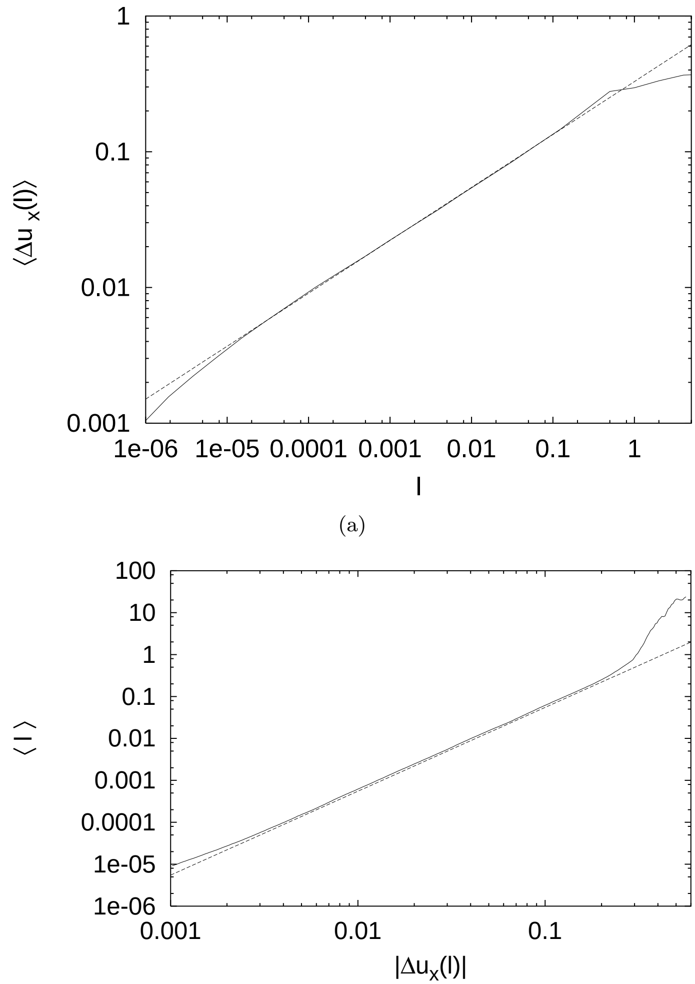
with boundary conditions $b_1 = b_N = c_1 = c_2 = a_{N-1} = a_N = 0$. f is an external, constant forcing, here on the forth mode. In order to construct a field in real space we apply a trick proposed by Vulpiani, to Fourier transform the shell amplitude to real space, see Ref. 13.

Equipped with a real space time dependent velocity field we start out with a test of this field by computing the standard velocity structure functions, given by Eq. (1). Indeed, the field exhibits nice scaling invariance as shown in Fig. 1a, where the first order velocity structure function is presented. We have extracted all the exponents with moment up to $q = 10$ and the corresponding results are shown in Fig. 2a. Having checked this we proceed to extract the inverted structure functions, Eq. (3). As the starting point we set $\mathbf{x} = \mathbf{0}$ and vary again along the coordinate axes. For a fixed value of $\Delta u_{\mathbf{0}}$, ℓ is increased until *for the first time* the velocity difference exceeds this fixed value: this defines $\ell(\Delta_{\mathbf{0}} u)$. Then $\Delta u_{\mathbf{0}}$ is increased by one more step and so on. Figure 1b presents the scaling of the first order inverse structure function and the corresponding exponent δ_1 is estimated to a rather good precision, $\delta_1 = 2.02 \pm 0.05$, with a scaling regime of 2 decades on the $\Delta_{\mathbf{0}} u$ axes and 4-5 decades on the ℓ axes. Note, the cut-off at low values of $\Delta_{\mathbf{0}} u$. This cut-off is related, both for values of velocity and distance, to the dissipative cut-off of the standard structure function, see Fig. 1a. The cut-off at large values of $\Delta_{\mathbf{0}} u$ is related to the velocity at the forcing scale. Figure 2b shows the multiscaling spectrum of δ_q.[8] We have included a straight line through the point $(1, \delta_1)$ in order to show the curved nature of the spectrum.

We have introduced the inverted structure functions defined for a velocity field in fully developed turbulence and the corresponding scaling exponents. These inverted structure functions (or exit time statistics) have been applied to study the intermediate dissipation regime in the very interesting paper by Biferale, Vulpiani and coworkers.[9] It has also been found for the forward enstrophy spectrum of two-dimensional turbulence, that even though the standard structure function exponents for the corresponding smooth velocity field are trivial (without multiscaling), the inverted structure functions may exhibit multiscaling.[14] The suggested method of inverse statistics can thus be a way to extract non-trivial information from seemingly simple data sets.

2. Inverse Statistiscs of Financial Data

Financial time series have been recorded and studied for many decades. With the appearance of the computer, this development has accelerated, and today large amounts of financial data are recorded daily. These data are used in the financial industry for statistical studies and for benchmarking. In particular, they can be used to measure the performance of a financial instrument. Traditionally this has been done by studying the distribution of returns[15–17] calculated over a *fixed* time period Δt. Such distributions measure how much an initial investment, made at time t, has gained or lost by the time $t + \Delta t$. Numerous empirical studies have



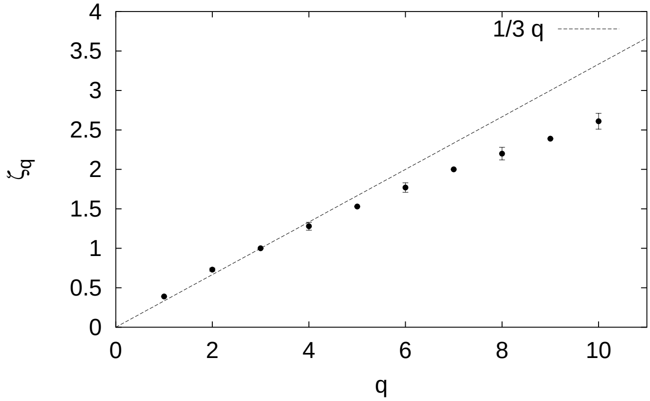
(a)

(b)

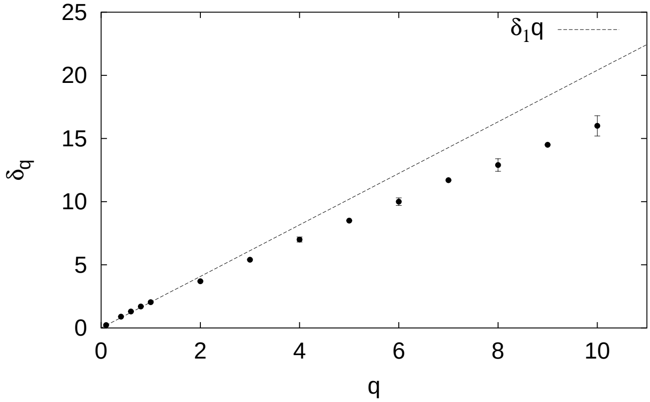
Fig. 1. a): The velocity structure function of order one. The line has a slope of 0.39. b): The inverse structure function of order one. The line has a slope of 2.02. Note the inner cut-off related to the dissipative cut-off in a), and the outer cut-off given by velocity of the forcing scale.[8]

demonstrated that for not too large Δt's, say from a few seconds to weeks, the corresponding (return) distributions are characterized by so-called fat tails.[15–18] This is to say that the probability for large price changes are much larger then what is to be expected from Gaussian statistics, an assumption typically made in theoretical and mathematical finance.[15–17] However, as Δt is increased even further, the distribution of returns gradually converge to the Gaussian distribution.

In the context of economics, it was recently suggested,[19] partly inspired by earlier work in turbulence,[20] to alternatively study the distribution of waiting times needed to reach a *fixed* level of return. These waiting times, for reasons to be clarified



(a)



(b)

Fig. 2. a): The exponents ζ_q for the velocity structure functions, with selected error bars. The line corresponds to Kolmogorov theory. b): The exponents δ_q for the inverted structure functions. The line is adjusted to pass through the value of the first order exponent $(1, \delta_1)$.[8]

in the discussion below, were termed *investment horizons*, and the corresponding distributions the *investment horizon distributions*. Furthermore, it was shown for positive levels of return, that the distributions of investment horizons had a well-defined maximum followed by a power-law tail scaling like[a]

$$p(t) \sim t^{-3/2} \,.$$

The maximum of this distribution signifies the *optimal investment horizon* for an investor aiming for a given return.

In order to present the method, let us start by letting $S(t)$ denote the asset price. Then the logarithmic return at time t, calculated over a time interval Δt, is

[a]Notice that this scaling behavior implies that the first (average investment horizon), and higher, moments of this distribution do not exist.

defined as[15–17]

$$r_{\Delta t}(t) = s(t+\Delta t) - s(t)\,, \tag{5}$$

where $s(t) = \ln S(t)$. Hence the log-return is nothing but the log-price change of the asset. We consider a situation where an investor is aiming for a given return level denoted ρ, which may be both positive (being "long" on the market) or negative (being "short" on the market). If the investment is made at time t, then the investment horizon is defined as the time $\tau_\rho(t) = \Delta t$ so that the inequality $r_{\Delta t}(t) \geq \rho$ when $\rho \geq 0$, or $r_{\Delta t}(t) \leq \rho$ when $\rho < 0$, is satisfied for the *first* time. The investment horizon distribution, $p(\tau_\rho)$, is then the distribution of investment horizons τ_ρ (see Fig. 4) averaged over the data.

A classic assumption made in theoretical finance is that the asset prices follow a geometrical Brownian motion, i.e. $s(t) = \ln S(t)$ is just a Brownian motion. For a Brownian motion, the investment horizon (first passage time) problem is known analytically.[21,22] It can be shown that the investment horizon distribution is given by the Gamma-distribution:

$$p(t) = |a| \exp(-a^2/t)/(\sqrt{\pi} t^{3/2})\,,$$

where $a \propto \rho$. Note, that in the limit of large (waiting) times, one recovers the well-known first return probability $p(t) \sim t^{-3/2}$. As the empirical logarithmic stock price process is known not to be Brownian,[15–18] we instead suggest to use a generalized (shifted) Gamma distribution of the form:

$$p(t) = \frac{\nu}{\Gamma(\frac{\alpha}{\nu})} \frac{|\beta|^{2\alpha}}{(t+t_0)^{\alpha+1}} \exp\left\{ -\left(\frac{\beta^2}{t+t_0} \right)^{\nu} \right\}\,, \tag{6}$$

as a basis for fitting the empirical investment horizon distributions. It will be seen below, that this form parametrize the empirically data excellently. Note, that the distribution, Eq. (6), reduces to the Gamma-distribution (given above) in the limit of $\alpha = 1/2$, $\beta = a$, $\nu = 1$, and $t_0 = 0$. Furthermore, the maximum of this distribution, i.e. the *optimal investment horizon*, is located at

$$\tau_\rho^* = \beta^2 (\nu/(\alpha+1))^{1/\nu} - t_0$$

for a given level of return ρ. If the underlying asset price process is geometric Brownian, then one would have $\tau_\rho^* \sim \rho^2$ for *all* values of ρ. We will later see that this is far from what is observed empirically.

It is well-known that many historic financial time series posses an (often close to exponential) positive drift over long time scales. If such a drift is present in the analyzed time series, one can obviously not compare directly the histograms for positive and negative levels of return. Since we in this paper mainly will be interested in making such a comparison, one has to be able to reduce the effect of the drift significantly. One possibility for detrending the data is to use deflated asset prices. However, in the present study we have chosen an alternative strategy for drift removal based on the use of wavelets,[23] which has the advantages of being non-parametric and does not rest on any economic theory whatsoever. This technique has been described in detail elsewhere,[19] and will therefore not be repeated here. It suffices to say that this wavelet technique enables a separation of the original time series into a short scale (detrended) time series $\tilde{s}(t)$ and a drift term $d(t)$ so that $s(t) = \tilde{s}(t) + d(t)$. In Fig. 3, we see the effect of this procedure on the whole history of one of the major US economical indicators, namely the Dow Jones Industrial Average (DJIA). In this particular example, which is the one used in the analysis, the separation is set to 1000 trading days, corresponding to roughly 4 calendar years.

Based on $\tilde{s}(t)$ for the DJIA, the empirical investment horizon distributions, $p(\tau_\rho)$, can easily be calculated for various levels of return ρ. In Fig. 4 these empirical distributions for $\rho = 0.05$ (open circles) and $\rho = -0.05$ (open squares) are presented. The solid lines in this figure are the maximum likelihood fits of the empirical data to the functional form (6). It is indeed observed that the generalized Gamma distribution, Eq. (6), fits the empirical data well for both positive and negative levels of return. It has been checked separately that the quality of the fits are of comparable quality for other values of ρ. However, as $|\rho|$ becomes large, the empirical distributions are hampered by low statistics that makes the fitting procedure more difficult.

The most interesting feature that can be observed from Fig. 4, is the apparent asymmetry between the empirical investment horizon distributions for $\rho = \pm 0.05$. In particular, for $\rho = -0.05$ there is a higher probability, as compared to what is observed for $\rho = 0.05$, to find short investment horizons, or in other words, *draw-downs* are faster then *draw-ups*. Consequently, one might say that there exists a gain-loss asymmetry! This result is in agreement with the drawdown/drawup

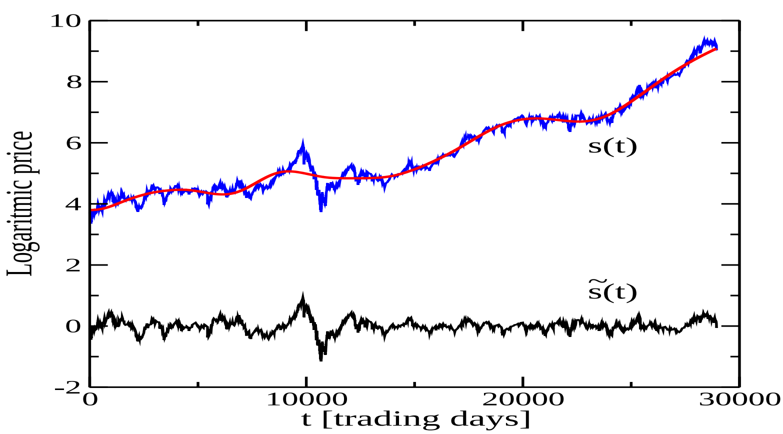


Fig. 3. The historic daily logarithmic closure prices, $S(t)$, of the Dow Jones Industrial Average (DJIA) over the period from May 26, 1896 to June 5, 2001. The upper curly curve is the raw logarithmic DJIA price $s(t) = \ln S(t)$, while the smooth curve represents the drift on a scale larger then 1000 trading days. The lower curly curve represents the wavelet filtered logarithmic DJIA data, $\tilde{s}(t)$, defining the fluctuations of $s(t)$ around the drift.

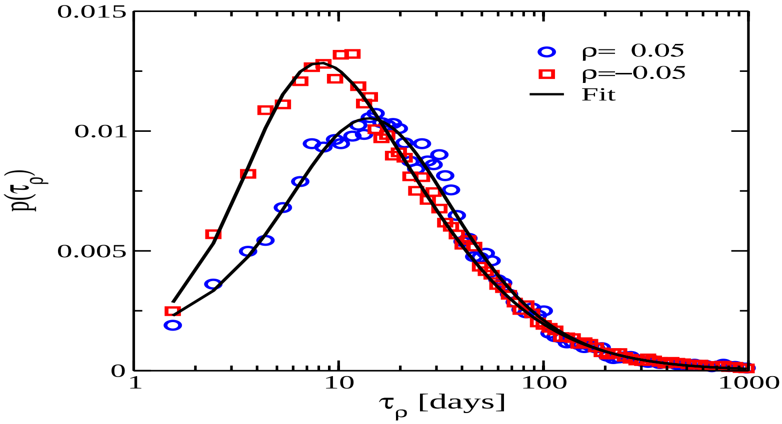


Fig. 4. The investment horizon distributions for the DJIA closing prices at a return level $|\rho| = 0.05$. The open symbols correspond to the empirical distributions, while the solid lines represents the maximum likelihood fit of these distributions to the functional form given by Eq. (6). The fitting parameters used to obtain these fits are for $\rho = 0.05$: $\alpha = 0.50$, $\beta = 4.5$ days$^{1/2}$, $\nu = 2.4$, and $t_0 = 11.2$ days; and for $\rho = -0.05$: $\alpha = 0.50$, $\beta = 5.0$ days$^{1/2}$, $\nu = 0.7$, and $t_0 = 0.6$ days.

analysis presented in Ref. 24. Similar results to those presented here have also been obtained for SP500 and NASDAQ.

Figure 5 depicts the optimal investment horizon *vs* level of return. From this figure it is observed that the asymmetry feature found for a return level of 5% is not unique. For the smallest levels considered, $|\rho| \sim 10^{-3}$, no asymmetry can be detected. However, as $|\rho|$ is gradually increased, the asymmetry starts to emerge at $|\rho| \sim 10^{-2}$. By further increasing the level of return, a state of saturation for the asymmetry appears to be reached. In this state the asymmetry in the optimal investment horizon for the DJIA is almost 200 trading days.

These findings in fact confirms the saying in the financial industry that *it takes time to drive up prices*. From this analysis, one may add *compared to driving them down*, a result that coincides with the common believe that the market reacts more violently to negative information than to positive. To our knowledge, this is the first time that such statements have been founded in a quantitative analysis.

We have considered inverse statistics in economics. It is argued that the natural candidate for such statistics is what we call the investment horizon distribution. Such a distribution, obtained from the historic data of a given market, indicates the time span an investor historically has to wait in order to obtain a predefined level of return. The distributions are parametrized excellently by a shifted generalized Gamma distributions for which the first moment does not exist. The typical waiting time, for a given level of return ρ, can therefore be characterized by e.g. the time position of the maximum of the distribution, i.e. by the *optimal* investment horizon. By studying the behaviour of this quantity for positive (gain) and negative (loss) levels of return, a very interesting and pronounced gain-loss asymmetry emerges.

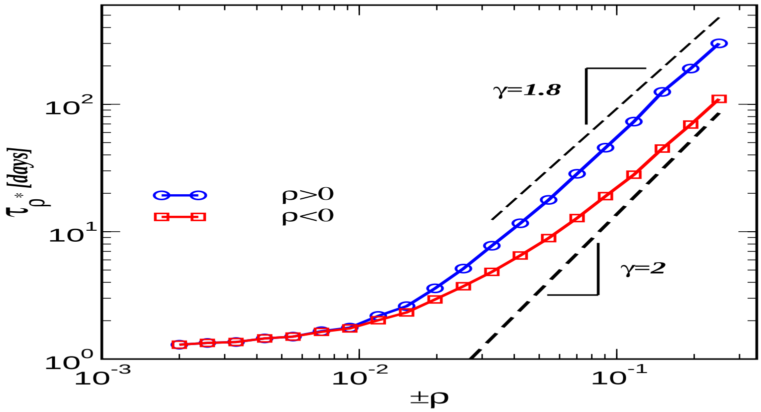


Fig. 5. The optimal investment horizon τ_ρ^* for positive (open circles) and negative (open squares) levels of return $\pm\rho$. In the case where $\rho < 0$ one has used $-\rho$ on the abscissa for reasons of comparison. If a geometrical Brownian price process is assumed, one will have $\tau_\rho^* \sim \rho^\gamma$ with $\gamma = 2$ for all values of ρ. Such a scaling behaviour is indicated by the lower dashed line in the graph. Empirically one finds $\gamma \simeq 1.8$ (upper dashed line), only for large values of the return.

As we have seen, the inverse statistics provides a useful tool to extract new information of time series in turbulence as well as finance. We believe the same methods can be applied to a range of data from many other systems. In many such cases new knowledge of the particular data set could be gained.

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