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# Surface electromagnetic waves on two-dimensional rough perfectly conducting surfaces

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A planar perfectly conducting surface does not support a surface electromagnetic wave. However, a structured perfectly conducting surface can support such a wave. By means of a Rayleigh equation for the electric field in the vacuum above the two-dimensional rough surface of a semi-infinite perfect conductor we calculate the dispersion relation for surface electromagnetic waves on both a doubly periodic and a randomly rough surface. In the former case, if the periodic surface modulation is weak, the dispersion relation possesses a single branch within the non-radiative region of frequency and wave vector values. In the case of a randomly rough surface, in the small roughness approximation, the binding of the surface wave to the surface is weak, but nonzero. Thus, periodically or randomly structured perfectly conducting surfaces constitute a new type of optical metamaterial. The implications of these results for the analysis of experimental results for the propagation of surface plasmon polaritons on metal surfaces in the far infrared frequency range are discussed.

*Keywords:* Perfect conductor; bigrating; random surface roughness; surface electromagnetic waves; dispersion relation.

En una superficie plana perfectamente conductora no es posible excitar ondas electromagnéticas de superficie. Sin embargo, este tipo de ondas si pueden existir en una superficie estructurada perfectamente conductora. Por medio de una ecuación de Rayleigh para el campo eléctrico en el vacío sobre la superficie rugosa bidimensional de un conductor perfecto, calculamos la relación de dispersión para las ondas electromagnéticas superficiales en los casos de una superficie con doble periodo y una superficie con rugosidad aleatoria. ÊEn el primer caso, si la modulación superficial periódica es débil, la relación de dispersión posee una sola rama dentro de la región no radiativa de valores de frecuencia y vector de onda. En el caso de una superficie con rugosidad aleatoria, en la aproximación de pequeña rugosidad, la liga de la onda superficial a la superficie es débil, pero diferente de cero. Así, las superficies perfectamente conductoras periódica o aleatoriamente estructuradas constituyen un nuevo tipo de metamaterial óptico. Se discuten las implicaciones de estos resultados en el análisis de resultados experimentales sobre la propagación de plasmones polaritones de superficie en superficies metálicas en el intervalo de frecuencias del infrarrojo lejano.

*Descriptores:* Perfectamente conductora; birejilla; rugosidad aleatoria; ondas electromagnéticas de superficie; relación de dispersión.

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## 1. Introduction

If a metamaterial can be defined as a deliberately structured material that possesses physical properties that are not possible in naturally occurring materials, surely deliberately structured surfaces that possess optical properties not found in naturally occurring surfaces can be considered to be optical metamaterials. Such surfaces can be periodically or randomly structured.

It is well known that the planar surface of a semi-infinite perfect conductor does not support a surface electromagnetic wave. This is readily seen. Let us consider the propagation of a surface plasmon polariton of frequency $\omega$ on the planar surface of a semi-infinite metal characterized by a dielectric function $\epsilon(\omega)$ that, for simplicity, we assume to be real. The metal, which occupies the region $x_3 < 0$, is in contact with vacuum, which occupies the region $x_3 > 0$. With no loss of generality we can assume that the surface plasmon polariton is propagating in the $x_1$ direction. The single nonzero component of the magnetic field of this surface wave in the vacuum region is given by

$$H_2^{>}(x_1, x_3|\omega) = \exp[ik(\omega)x_1 - \beta_0(\omega)x_3], \qquad (1)$$

where

$$k(\omega) = \frac{\omega}{c}\left[\frac{|\epsilon(\omega)|}{|\epsilon(\omega)| - 1}\right]^{1/2} \qquad (2)$$

$$\beta_0(\omega) = \frac{\omega}{c}\left[\frac{1}{|\epsilon(\omega)| - 1}\right]^{1/2}, \qquad (3)$$

and we have taken into account that $\epsilon(\omega)$ is negative in the frequency range where surface plasmon polaritons exist. In the far infrared frequency range $|\epsilon(\omega)| \gg 1$, and Eqs. (2)-(3) become

$$k(\omega) \cong \frac{\omega}{c}\left[1 + \frac{1}{2|\epsilon(\omega)|}\right] \qquad (4)$$

$$\beta_0(\omega) \cong \frac{\omega}{c}|\epsilon(\omega)|^{-1/2}. \qquad (5)$$

In the limit of a perfect conductor, when $|\epsilon(\omega)| = \infty$, we see from Eqs. (4)–(5) that the magnetic field in the vacuum region becomes

$$H_2^>(x_1, x_3|\omega) = \exp[i(\omega/c)x_1], \qquad (6)$$

which is just a surface skimming bulk electromagnetic wave that is not bound to the surface ($\beta_0(\omega) \equiv 0$). Consequently, on a flat surface $1/|\epsilon(\omega)|$ must be nonzero for the surface plasmon polariton to bind to the surface.

The wave described by Eq. (6) is "unstable" [1], in the sense that even a slight change of the boundary condition on the perfectly conducting surface converts it into a surface wave or into a surface shape resonance, both of which are bound to the surface.

In this paper we study theoretically the propagation of surface electromagnetic waves on a structured two-dimensional perfectly conducting surface that is doubly periodically rough or randomly rough. This is a problem of some current interest. In two recent papers, Pendry and his colleagues [2,3] studied the existence of surface electromagnetic waves on corrugated surfaces of perfect conductors. Two cases were considered: a one-dimensional periodic array of rectangular grooves [2], and a square array of holes of square cross section of infinite [2,3] or finite depth [2]. By the use of an effective medium approach they showed that such surface electromagnetic waves exist, and calculated their dispersion curves.

In the work presented here we first derive the Rayleigh equation for the electric field in the vacuum region above a two-dimensional rough perfectly conducting surface. We then consider doubly periodic surfaces with more general surface profiles than those considered in Refs. 2 and 3, and use the Rayleigh equation, rather than an effective medium approach, to calculate the dispersion relation of the surface electromagnetic waves it supports. We also obtain the dispersion relation for surface electromagnetic waves on a randomly rough surface on the basis of the small roughness approximation to the Rayleigh equation. Thus, for both types of two-dimensional surface roughness we find that a perfectly conducting surface supports surface electromagnetic waves.

Because no real metal is a perfect conductor, even in the far infrared region of the optical spectrum, it might be thought that this investigation is a purely academic exercise. Not at all.

We have seen above that in the vacuum above a perfectly flat metal surface the electromagnetic fields of a surface plasmon polariton decay exponentially with increasing distance into the vacuum from the surface [Eq. (1)]. In the far infrared, the function $\beta_0(\omega)$ characterizing the rate of this decay is given by Eq. (5). Since the skin depth of the electromagnetic fields in the metal is $d(\omega) = (c/\omega)|\epsilon(\omega)|^{-1/2}$, in this frequency range we may write

$$\beta_0(\omega) = (\omega/c)^2 d(\omega). \qquad (7)$$

If the frequency of the surface plasmon polariton corresponds to a wavelength of $100\mu$m, at which the skin depth of the metal is $d(\omega) = 500\mathring{A}$, we find that $\beta_0(\omega)^{-1} = 5.07$ mm, *i.e.* approximately 50 wavelengths. Thus, in the far infrared the electromagnetic fields of the surface plasmon polariton extend far into the vacuum from the surface, *i.e.* it is weakly bound to the surface.

However, in experiments carried out several years ago by Stegeman and his colleagues [4], these authors excited surface plasmon polaritons on bare (and overcoated) silver surfaces in the far infrared (118.8 $\mu$m) by the use of a grating coupler, and found excitation efficiencies very much larger than expected on the basis of perturbation theory [5]. These results could be understood qualitatively if, for some reason, the surface plasmon polaritons were bound to the surface more tightly than predicted by the dielectric theory of a planar metal surface, *i.e.* if $\beta_0(\omega)$ is larger than predicted in Eq. (5).

This suggestion prompted a theoretical investigation [6] into the propagation of a surface plasmon polariton on a one-dimensional periodically or randomly corrugated metal surface in the far infrared frequency range. It was shown there that such surface roughness, even if its amplitude is quite small compared to the wavelength of the surface plasmon polariton, can increase the binding of that wave very substantially over that expected from the dielectric theory of the flat surface. This effect exists in any frequency range in which surface plasmon polaritons exist, but it can dominate the binding for waves on metal surfaces at far infrared frequencies where the binding provided by the dielectric response of the metal is so very weak. It was shown in Ref. 6 that in the limit of infinite conductivity ($|\epsilon(\omega)| \to \infty$) the surface waves still bind to the corrugated surface in the absence of field penetration into the substrate. This result follows directly from the literature [7–10] on a mathematically isomorphic problem. The planar surface of a semi-infinite isotropic elastic medium does not support surface acoustic waves of shear horizontal polarization. However, shear horizontal waves do bind to the surface if a periodic grating is ruled on it, as was first noted by Auld, *et al.* [7], or if a one-dimensional random profile is ruled on it [11,12]. If one formulates the surface plasmon polariton problem in terms of the magnetic field in the wave, which is parallel to the surface for propagation normal to the grooves and ridges on it, then in the limit of infinite conductivity the wave equation and boundary condition at the surface become identical to those occurring in the discussion of shear horizontal surface acoustic waves on the surface of an isotropic elastic medium.

The speed of transverse sound waves is replaced by that of light in vacuum.

These results prompted the conclusion that many experimental studies of surface plasmon propagation on corrugated metal surfaces in the far infrared frequency range have been interpreted within the framework of the incorrect zero-order picture, in which it is assumed that the dominant role in the binding of these waves is played by the dielectric response of the metal. The correct zero-order picture in many cases is that of a wave on a surface whose conductivity is infinite; the dominant role in the binding is played by the interaction of the surface wave fields with the corrugations of the surface, with the dielectric response of the metal playing a minor role.

Our aim in this paper is to provide this correct zero-order picture in the case that the perfectly conducting surface is defined either by doubly periodic or two-dimensional random surface profile functions of rather general form, extending thereby the approaches used and the results obtained in Refs. 2 and 3.

## 2. The Rayleigh equation

The two-dimensional perfectly conducting rough surface we consider is defined by the equation $x_3 = \zeta(\mathbf{x}_\parallel)$, where $\mathbf{x}_\parallel = (x_1, x_3, 0)$ is an arbitrary vector in the plane $x_3 = 0$. The region $x_3 > \zeta(x_1)$ is vacuum, the region $x_3 < \zeta(\mathbf{x}_\parallel)$ is the perfect conductor. The surface profile function $\zeta(\mathbf{x}_\parallel)$ is assumed to be a single-valued function of $\mathbf{x}_\parallel$ that is differentiable with respect to $x_1$ and $x_2$.

The electric field in the vacuum region $x_3 > \zeta(\mathbf{x}_\parallel)$ can be written as

$$\mathbf{E}(\mathbf{x}|\omega) = \int \frac{d^2q_\parallel}{(2\pi)^2}\left\{\frac{c}{\omega}[\hat{\mathbf{q}}_\parallel \alpha_0(q_\parallel) - \hat{\mathbf{x}}_3 q_\parallel] A_p(\mathbf{q}_\parallel) + (\hat{\mathbf{x}}_3 \times \hat{\mathbf{q}}_\parallel) A_s(\mathbf{q}_\parallel)\right\} \exp[i\mathbf{q}_\parallel \cdot \mathbf{x}_\parallel + i\alpha_0(q_\parallel)x_3], \tag{8}$$

where $\hat{\mathbf{q}}_\parallel = \mathbf{q}_\parallel / q_\parallel$, $q_\parallel = |\mathbf{q}_\parallel|$, and

$$\alpha_0(q_\parallel) = [(\omega/c)^2 - q_\parallel^2]^{1/2},$$

$$\mathrm{Re}\alpha_0(q_\parallel) > 0, \quad \mathrm{Im}\alpha_0(q_\parallel) > 0. \tag{9}$$

The coefficients $A_p(\mathbf{q}_\parallel)$ and $A_s(\mathbf{q}_\parallel)$ are the amplitudes of the $p-$ and $s-$polarized components of this field with respect to the local sagittal plane defined by the vectors $\hat{\mathbf{q}}_\parallel$ and $\hat{\mathbf{x}}_3$.

Let us define the vector

$$\mathbf{J}_E(\mathbf{x}_\parallel|\omega) = \mathbf{n} \times \mathbf{E}(\mathbf{x}|\omega)\Big|_{x_3=\zeta(\mathbf{x}_\parallel)}, \tag{10}$$

where

$$\mathbf{n} = (-\zeta_1(\mathbf{x}_\parallel), -\zeta_2(\mathbf{x}_\parallel), 1) \tag{11}$$

is a vector normal to the surface $x_3 = \zeta(\mathbf{x}_\parallel)$ at each point on it, directed from the perfect conductor into the vacuum, and we have introduced the definition

$$\zeta_\alpha(\mathbf{x}_\parallel) \equiv \frac{\partial}{\partial x_\propto}\zeta(\mathbf{x}_\parallel) \qquad \alpha = 1, 2. \tag{12}$$

The boundary condition satisfied by the field (8) on the surface $x_3 = \zeta(\mathbf{x}_\parallel)$ is then given by

$$\mathbf{J}_E(\mathbf{x}_\parallel|\omega) = 0. \tag{13}$$

This vector equation constitutes a set of three equations,

$$J_E(\mathbf{x}_\parallel|\omega)_1 = 0 \tag{14}$$

$$J_E(\mathbf{x}_\parallel|\omega)_2 = 0 \tag{15}$$

$$J_E(\mathbf{x}_\parallel|\omega)_3 = 0. \tag{16}$$

However, we have only two unknown functions, namely $A_p(\mathbf{q}_\parallel)$ and $A_s(\mathbf{q}_\parallel)$. To obtain a pair of equations for these two functions we note that Eqs. (14)-(16) are not independent. Because the vector $\mathbf{n}$ is perpendicular to the vector $\mathbf{J}_E(\mathbf{x}_\parallel(\omega)$, we have that

$$\mathbf{n} \cdot \mathbf{J}_E(\mathbf{x}_\parallel|\omega = 0, \tag{17}$$

or

$$-\zeta_1(\mathbf{x}_\parallel)J_E(\mathbf{x}_\parallel|\omega)_1 - \zeta_2(\mathbf{x}_\parallel)J_E(\mathbf{x}_\parallel|\omega)_2 + J_E(\mathbf{x}_\parallel|\omega)_3 = 0. \tag{18}$$

Thus, the satisfaction of any two of Eqs. (14)-(16) guarantees the satisfaction of the third. We will take as the pair of independent equations Eqs. (14) and (15), which written out explicitly are

$$\int \frac{d^2q_\parallel}{(2\pi)^2}\left\{\left[\frac{c}{\omega}q_\parallel\zeta_2(\mathbf{x}_\parallel) - \frac{c}{\omega}\hat{q}_2\alpha_0(q_\parallel)\right] A_p(\mathbf{q}_\parallel) - \hat{q}_1 A_s(\mathbf{q}_\parallel)\right\} \exp[i\mathbf{q}_\parallel \cdot \mathbf{x}_\parallel + i\alpha_0(q_\parallel)\zeta(\mathbf{x}_\parallel)] = 0 \tag{19}$$

$$\int \frac{d^2q_\parallel}{(2\pi)^2}\left\{\left[\frac{c}{\omega}\hat{q}_1\alpha_0(q_\parallel) - \frac{c}{\omega}q_\parallel\zeta_1(\mathbf{x}_\parallel)\right] A_p(\mathbf{q}_\parallel) - \hat{q}_2 A_s(\mathbf{q}_\parallel)\right\}\exp[i\mathbf{q}_\parallel \cdot \mathbf{x}_\parallel + i\alpha_0(q_\parallel)\zeta(\mathbf{x}_\parallel)] = 0. \tag{20}$$

We now introduce the representation

$$\exp[i\alpha_0(q_\parallel)\zeta(\mathbf{x}_\parallel)] = \int \frac{d^2Q_\parallel}{(2\pi)^2} I(\alpha_0(q_\parallel)|\mathbf{Q}_\parallel) \exp(i\mathbf{Q}_\parallel \cdot \mathbf{x}_\parallel), \tag{21}$$

so that

$$I(\alpha_0(q_\parallel)|\mathbf{Q}_\parallel) = \int d^2x_\parallel \exp[-i\mathbf{Q}_\parallel \cdot \mathbf{x}_\parallel)] \exp[i\alpha_0(q_\parallel)\zeta(\mathbf{x}_\parallel)]. \quad (22)$$

By differentiating both sides of Eq. (21) with respect to $x_\mu$ ($\mu = 1, 2$), we obtain a second useful result,

$$\zeta_\mu(\mathbf{x}_\parallel) \exp[i\alpha_0(q_\parallel)\zeta(\mathbf{x}_\parallel)] = \int \frac{d^2Q_\parallel}{(2\pi)^2} \frac{Q_\mu}{\alpha_0(q_\parallel)} I(\alpha_0(q_\parallel)|\mathbf{Q}_\parallel) \exp(i\mathbf{Q}_\parallel \cdot \mathbf{x}_\parallel). \quad (23)$$

When we introduce Eqs. (21) and (23) into Eqs. (19) and (20), then, on multiplying the resulting equations by $\exp(-i\mathbf{k}_\parallel \cdot \mathbf{x}_\parallel)$, where $\mathbf{k}_\parallel$ is an arbitrary two-dimensional wave vector, and integrating on $\mathbf{x}_\parallel$, we obtain as the equations for the amplitudes $A_p(\mathbf{q}_\parallel)$ and $A_s(\mathbf{q}_\parallel)$

$$\int \frac{d^2q_\parallel}{(2\pi)^2} I(\alpha_0(q_\parallel)|\mathbf{k}_\parallel - \mathbf{q}_\parallel) \times \left\{ \frac{c}{\omega} \frac{q_\parallel k_\parallel \hat{k}_2 - (\omega^2/c^2)\hat{q}_2}{\alpha_0(q_\parallel)} A_p(\mathbf{q}_\parallel) - \hat{q}_1 A_s(\mathbf{q}_\parallel) \right\} = 0 \quad (24)$$

$$\int \frac{d^2q_\parallel}{(2\pi)^2} I(\alpha_0(q_\parallel)|\mathbf{k}_\parallel - \mathbf{q}_\parallel) \times \left\{ -\frac{c}{\omega} \frac{q_\parallel k_\parallel \hat{k}_1 - (\omega^2/c^2)\hat{q}_1}{\alpha_0(q_\parallel)} A_p(\mathbf{q}_\parallel) - \hat{q}_2 A_s(\mathbf{q}_\parallel) \right\} = 0. \quad (25)$$

Although one can use Eqs. (24)-(25) to study electromagnetic excitations at the rough perfectly conducting surface, it is convenient to transform them into a pair of equations that resemble the reduced Rayleigh equations arising in the study of electromagnetic excitations at a penetrable, e.g. finitely conducting metal, surface [13]. To obtain this pair of equations we multiply Eq. (24) by $\hat{k}_2$, Eq. (25) by $-\hat{k}_1$, and add the resulting equations. The result is

$$\int \frac{d^2q_\parallel}{(2\pi)^2} I(\alpha_0(q_\parallel)|\mathbf{k}_\parallel - \mathbf{q}_\parallel) \times \left\{ \frac{c}{\omega} \frac{k_\parallel q_\parallel - (\omega/c)^2(\hat{\mathbf{k}}_\parallel \cdot \hat{\mathbf{q}}_\parallel)}{\alpha_0(q_\parallel)} A_p(\mathbf{q}_\parallel) + (\hat{\mathbf{k}}_\parallel \times \hat{\mathbf{q}}_\parallel)_3 A_s(\mathbf{q}_\parallel) \right\} = 0. \quad (26)$$

We next multiply Eq. (24) by $\hat{k}_1$ and Eq. (25) by $\hat{k}_2$, and add the resulting equations. The result is

$$\int \frac{d^2q_\parallel}{(2\pi)^2} I(\alpha_0(q_\parallel)|\mathbf{k}_\parallel - \mathbf{q}_\parallel) \times \left\{ \frac{\omega}{c} \frac{(\hat{\mathbf{k}}_\parallel \times \hat{\mathbf{q}}_\parallel)_3}{\alpha_0(q_\parallel)} A_p(\mathbf{q}_\parallel) + (\hat{\mathbf{k}}_\parallel \cdot \hat{\mathbf{q}}_\parallel) A_s(\mathbf{q}_\parallel) \right\} = 0. \quad (27)$$

It should be noted that Eqs. (26)-(27) are not obtained simply by letting $\epsilon(\omega) \to -\infty$ in the corresponding equations in Ref. 13.

We now apply Eqs. (26)-(27) to a study of electromagnetic surface waves on two-dimensional periodic and randomly rough surfaces.

## 3. A bigrating

We assume here that the surface profile function $\zeta(\mathbf{x}_\parallel)$ is a doubly periodic function of $\mathbf{x}_\parallel$,

$$\zeta(\mathbf{x}_\parallel + \mathbf{x}_\parallel(\ell)) = \zeta(\mathbf{x}_\parallel), \quad (28)$$

where

$$\mathbf{x}_\parallel(\ell) = \ell_1\mathbf{a}_1 + \ell_2\mathbf{a}_2. \quad (29)$$

In Eq. (29) $\mathbf{a}_1$ and $\mathbf{a}_2$ are the two noncollinear primitive translation vectors of a two-dimensional Bravais lattice, while $\ell_1$ and $\ell_2$ are any positive or negative integers or zero, which we denote collectively by $\ell$. The area of the primitive unit cell of the two-dimensional lattice defined by Eq. (29) is $a_c = |\mathbf{a}_1 \times \mathbf{a}_2|$.

We also need the lattice reciprocal to the one defined by Eq. (29), whose sites are given by

$$\mathbf{G}_\parallel(h) = h_1\mathbf{b}_1 + h_2\mathbf{b}_2, \quad (30)$$

where the primitive translation vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ satisfy the conditions

$$\mathbf{a}_i \cdot \mathbf{b}_j = 2\pi\delta_{ij}, \quad (31)$$

and $h_1$ and $h_2$ are any positive and negative integers and zero that we denote collectively by $h$.

For a surface of the type defined by Eq. (28) the function $I(\alpha_0(q_\parallel)|\mathbf{Q}_\parallel)$ becomes

$$I(\alpha_0(q_\parallel)|\mathbf{Q}_\parallel) = \sum_{\mathbf{G}_\parallel} (2\pi)^2 \delta(\mathbf{Q}_\parallel - \mathbf{G}_\parallel) \times \frac{1}{a_c} \int_{a_c} d^2x_\parallel \exp(-i\mathbf{G}_\parallel \cdot \mathbf{x}_\parallel) \exp[i\alpha_0(q_\parallel)\zeta(\mathbf{x}_\parallel)]. \quad (32)$$

In obtaining this expression we have used the result

$$\sum_\ell e^{-i\mathbf{Q}_\parallel \cdot \mathbf{x}_\parallel(\ell)} = \frac{(2\pi)^2}{a_c} \sum_{\mathbf{G}_\parallel} \delta(\mathbf{Q}_\parallel - \mathbf{G}_\parallel). \quad (33)$$

We also introduce the expansions

$$A_{p,s}(\mathbf{q}_\parallel) = \sum_{\mathbf{G}_\parallel} (2\pi)^2 \delta(\mathbf{q}_\parallel - \mathbf{k}_\parallel - \mathbf{G}_\parallel) a^{(p,s)}_{\mathbf{k}_\parallel}(\mathbf{G}_\parallel), \quad (34)$$

where $\mathbf{k}_\parallel$ is the two-dimensional wave vector of the surface electromagnetic wave, in order that the electric field in the vacuum region satisfy the Bloch-Floquet theorem. When the results given by Eqs. (32) and (34) are substituted into Eqs. (26)-(27) we obtain as the equation for the coefficients $\{a^{(p,s)}_{\mathbf{k}_\parallel}(\mathbf{G}_\parallel)\}$

$$\sum_{\mathbf{G}'_\parallel} \mathcal{I}(\alpha_0(K'_\parallel)|\mathbf{K}_\parallel - \mathbf{K}'_\parallel)$$

$$\times \begin{pmatrix} \frac{c}{\omega} \quad \frac{K_\parallel K'_\parallel - (\omega/c)^2 \hat{\mathbf{K}}_\parallel \cdot \hat{\mathbf{K}}'_\parallel}{\alpha_0(K'_\parallel)} & (\hat{\mathbf{K}}_\parallel \times \hat{\mathbf{K}}'_\parallel)_3 \\ \frac{\omega}{c} \quad \frac{(\hat{\mathbf{K}}_\parallel \times \hat{\mathbf{K}}'_\parallel)_3}{\alpha_0(K'_\parallel)} & (\hat{\mathbf{K}}_\parallel \cdot \hat{\mathbf{K}}'_\parallel) \end{pmatrix}$$

$$\times \begin{pmatrix} a^{(p)}_{\mathbf{k}_\parallel}(\mathbf{G}'_\parallel) \\ a^{(s)}_{\mathbf{k}_\parallel}(\mathbf{G}'_\parallel) \end{pmatrix} = 0, \quad (35)$$

where

$$\mathcal{I}(\alpha_0(K'_\parallel)|\mathbf{K}_\parallel - \mathbf{K}'_\parallel) = \frac{1}{a_c}$$

$$\times \int_{a_c} d^2x_\parallel \exp[-i(\mathbf{K}_\parallel - \mathbf{K}'_\parallel)\cdot \mathbf{x}_\parallel]$$

$$\times \exp[i\alpha_0(K'_\parallel)\zeta(\mathbf{x}_\parallel)] \quad (36)$$

and, to simplify the notation, we have introduced the vectors $\mathbf{K}_\parallel = \mathbf{k}_\parallel + \mathbf{G}_\parallel$ and $\mathbf{K}'_\parallel = \mathbf{k}_\parallel + \mathbf{G}'_\parallel$ in writing Eq. (35). The dispersion relation for surface electromagnetic waves on a perfectly conducting bigrating is then obtained by equating to zero the determinant of the matrix of coefficients in Eq. (35).

The solutions of this equation are even functions of $\mathbf{k}_\parallel$, $\omega_s(-\mathbf{k}_\parallel) = \omega_s(\mathbf{k}_\parallel)$, where $s$ labels the solutions (bands) for a given $\mathbf{k}_\parallel$ in the order of increasing magnitude. They are also periodic functions of $\mathbf{k}_\parallel$ with a period that is the first Brillouin zone of the bigrating, $\omega_s(\mathbf{k}_\parallel + \mathbf{G}_\parallel) = \omega_s(\mathbf{k}_\parallel)$. The solutions can therefore be sought for values of $\mathbf{k}_\parallel$ inside this first Brillouin zone, and inside the nonradiative region defined by $|\mathbf{k}_\parallel| > (\omega/c)$.

In the present work the surface profile function $\zeta(\mathbf{x}_\parallel)$ is represented by a square array of hemiellipsoids on an otherwise planar surface of a perfect conductor. The primitive translation vectors of the square lattice are given by

$$\mathbf{a}_1 = a(1,0), \quad \mathbf{a}_2 = a(0,1). \quad (37)$$

The primitive translation vectors of the lattice reciprocal to the square lattice defined by Eqs. (29) and (37), defined by Eq. (31), are

$$\mathbf{b}_1 = \frac{2\pi}{a}(1,0), \quad \mathbf{b}_2 = \frac{2\pi}{a}(0,1). \quad (38)$$

The profile function describing a square array of hemiellipsoids of radius $R$ $(< a)$ and height $HR$ (Fig. 1) can be written in the form

$$\zeta(\mathbf{x}_\parallel) = \sum_{\ell} s(\mathbf{x}_\parallel - \mathbf{x}_\parallel(\ell)), \quad (39)$$

where

$$s(\mathbf{x}_\parallel) = \begin{cases} H\sqrt{R^2 - x^2_\parallel} & |\mathbf{x}_\parallel| < R \\ 0 & |\mathbf{x}_\parallel| > R. \end{cases} \quad (40)$$

Because of the circular cross section of these hemiellipsoids, the dispersion relation satisfies the relation

$$\omega(\overleftrightarrow{S}(\mathbf{k}_\parallel)) = \omega(\mathbf{k}_\parallel) \quad (41)$$

for each band, where $\overleftrightarrow{S}$ is a $2 \times 2$ real orthogonal matrix representative of any of the point group operations that leave the square lattice invariant. In the present case this is the point group $C_{4v}$. The property (41) has the consequence that all of the independent solutions of the dispersion relation are obtained if we restrict the wave vector $\mathbf{k}_\parallel$ to the *irreducible element* of the two-dimensional first Brillouin zone. This is that region of the Brillouin zone that generates the entire zone when transformed by the application of the operations of the point group $C_{4v}$ to it.

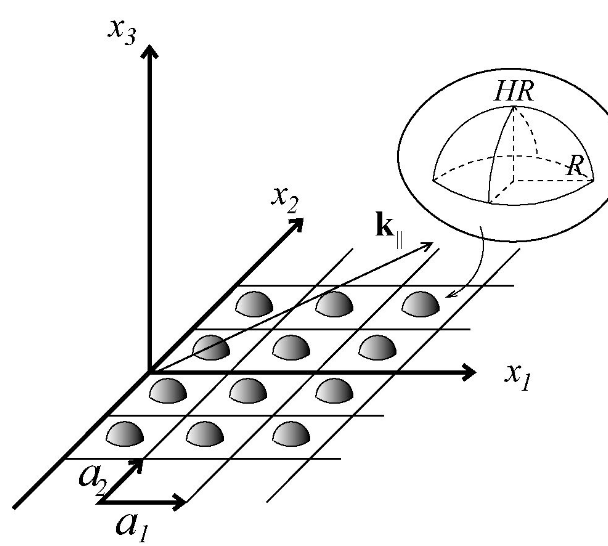


FIGURE 1. A section of an infinitely extended bigrating whose primitive translation vectors are $\mathbf{a}_1$ and $\mathbf{a}_2$. The hemiellipsoids forming it have radius $R$ and amplitude $HR$. The wave vector of the surface electromagnetic wave is denoted by $\mathbf{k}_\parallel$.

The function $\mathcal{I}(\gamma|\mathbf{G}_{\parallel})$ for the profile function (39)-(40) can be evaluated analytically, with the result

$$\mathcal{I}(\gamma|\mathbf{G}_{\parallel}) = \frac{1}{a_c}\int_{a_c} d^2x_{\parallel}\exp(-i\mathbf{G}_{\parallel}\cdot\mathbf{x}_{\parallel})\exp[i\gamma\zeta(\mathbf{x}_{\parallel})]$$

$$= 1+\frac{2\pi R^2}{a^2}\sum_{n=1}^{\infty}\frac{(i\gamma HR)^n}{(n+2)n!} \quad \mathbf{G}_{\parallel}=0 \tag{42}$$

$$= \frac{2\pi R^2}{a^2}\sum_{n=1}^{\infty}\frac{(i\gamma HR)^n}{n!}\frac{2^{\frac{n}{2}}\Gamma(\frac{n}{2}+1)}{(G_{\parallel}R)^{\frac{n}{2}+1}}J_{\frac{n}{2}+1}(G_{\parallel}R)$$

$$\mathbf{G}_{\parallel}\neq 0. \tag{43}$$

For the values of the parameters used in the numerical calculations based on the results obtained in this section, namely, it was necessary to take into account only the first few terms ($\approx 15$) in these rapidly convergent series.

In order to solve the dispersion relation obtained from Eq. (35) the infinite sum had to be truncated. This was done by restricting the reciprocal lattice vectors $\mathbf{G}_{\parallel}(h) = h_1\mathbf{b}_1 + h_2\mathbf{b}_2$ and $\mathbf{G}'_{\parallel}(h) = h'_1\mathbf{b}_1 + h'_2\mathbf{b}_2$ to those that satisfied the conditions $[h_1^2+h_2^2]^{1/2}\leq N_{max}$ and $[h_1'^2+h_2'^2]^{1/2}\leq N_{max}$ for some integer $N_{max}$. The convergence of a solution was tested by increasing $N_{max}$ systematically until it stopped changing.

The solution of the dispersion relation was based on the search for real zeros of a real-valued determinant. For a fixed $\mathbf{k}_{\parallel}$ inside or on the boundary of the irreducible element of the first Brillouin zone the interval $0\leq\omega\leq ck_{\parallel}$ was sampled for changes of the sign of the determinant. The frequencies at which these occurred were labeled in the order of increasing magnitude. This collection constitutes the different branches of the dispersion curve.

In Fig. 2a we present dispersion curves along the symmetry directions in the first Brillouin zone, *i.e.* for wave vectors $\mathbf{k}_{\parallel}$ on the boundary of the irreducible element of the Brillouin zone. The values of the parameters assumed in obtaining these results were $R/a = 0.375$, $H = 0.5$, and $N_{max} = 6$. The surface thus consists of a square lattice of hemiellipsoidal protuberances on an otherwise planar surface. For these values of the parameters defining the surface the dispersion curve consists of only a single branch within the nonradiative region of $\omega$ and $\mathbf{k}_{\parallel}$ values, as in the case of a square array of holes of square cross section and finite [2] or infinite depth [2, 3] on a perfect conductor. All frequencies above the maximum frequency of this branch constitute a stop band. In Fig. 2b we present the dispersion curves calculated for the values $R/a = 0.375$, $H = -0.5$, and $N_{max} = 6$. In this case the surface consists of a square lattice of hemiellipsoidal indentations ("dimples"). For this surface the dispersion curve also consists of a single branch within the nonradiative region. However, it bends away from the vacuum light line into the nonradiative region more weakly than the dispersion curve corresponding to the lattice of protuberances.

In Figs. 3a and 3b we present the dispersion curves calculated for the values of the parameters given by $R/a = 0.375$, $H = 1$, and $R/a = 0.375$, $H = -1$, respectively. The value of $N_{max} = 10$ was used in calculating both figures. For both the surface formed from protuberances and the surface formed from indentations the dispersion curves bend away from the vacuum light line into the nonradiative region more strongly than the dispersion curves presented in Fig. 2a and 2b.

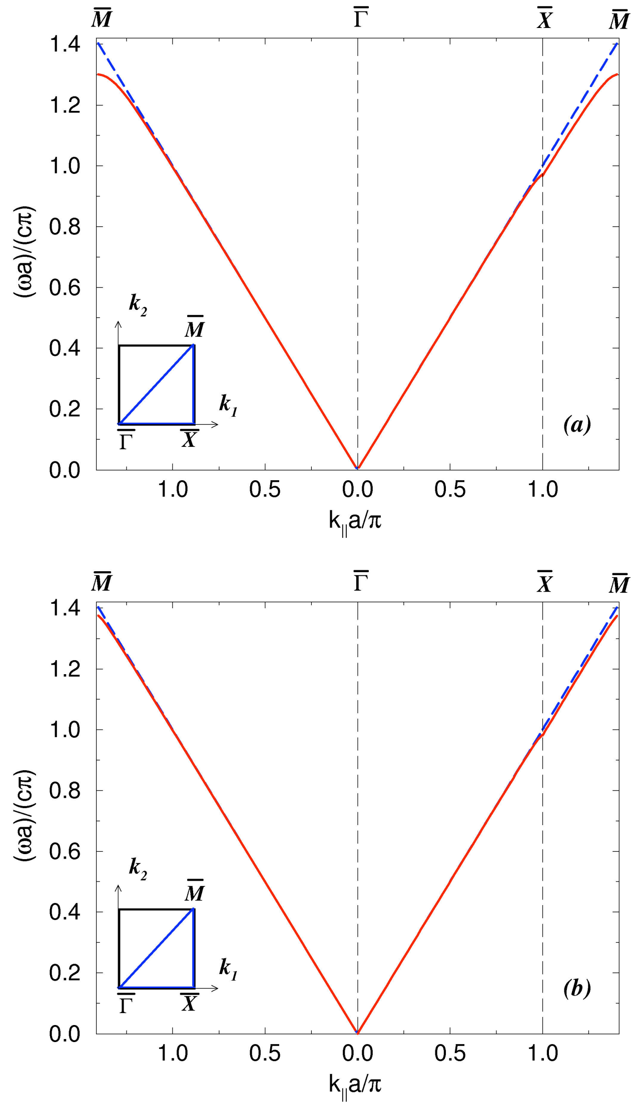


FIGURE 2. Dispersion curve for surface electromagnetic waves on a square lattice of hemiellipsoids on the planar surface of a perfect conductor. The single branch of the dispersion curve in the nonradiative region of $\omega$ and $\mathbf{k}_{\parallel}$ values is plotted as a function of $\mathbf{k}_{\parallel}$ along the boundary of the irreducible element of the Brillouin zone (depicted in the insets). The dotted lines represent the vacuum light line $\omega = ck_{\parallel}$. The values of the parameters assumed in obtaining these results are: (a) $R/a = 0.375$, $H = 0.5$, $N_{max} = 6$; (b) $R/a = 0.375$, $H = -0.5$, $N_{max} = 6$.

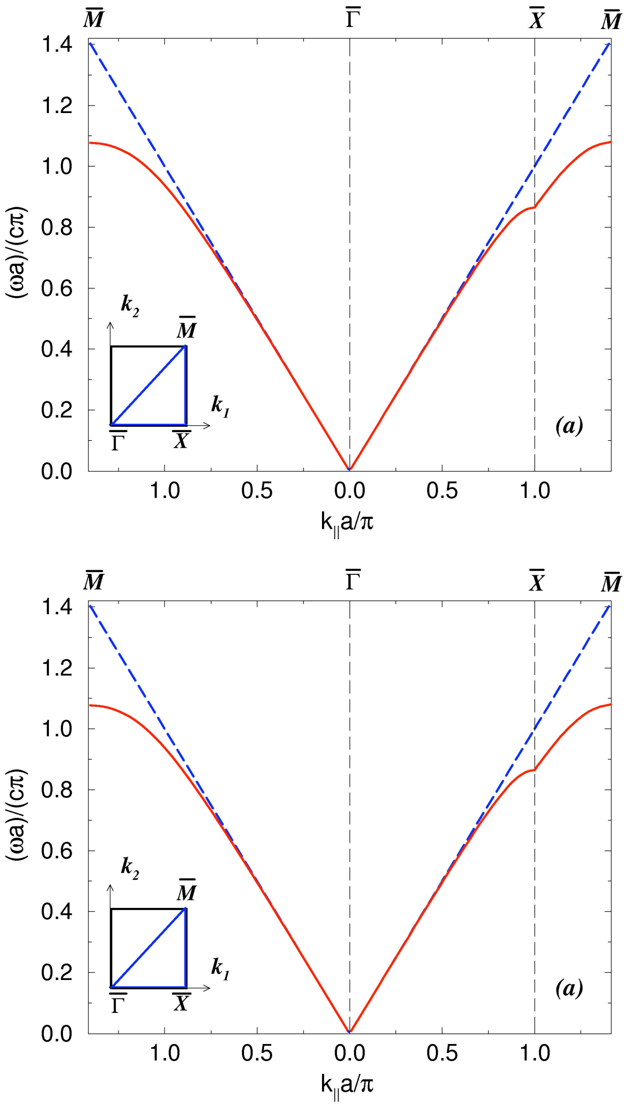


FIGURE 3. The same as Fig. 2, but for the following values of the parameters: (a) $R/a = 0.375$, $H = 1$, $N_{max} = 10$; (b) $R/a = 0.375$, $H = -1$, $N_{max} = 10$.

The results presented in Figs. 2 and 3 also illustrate that the dispersion curves of the surface electromagnetic waves on a doubly periodic surface of a perfect conductor can be varied by changing the parameters characterizing its roughness.

## 4. A randomly rough surface

Let us now assume that the surface profile function $\zeta(\mathbf{x}_\parallel)$, in addition to being a single-valued function of $\mathbf{x}_\parallel$, and differentiable with respect to $x_1$ and $x_2$, constitutes a stationary, zero-mean, isotropic, Gaussian random process, defined by

$$\langle \zeta(\mathbf{x}_\parallel)\zeta(\mathbf{x}'_\parallel)\rangle = \delta^2 W(|\mathbf{x}_\parallel - \mathbf{x}'_\parallel|). \tag{44}$$

In this equation the angle brackets denote an average over the ensemble of realizations of the surface profile function, while $\delta = \langle \zeta^2(\mathbf{x}_\parallel)\rangle^{1/2}$ is the rms height of the surface.

We also need the Fourier integral representation of $\zeta(\mathbf{x}_\parallel)$,

$$\zeta(\mathbf{x}_\parallel) = \int \frac{d^2Q_\parallel}{(2\pi)^2}\hat{\zeta}(\mathbf{Q}_\parallel)\exp(i\mathbf{Q}_\parallel \cdot \mathbf{x}_\parallel). \tag{45}$$

The Fourier coefficient $\hat{\zeta}(\mathbf{Q}_\parallel)$ is also a zero-mean Gaussian random process that is defined by

$$\langle \hat{\zeta}(\mathbf{Q}_\parallel)\hat{\zeta}(\mathbf{Q}'_\parallel)\rangle = (2\pi)^2\delta(\mathbf{Q}_\parallel + \mathbf{Q}'_\parallel)\delta^2 g(|\mathbf{Q}_\parallel|). \tag{46}$$

In this result $g(|\mathbf{Q}_\parallel|)$ is the power spectrum of the surface roughness, and is given by

$$g(|\mathbf{Q}_\parallel|) = \int d^2x_\parallel W(|\mathbf{x}_\parallel|)\exp(-i\mathbf{Q}_\parallel \cdot \mathbf{x}_\parallel). \tag{47}$$

To obtain the dispersion relation for surface electromagnetic waves on a randomly rough perfectly conducting surface we will use the small roughness approximation. This consists of expanding the function $I(\alpha_0(q_\parallel)|\mathbf{k}_\parallel - \mathbf{q}_\parallel)$ in Eqs. (26)-(27) in powers of the surface profile function and retaining only the first two terms:

$$I(\alpha_0(q_\parallel)|\mathbf{k}_\parallel - \mathbf{q}_\parallel) = (2\pi)^2\delta(\mathbf{k}_\parallel - \mathbf{q}_\parallel) + i\alpha_0(q_\parallel)\hat{\zeta}(\mathbf{k}_\parallel - \mathbf{q}_\parallel) + O(\zeta^2). \tag{48}$$

Equations (26)-(27) can then be rewritten in the form $(\alpha, \beta{=}p, s)$

$$a_\alpha(k_\parallel)A_\alpha(\mathbf{k}_\parallel) = \sum_\beta \int \frac{d^2q_\parallel}{(2\pi)^2} V_{\alpha\beta}(\mathbf{k}_\parallel|\mathbf{q}_\parallel)A_\beta(\mathbf{q}_\parallel), \tag{49}$$

where

$$a_p(k_\parallel) = -\frac{c}{\omega}\alpha_0(k_\parallel), \quad a_s(k_\parallel) = 1, \tag{50}$$

and

$$V_{\alpha\beta}(\mathbf{k}_\parallel|\mathbf{q}_\parallel) = -i\hat{\zeta}(\mathbf{k}_\parallel - \mathbf{q}_\parallel)\alpha_0(q_\parallel)U_{\alpha\beta}(\mathbf{k}_\parallel|\mathbf{q}_\parallel), \tag{51}$$

with

$$U_{pp}(\mathbf{k}_\parallel|\mathbf{q}_\parallel) = \frac{c}{\omega}\frac{k_\parallel q_\parallel - (\omega/c)^2(\hat{\mathbf{k}}_\parallel \cdot \hat{\mathbf{q}}_\parallel)}{\alpha_0(q_\parallel)} \tag{52}$$

$$U_{ps}(\mathbf{k}_\parallel|\mathbf{q}_\parallel) = (\hat{\mathbf{k}}_\parallel \times \hat{\mathbf{q}}_\parallel)_3 \tag{53}$$

$$U_{sp}(\mathbf{k}_\parallel|\mathbf{q}_\parallel) = \frac{\omega}{c}\frac{(\hat{\mathbf{k}}_\parallel \times \hat{\mathbf{q}}_\parallel)_3}{\alpha_0(q_\parallel)} \tag{54}$$

$$U_{ss}(\mathbf{k}_\parallel|\mathbf{q}_\parallel) = (\hat{\mathbf{k}}_\parallel \cdot \hat{\mathbf{q}}_\parallel). \tag{55}$$

The surface profile function $\zeta(\mathbf{x}_\parallel)$ entering Eq. (49) is a random process. Consequently, the solutions $A_p(\mathbf{k}_\parallel)$ and $A_s(\mathbf{k}_\parallel)$ of this equation are random functions. Just as $\zeta(\mathbf{x}_\parallel)$ is defined by the moments of its probability density function, so can these solutions be defined by the moments of their probability density functions. Of these, a particularly important one is the first moment, $\langle A_{p,s}(\mathbf{k}_\parallel)\rangle$, which describes the propagation of the mean wave across the randomly rough surface.

To obtain the equations satisfied by $\langle A_p(\mathbf{k}_\parallel)\rangle$ and $\langle A_s(\mathbf{k}_\parallel)\rangle$, we apply the smoothing operator $P$, which averages every function to which it is applied over the ensemble of realizations of the surface profile function, and its complementary operator $Q = 1 - P$, in turn to Eq. (49), with the results

$$a_\alpha(k_\parallel) P A_\alpha(\mathbf{k}_\parallel) = \sum_\beta \int \frac{d^2q_\parallel}{(2\pi)^2} P V_{\alpha\beta}(\mathbf{k}_\parallel|\mathbf{q}_\parallel) \times [P A_\beta(\mathbf{q}_\parallel) + Q A_\beta(\mathbf{q}_\parallel)] \tag{56}$$

$$a_\beta(q_\parallel) Q A_\beta(\mathbf{q}_\parallel) = \sum_\gamma \int \frac{d^2r_\parallel}{(2\pi)^2} Q V_{\beta\gamma}(\mathbf{q}_\parallel|\mathbf{r}_\parallel) \times [P A_\gamma(\mathbf{r}_\parallel) + Q A_\gamma(\mathbf{r}_\parallel)]. \tag{57}$$

When we use the results that $PV_{\alpha\beta}(\mathbf{k}_\parallel|\mathbf{q}_\parallel) = 0$, since $\zeta(\mathbf{x}_\parallel)$ is a zero-mean random process, and that $QA_\gamma(\mathbf{r}_\parallel)$ is of order $\zeta$, Eqs. (56)-(57) simplify to

$$a_\alpha(k_\parallel) P A_\alpha(\mathbf{k}_\parallel) = \sum_\beta \int \frac{d^2q_\parallel}{(2\pi)^2} P V_{\alpha\beta}(\mathbf{k}_\parallel|\mathbf{q}_\parallel) Q A_\beta(\mathbf{q}_\parallel) \tag{58}$$

$$a_\beta(q_\parallel) Q A_\beta(\mathbf{q}_\parallel) = \sum_\gamma \int \frac{d^2r_\parallel}{(2\pi)^2} V_{\beta\gamma}(\mathbf{q}_\parallel|\mathbf{r}_\parallel) P A_\gamma(\mathbf{r}_\parallel). \tag{59}$$

On combining Eqs. (58) and (59) we obtain as the equation satisfied by the mean amplitude $\langle A_\alpha(\mathbf{k}_\parallel)\rangle$

$$a_\alpha(k_\parallel)\langle A_\alpha(\mathbf{k}_\parallel)\rangle = \sum_\beta \sum_\gamma \int \frac{d^2q_\parallel}{(2\pi)^2} \int \frac{d^2r_\parallel}{(2\pi)^2} \times \frac{\langle V_{\alpha\beta}(\mathbf{k}_\parallel|\mathbf{q}_\parallel) V_{\beta\gamma}(\mathbf{q}_\parallel|\mathbf{r}_\parallel)\rangle}{a_\beta(q_\parallel)} \langle A_\gamma(\mathbf{r}_\parallel)\rangle. \tag{60}$$

The result that

$$\langle V_{\alpha\beta}(\mathbf{k}_\parallel|\mathbf{q}_\parallel) V_{\beta\gamma}(\mathbf{q}_\parallel|\mathbf{r}_\parallel)\rangle = -(2\pi)^2 \delta(\mathbf{k}_\parallel - \mathbf{r}_\parallel) \delta^2 g(|\mathbf{k}_\parallel - \mathbf{q}_\parallel|) \times \alpha_0(q_\parallel)\alpha_0(p_\parallel) U_{\alpha\beta}(\mathbf{k}_\parallel|\mathbf{q}_\parallel) U_{\beta\gamma}(\mathbf{q}_\parallel|\mathbf{k}_\parallel) \tag{61}$$

reduces Eq. (60) to

$$a_\alpha(k_\parallel)\langle A_\alpha(\mathbf{k}_\parallel)\rangle = -\sum_\beta \sum_\gamma \int \frac{d^2q_\parallel}{(2\pi)^2} \delta^2 g(|\mathbf{k}_\parallel - \mathbf{q}_\parallel|) \frac{1}{a_\beta(q_\parallel)} \times U_{\alpha\beta}(\mathbf{k}_\parallel|\mathbf{q}_\parallel)\alpha_0(q_\parallel) U_{\beta\gamma}(\mathbf{q}_\parallel|\mathbf{k}_\parallel)\alpha_0(k_\parallel)\langle A_\gamma(\mathbf{k}_\parallel)\rangle. \tag{62}$$

However, the integral over the azimuthal angle of the vector $\mathbf{q}_\parallel$ vanishes unless the summation index $\gamma$ equals $\alpha$. This is a consequence of our assumption that $\zeta(\mathbf{x}_\parallel)$ is an isotropic random process. Thus, we obtain the pair of uncoupled equations as the dispersion relations of surface waves of p or s polarization:

$$a_\alpha(k_\parallel) = -\sum_\beta \int \frac{d^2q_\parallel}{(2\pi)^2} \delta^2 g(|\mathbf{k}_\parallel - \mathbf{q}_\parallel|) \times \frac{\alpha_0(k_\parallel)\alpha_0(q_\parallel)}{a_\beta(q_\parallel)} U_{\alpha\beta}(\mathbf{k}_\parallel|\mathbf{q}_\parallel) U_{\beta\alpha}(\mathbf{q}_\parallel|\mathbf{k}_\parallel). \tag{63}$$

We now focus our attention on the p-polarized surface electromagnetic waves. With the aid of Eqs. (50) and (52)-(55) we find that their dispersion relation is

$$\alpha_0(k_\parallel) = -\delta^2 \int \frac{d^2q_\parallel}{(2\pi)^2} \frac{g(|\mathbf{k}_\parallel - \mathbf{q}_\parallel|)}{\alpha_0(q_\parallel)} \{[k_\parallel q_\parallel - (\omega/c)^2(\hat{\mathbf{k}}_\parallel \cdot \hat{\mathbf{q}}_\parallel)]^2 + (\omega/c)^2[(\hat{\mathbf{k}}_\parallel \times \hat{\mathbf{q}}_\parallel)_3]^2 \alpha_0^2(q_\parallel)\}. \tag{64}$$

In the numerical calculations that follow we will assume that the surface height autocorrelation function $W(|\mathbf{x}_\parallel|)$ has the Gaussian form

$$W(|\mathbf{x}_\parallel|) = \exp(-x_\parallel^2/a^2), \tag{65}$$

so that the power spectrum of the surface roughness also has a Gaussian form,

$$\begin{aligned} g(|\mathbf{k}_\parallel - \mathbf{q}_\parallel|) &= \pi a^2 \exp[-(a^2/4)(\mathbf{k}_\parallel - \mathbf{q}_\parallel)^2] \\ &= \pi a^2 \exp[-(a^2/4)(k_\parallel^2 + q_\parallel^2)] \\ &\times \left[ I_0(\frac{1}{2}a^2 k_\parallel q_\parallel) + 2\sum_{n=1}^{\infty} I_n(\frac{1}{2}a^2 k_\parallel q_\parallel) \times \cos n(\phi_k - \phi_q) \right], \end{aligned} \tag{66}$$

where $I_n(z)$ is the modified Bessel function of order $n$, and $\phi_k$ and $\phi_q$ are the azimuthal angles of the vectors $\mathbf{k}_\parallel$ and $\mathbf{q}_\parallel$, respectively. When Eq. (66) is substituted into Eq. (64), and the angular integrations are carried out, the latter equation becomes

$$\alpha_0(k_\parallel) = -\frac{1}{2}\delta^2 a^2 \exp(-a^2 k_\parallel^2/4) \left\{ \int\limits_0^{\omega/c} dq_\parallel \frac{q_\parallel}{[(\omega/c)^2 - q_\parallel^2]^{1/2}} - i \int\limits_{\omega/c}^{\infty} dq_\parallel \frac{q_\parallel}{[q_\parallel^2 - (\omega/c)^2]^{1/2}} \right\}$$

$$\times \exp(-a^2 q_\parallel^2/4) \left[ A I_0\left(\frac{1}{2}a^2 k_\parallel q_\parallel\right) + B I_1\left(\frac{1}{2}a^2 k_\parallel q_\parallel\right) + C I_2\left(\frac{1}{2}a^2 k_\parallel q_\parallel\right)\right], \tag{67}$$

where

$$A = k_\parallel^2 q_\parallel^2 - \frac{1}{2}(\omega/c)^2 q_\parallel^2 + (\omega/c)^4 \tag{68}$$

$$B = -2k_\parallel q_\parallel (\omega/c)^2 \tag{69}$$

$$C = \frac{1}{2}(\omega/c)^2 q_\parallel^2. \tag{70}$$

To solve this dispersion relation we make the *Ansatz*

$$\frac{\omega^2}{c^2} = k_\parallel^2 - \delta^4 \Delta^2(k_\parallel, \omega). \tag{71}$$

It then follows from Eqs. (9) and (71) that

$$\alpha_0(k_\parallel) = i\delta^2 \Delta(k_\parallel, \omega) \tag{72}$$

where

$$\Delta(k_\parallel, \omega) = \frac{a^2}{2}\exp(-a^2 k_\parallel^2/4) \Bigg\{ \int\limits_{\omega/c}^{\infty} dq_\parallel \frac{q_\parallel}{[q_\parallel^2 - (\omega/c)^2]^{1/2}}$$

$$+ i \int\limits_0^{\omega/c} dq_\parallel \frac{q_\parallel}{[(\omega/c)^2 - q_\parallel^2]^{1/2}} \Bigg\} \exp(-a^2 q_\parallel^2/4) \times \left[ A I_0\left(\frac{1}{2}a^2 k_\parallel q_\parallel\right) + B I_1\left(\frac{1}{2}a^2 k_\parallel q_\parallel\right) + C I_2\left(\frac{1}{2}a^2 k_\parallel q_\parallel\right)\right]. \tag{73}$$

Since the deviation of $(\omega/c)$ from $k_\parallel$ is of $O(\delta^4)$, we can replace $(\omega/c)$ in the integrals in Eq. (67) by $k_\parallel$, with the result that

$$\Delta(k_\parallel, \omega) \approx \Delta(k_\parallel, ck_\parallel) = \Delta_1(k_\parallel) + i\Delta_2(k_\parallel), \tag{74}$$

where

$$\Delta_1(k_\parallel) \cong \frac{a^2}{2}\exp(-a^2 k_\parallel^2/4) \int\limits_{k_\parallel}^{\infty} dq_\parallel \frac{q_\parallel}{(q_\parallel^2 - k_\parallel^2)^{1/2}} \exp(-a^2 q_\parallel^2/4)$$

$$\times \left\{ \left[\frac{1}{2}k_\parallel^2 q_\parallel^2 + k_\parallel^4\right] I_0\left(\frac{1}{2}a^2 k_\parallel q_\parallel\right) - 2k_\parallel^3 q_\parallel I_1\left(\frac{1}{2}a^2 k_\parallel q_\parallel\right) + \frac{1}{2}k_\parallel^2 q_\parallel^2 I_2\left(\frac{1}{2}a^2 k_\parallel q_\parallel\right)\right\} \tag{75}$$

$$\Delta_2(k_\parallel) = \frac{a^2}{2}\exp(-a^2 k_\parallel^2/4) \int\limits_0^{k_\parallel} dq_\parallel \frac{q_\parallel}{(k_\parallel^2 - q_\parallel^2)^{1/2}} \exp(-a^2 q_\parallel^2/4)$$

$$\times \left\{ \left[\frac{1}{2}k_\parallel^2 q_\parallel^2 + k_\parallel^4\right] I_0\left(\frac{1}{2}a^2 k_\parallel q_\parallel\right) - 2k_\parallel^3 q_\parallel I_1\left(\frac{1}{2}a^2 k_\parallel q_\parallel\right) + \frac{1}{2}k_\parallel^2 q_\parallel^2 I_2\left(\frac{1}{2}a^2 k_\parallel q_\parallel\right)\right\}. \tag{76}$$

From Eq. (8) we see that the imaginary part of $\alpha_0(q_\parallel)$ must be positive for the wave to be bound to the surface. From Eqs. (72) and (74) we obtain

$$\alpha_0(k_\parallel) = \delta^2[-\Delta_2(k_\parallel) + i\Delta_1(k_\parallel)], \tag{77}$$

from which it follows that $\Delta_1(k_\parallel)$ must be positive.

With the definitions

$$x = k_\parallel a, \qquad y = q_\parallel a, \tag{78}$$

we obtain from Eqs. (75)–(76)

$$\Delta_{1,2}(k_\parallel) = \frac{x^5}{2a^3} d_{1,2}(x), \tag{79}$$

where $d_1(x)$ and $d_2(x)$ are universal functions given by

$$d_1(x) = \exp(-x^2/4) \int_0^\infty d\theta \cosh\theta \exp[-(x^2/4)\cosh^2\theta]$$
$$\times \left[ \left(\frac{1}{2}\cosh^2\theta + 1\right) I_0\left(\frac{1}{2}x^2\cosh\theta\right) \right.$$
$$- 2\cosh\theta I_1\left(\frac{1}{2}x^2\cosh\theta\right)$$
$$\left. + \frac{1}{2}\cosh^2\theta I_2\left(\frac{1}{2}x^2\cosh\theta\right)\right] \tag{80}$$
$$= \frac{\sqrt{\pi}}{2}\left(\frac{2}{x^3} - \frac{3}{2x} + O(x)\right) \qquad x \to 0 \tag{81}$$

$$d_2(x) = \exp(-x^2/4) \int_0^{\frac{\pi}{2}} d\theta \sin\theta \exp[-(x^2/4)\sin^2\theta]$$
$$\times \left[ \left(1 + \frac{1}{2}\sin^2\theta\right) I_0\left(\frac{1}{2}x^2\sin\theta\right) \right.$$
$$- 2\sin\theta I_1\left(\frac{1}{2}x^2\sin\theta\right)$$
$$\left. + \frac{1}{2}\sin^2\theta\left(\frac{1}{2}x^2\sin\theta\right)\right] \tag{82}$$
$$= \frac{4}{3} - \frac{9}{10}x^2 + O(x^4) \qquad x \to 0. \tag{83}$$

The further changes of variables $y{=}x\cosh\theta$ and $y{=}x\sin\theta$ were made in obtaining Eqs. (80) and (82), respectively.

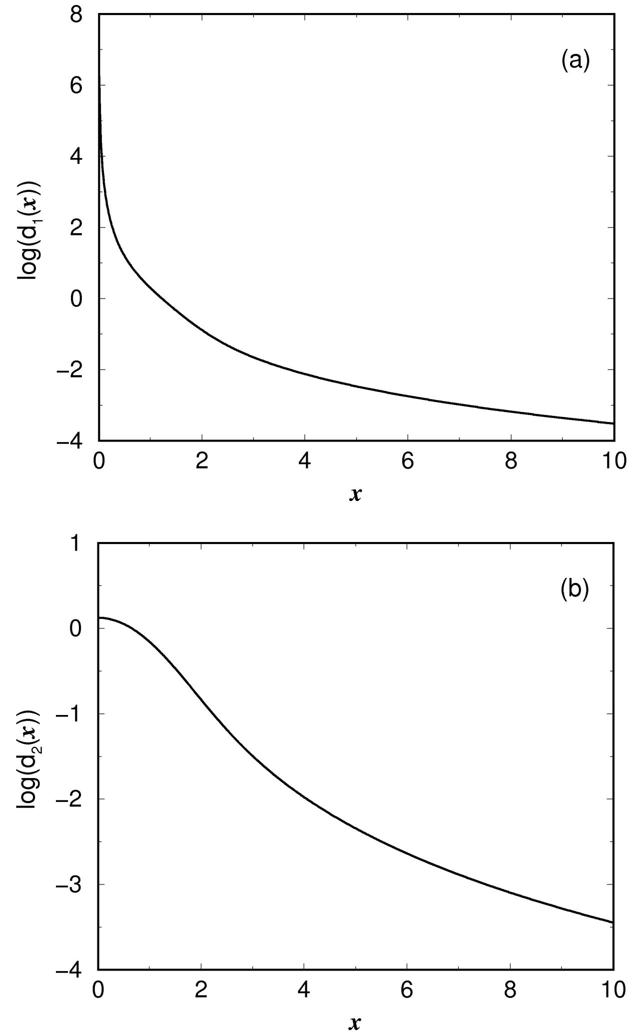


FIGURE 4. (a) The function $d_1(x)$ defined by Eq. (80) of the text; (b) the function $d_2(x)$ defined by Eq. (82) of the text.

On combining Eqs. (71), (74), and (79) we can write the dispersion relation for surface electromagnetic waves on a two-dimensional, randomly rough, perfectly conducting surface in the form

$$\omega(k_\parallel) = ck_\parallel[1 - \frac{\delta^4}{a^4}\omega_1(x) - i\frac{\delta^4}{a^4}\omega_2(x)], \tag{84}$$

where

$$\omega_1(x) = \frac{x^8}{8}[d_1^2(x) - d_2^2(x)] \tag{85}$$
$$= \frac{\pi}{8}[x^2 - \frac{3}{2}x^4 + O(x^6)] \qquad x \to 0 \tag{86}$$
$$\omega_2(x) = \frac{x^8}{4} d_1(x) d_2(x) \tag{87}$$
$$= \frac{\sqrt{\pi}}{3}x^5[1 - \frac{57}{40}x^2 + O(x^4)] \qquad x \to 0. \tag{88}$$

At the same time, from Eqs. (77) and (79) we find that the function $\alpha_0(k_\parallel)$ is given by

$$\alpha_0(k_\parallel) = \frac{\delta^2}{2a^3} x^5 [-d_2(x) + i d_1(x)] \tag{89}$$

$$= \frac{\delta^2}{2a^3} [-\frac{4}{3} x^5 + i\sqrt{\pi} x^2] \qquad x \to 0. \tag{90}$$

We recall that as long as the imaginary part of $\alpha_0(k_\parallel)$ is positive the surface electromagnetic wave is bound to the surface. We see from Eq. (90) that in the long wavelength limit the decay length of the surface wave into the vacuum is proportional to the square of its wavelength parallel to the surface. The wave is therefore weakly bound to the surface in this limit, but it is bound nevertheless.

In Fig. 4 we have plotted $d_1(x)$ and $d_2(x)$ as functions of $x$, while the functions $\omega_1(x)$ and $\omega_2(x)$ are plotted in Fig. 5. We see from Fig. 4a that $d_1(x)$ is a positive function of $x$, so that the surface wave whose dispersion relation is given by Eq. (84) is indeed bound to the surface. This wave also displays the phenomenon of wave slowing ($\omega_1(x) > 0$) for small $x$.

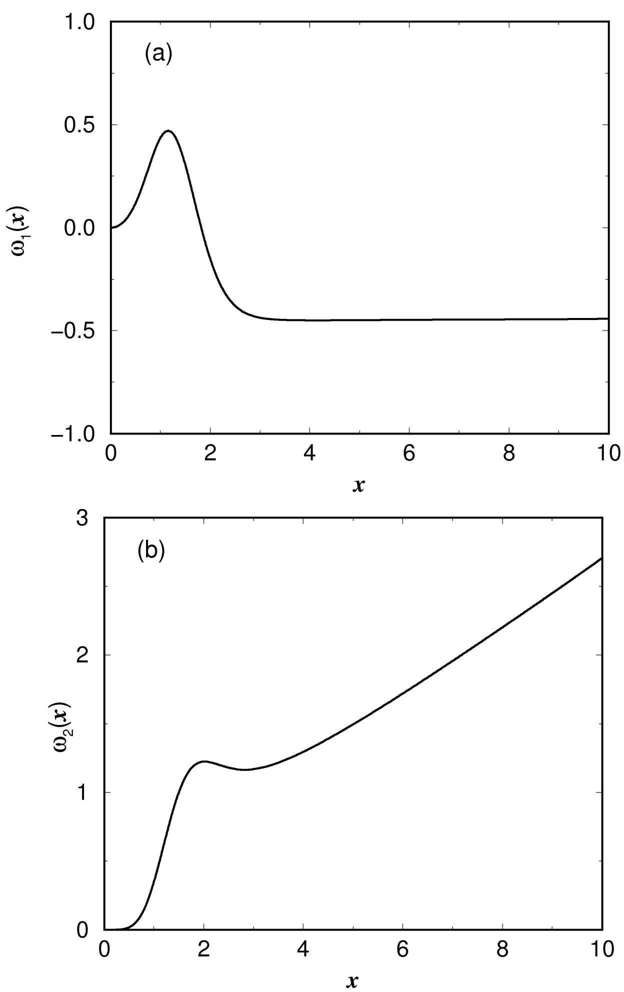


FIGURE 5. (a) The function $\omega_1(x)$ defined by Eq. (85) of the text; (b) the function $\omega_2(x)$ defined by Eq. (87) of the text.

From Eqs. (84) and (88) we see that in the long wavelength limit the attenuation rate of the surface electromagnetic wave due to its roughness-induced conversion into volume electromagnetic waves in the vacuum is proportional to $(k_\parallel a)^6$. The explanation for this dependence is that the frequency dependence of Rayleigh scattering in $d$ dimensions is $\omega^{d+1}$. The protuberances and indentations of the surface responsible for the scattering of the wave in the present case are three dimensional, since they are defined by the equation $x_3 = \zeta(x_1, x_2)$. Thus the Rayleigh scattering law in the present case gives us an $\omega^4$ frequency dependence of the scattering rate in the low frequency, long-wavelength limit. The remaining factor of $\omega^2$ arises because the amplitude of the surface wave decays exponentially with distance into the vacuum region, reducing the volume within which its interaction with the surface roughness occurs thereby. It just has to be remembered that the decay length of the wave in the vacuum region is proportional to the square of its wavelength parallel to the surface [Im$\alpha_0(k_\parallel)$ is proportional to $(k_\parallel a)^2$ in this case: see Eq. (90)].

We conclude this section by pointing out that in order to obtain a dispersion relation that is correct to $O(\delta^2)$ it suffices to expand the function $I(\alpha_0(q_\parallel)|\mathbf{k}_\parallel - \mathbf{q}_\parallel)$ in Eqs. (26)-(27) only to first order in the surface profile function, as was done in Eq. (48). The term of $O(\zeta^2)$ in this expansion, which at first glance might be expected to contribute to the dispersion relation in $O(\delta^2)$, in fact contributes to the dispersion relation in a higher order in $\delta^2$. Indeed, if the term of the second order in $\zeta(\mathbf{x}_\parallel)$ is retained in the expansion of $I(\alpha_0(q_\parallel)|\mathbf{k}_\parallel|\mathbf{q}_\parallel)$ in Eq. (48), the end result is that the left hand side of Eq. (63) becomes

$$\left[1 - \frac{1}{2}\delta^2 \alpha_0^2(k_\parallel)\right] a_\alpha(k_\parallel).$$

The presence of the term

$$-\frac{1}{2}\delta^2 \alpha_0^2(k_\parallel)$$

in this expression leads to a correction to the equation (63) for $a_\alpha(k_\parallel)$ of $O(\delta^4)$, which we have neglected.

## 5. Conclusions

In this paper we have studied theoretically the existence of surface electromagnetic waves on the two-dimensional rough surface of a perfect conductor. We have shown that surface waves do exist on such a surface. Since surface electromagnetic waves do not exist on a flat perfectly conducting surface, the binding of these waves to a perfectly conducting surface is due entirely to its structuring. We have derived the dispersion relations for surface waves on a doubly periodically corrugated surface with a rather general form of its surface profile function, and have illustrated this result by calculating the dispersion curve of these waves in the case where the surface is represented as a square array of hemiellipsoids. However, we have also shown that the structuring of the surface need not be doubly periodic for it to bind a surface electromagnetic wave: a randomly rough surface is sufficient. In the case of a doubly periodic surface we have shown that it is possible to modify the dispersion curves of the surface waves by varying the lattice parameter of the surface structure and the size and shape of the units forming that structure. In the case of a two-dimensional randomly rough surface, in the small roughness approximation, the surface wave dispersion curve can be modified only by varying the ration $(\delta/a)$, since the functions $\omega_1(x)$ and $\omega_2(x)$ are universal functions of $x$.

The results we have obtained in this work support the conclusion reached in Ref. 6 on the basis of calculations carried out for one-dimensional periodic or randomly rough surfaces that the analysis of experimental data on surface plasmon polariton propagation on metal surfaces in the far infrared frequency range must take into account the possibility that the spatial profile of the wave may differ substantially from the one corresponding to a planar ideal surface. A small-amplitude bigrating or random surface roughness can give rise to a wave more tightly bound to the surface than predicted by dielectric theory applied to a perfectly flat surface.

Finally, since they give rise to an optical effect, namely the binding of a surface electromagnetic wave, that is not possible on a flat perfectly conducting surface, doubly periodic or randomly rough perfectly conducting surfaces can indeed be regarded as optical metamaterials.

## Acknowledgement

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