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The angular intensity correlation functions $C^{(1)}$ and $C^{(10)}$ for the scattering of light from randomly rough dielectric and metal surfaces

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Abstract

We study the statistical properties of the scattering matrix $S(q|k)$ for the problem of the scattering of light of frequency ω from a randomly rough one-dimensional surface, defined by the equation $x_3 = \zeta(x_1)$, where the surface profile function $\zeta(x_1)$ constitutes a zero-mean, stationary, Gaussian random process. This is done by studying the effects of $S(q|k)$ on the angular intensity correlation function $C(q, k|q', k') = \langle I(q|k)I(q'|k')\rangle - \langle I(q|k)\rangle\langle I(q'|k')\rangle$, where the intensity $I(q|k)$ is defined in terms of $S(q|k)$ by $I(q|k) = L_1^{-1}(\omega/c)|S(q|k)|^2$, with L_1 the length of the x_1 axis covered by the random surface. We focus our attention on the $C^{(1)}$ and $C^{(10)}$ correlation functions, which are the contributions to $C(q, k|q', k')$ proportional to $\delta(q - k - q' + k')$ and $\delta(q - k + q' - k')$, respectively. The existence of both of these correlation functions is consistent with the amplitude of the scattered field obeying complex Gaussian statistics in the limit of a long surface and in the presence of weak surface roughness. We show that the deviation of the statistics of the scattering matrix from complex circular Gaussian statistics and the $C^{(10)}$ correlation function are determined by exactly the same statistical moment of $S(q|k)$. As the random surface becomes rougher, the amplitude of the scattered field no longer obeys complex Gaussian statistics but obeys complex circular Gaussian statistics instead. In this case the $C^{(10)}$ correlation function should therefore vanish. This result is confirmed by numerical simulation calculations.

1. Introduction

The scattering of light from randomly rough surfaces has attracted attention over many years. The majority of the theoretical and experimental studies of such scattering has been devoted to coherent interference effects occurring in the multiple scattering of electromagnetic waves from randomly rough surfaces and the related backscattering enhancement phenomenon. These effects are contained in the angular distribution of the intensity of the light scattered incoherently, i.e. in the second moment of the scattered field.

Recently, attention has begun to be directed toward theoretical [1–12] and experimental [2, 7, 8, 12, 13] studies of multiple-scattering effects on higher moments of the scattered field, in particular on angular intensity correlation functions. These correlation functions describe how the speckle pattern, formed through the interference of randomly scattered waves, changes when one or more parameters of the scattering system are varied.

The interest in these correlations has been stimulated by the expectation that, just as the inclusion of multiple-scattering processes in the calculation of the angular dependence of the intensity of the light that has been scattered incoherently from, or incoherently through, a randomly rough surface, led to the prediction of enhanced backscattering [14] and enhanced transmission [15], their inclusion in the calculation of higher-order moments of the scattered or transmitted field would also lead to the prediction of new physical effects. This expectation was prompted by the results of earlier theoretical [16, 17] and experimental [18–20] investigations of angular intensity correlation functions in the scattering of classical waves from volume disordered media. In a theoretical investigation [9] it was predicted that three types of correlations occur in such scattering, namely short-range correlations, long-range correlations and infinite-range correlations. These were termed the $C^{(1)}$, $C^{(2)}$ and $C^{(3)}$ correlations, respectively. The $C^{(1)}$ correlation function includes both the 'memory effect' and the 'reciprocal memory effect' [9, 10], so named because of the wavevector conservation conditions they satisfy. Both of these effects have now been observed in volume scattering experiments [16, 17]. The $C^{(2)}$ correlation function has also been observed in volume scattering experiments [18, 19], as has the $C^{(3)}$ correlation function [20].

Until recently, only the $C^{(1)}$ correlation function arising in the scattering of light from a randomly rough surface had been studied theoretically and experimentally [1–8]. In a recent series of papers devoted to theoretical studies of angular correlation functions of the intensity of light scattered from one-dimensional [9, 10] and two-dimensional [10] randomly rough metal surfaces the long-range $C^{(2)}$ and infinite-range $C^{(3)}$ correlation functions were calculated, and two additional types of correlation functions, a short-range correlation function, named $C^{(10)}$, and a long-range correlation function, named $C^{(1.5)}$, were predicted. In very recent experimental work [12] the envelopes of the $C^{(1)}$ and $C^{(10)}$ correlation functions were measured experimentally for the scattering of p-polarized light from weakly rough, one-dimensional gold surfaces. The $C^{(1.5)}$, $C^{(2)}$ and $C^{(3)}$ correlation functions have yet to be observed experimentally.

The question arises as to whether it possible to determine the relative magnitudes of the different correlation functions from a knowledge of the experimental parameters of the surface roughness and its statistical properties. This question had been raised earlier in [12, 16], but not answered definitively. We therefore address it here for the case of a one-dimensional random surface defined by the equation $x_3 = \zeta(x_1)$, on the basis of the single assumption that the surface profile function $\zeta(x_1)$ is a single-valued function of x_1 that constitutes a zero-mean, stationary, Gaussian random process. At the same time we address the question of how the statistical properties of the amplitude of the scattered field are reflected in the symmetry properties of the speckle pattern to which it gives rise.

The outline of this paper is as follows. In section 2 we introduce the angular intensity correlation function and analyse it in terms of the possible statistics of the scattering matrix. In section 3 we illustrate the conclusions of section 2 for the simple example of the scattering of light from the randomly rough surface of a perfect conductor. Finally, in section 4 we present the conclusions drawn from the results obtained in this paper.

2. The angular intensity correlation function

The general angular intensity correlation function $C(q, k|q', k')$ we study in this work is defined by

$$C(q, k|q', k') = \langle I(q|k)I(q'|k')\rangle - \langle I(q|k)\rangle\langle I(q'|k')\rangle, \tag{2.1}$$

where the angle brackets denote an average over the ensemble of realizations of the surface profile function. The intensity $I(q|k)$ entering this expression is defined in terms of the scattering matrix $S(q|k)$ for the scattering of light of frequency ω from a one-dimensional random surface by

$$I(q|k) = \frac{1}{L_1}\left(\frac{\omega}{c}\right)|S(q|k)|^2, \tag{2.2}$$

where L_1 is the length of the x_1 axis covered by the random surface, and the wavenumbers k and q are related to the angles of incidence and scattering, θ_0 and θ_s, measured counterclockwise and clockwise from the normal to the mean scattering surface, respectively, by $k = (\omega/c)\sin\theta_0$ and $q = (\omega/c)\sin\theta_s$.

From equations (2.1) and (2.2) we see that, because the correlation of $I(q|k)$ with itself should generally be stronger than the correlation of $I(q|k)$ with $I(q'|k')$ when $q' \neq q$ and $k' \neq k$, a peak in $C(q, k|q', k')$ is expected when $q' = q$ and $k' = k$. This peak is called the *memory effect peak* for the reason that is explained below. In addition, because $S(q|k)$ is reciprocal, $S(q|k) = S(-k| - q)$, a peak in $C(q, k|q', k')$ is also expected when $q' = -k$ and $k' = -q$. This peak is called the *reciprocal memory effect peak.*

In terms of the scattering matrix $S(q|k)$ the correlation function $C(q, k|q', k')$ becomes

$$\begin{aligned} C(q, k|q', k') = \frac{1}{L_1^2}\frac{\omega^2}{c^2}[&\langle S(q|k)S^*(q|k)S(q'|k')S^*(q'|k')\rangle \\ &- \langle S(q|k)S^*(q|k)\rangle\langle S(q'|k')S^*(q'|k')\rangle]. \end{aligned} \tag{2.3}$$

Since, due to the stationarity of the surface profile function, $\langle S(q|k)\rangle$ is diagonal in q and k, $\langle S(q|k)\rangle = 2\pi\delta(q - k)S(k)$, we introduce the incoherent part of the scattering matrix $\delta S(q|k) = S(q|k) - \langle S(q|k)\rangle$. Then, from the relations between averages of the products of random functions and the corresponding cumulant averages [21, 22] and omitting all terms proportional to $2\pi\delta(q-k)$ and/or $2\pi\delta(q'-k')$ as uninteresting specular effects, equation (2.3) can be rewritten in the form

$$\begin{aligned} C(q, k|q', k') = \frac{1}{L_1^2}\frac{\omega^2}{c^2}[&|\langle\delta S(q|k)\delta S^*(q'|k')\rangle|^2 + |\langle\delta S(q|k)\delta S(q'|k')\rangle|^2 \\ &+ \langle\delta S(q|k)\delta S^*(q|k)\delta S(q'|k')\delta S^*(q'|k')\rangle_c], \end{aligned} \tag{2.4}$$

where $\langle\cdots\rangle_c$ denotes the cumulant average.

Due to the stationarity of the surface profile function $\zeta(x_1)$, $\langle\delta S(q|k)\delta S^*(q'|k')\rangle$ is proportional to $2\pi\delta(q - k - q' + k')$. It gives rise to the contribution to $C(q, k|q', k')$ called $C^{(1)}(q, k|q', k')$ [9, 10] and describes the memory effect and the reciprocal memory effect. The property of a speckle pattern that is characterized by the presence of the factor $2\pi\delta(q-k-q'+k')$ in $C^{(1)}(q, k|q', k')$ is that, if we change the angle of incidence in such a way that k goes into $k' = k + \Delta k$, the entire speckle pattern shifts in such a way that any feature initially at q moves to $q' = q + \Delta k$. This is the reason why the $C^{(1)}$ correlation function was originally named the memory effect. In terms of the angles of incidence and scattering, we have that, if θ_0 is changed into $\theta_0' = \theta_0 + \Delta\theta_0$, any feature in the speckle pattern originally at θ_s is shifted to $\theta_s' = \theta_s + \Delta\theta_s$, where $\Delta\theta_s = \Delta\theta_0\ (\cos\theta_0/\cos\theta_s)$ to first order in $\Delta\theta_0$.

Similarly $\langle \delta S(q|k)\delta S(q'|k')\rangle$ is proportional to $2\pi\delta(q-k+q'-k')$ and contributes to the correlation function $C^{(10)}(q,k|q',k')$ to $C(q,k|q',k')$ [9,10]. The property of a speckle pattern that is characterized by the presence of the factor $2\pi\delta(q-k+q'-k')$ in $C^{(10)}(q,k|q',k')$ is that, if we change the angle of incidence in such a way that k goes into $k'=k+\Delta k$, a feature at $q=k-\Delta q$ will be shifted to $q'=k'+\Delta q$, i.e. to a point as much to one side of the new specular direction as the original point was on the other side of the original specular direction (in wavenumber space). For one and the same incident beam the $C^{(10)}$ correlation function therefore reflects the 'symmetry' of the speckle pattern with respect to the specular direction.

The third term on the right-hand side of equation (2.4), $\langle \delta S(q|k)\delta S^*(q|k)\delta S(q'|k') \delta S^*(q'|k')\rangle_c$, is proportional to $2\pi\delta(0)=L_1$, due to the stationarity of the surface profile function $\zeta(x_1)$ and gives rise to the long-range and infinite-range contributions to $C(q,k|q',k')$ given by the sum $C^{(1.5)}(q,k|q',k')+C^{(2)}(q,k|q',k')+C^{(3)}(q,k|q',k')$ [9,10]. Thus, we have separated out explicitly the contributions to $C(q,k|q',k')$ that have been called $C^{(1)}(q,k|q',k')$ and $C^{(10)}(q,k|q',k')$.

What is more, from equation (2.4) we can easily estimate the relative magnitudes of the different contributions to the general correlation function. Indeed, since $2\pi\delta(0)=L_1$, when the arguments of the δ functions vanish the $C^{(1)}(q,k|q',k')$ and $C^{(10)}(q,k|q',k')$ correlation functions are independent of the length of the surface L_1, because they contain $[2\pi\delta(0)]^2$. At the same time the remaining term in equation (2.4), which yields the sum $C^{(1.5)}(q,k|q',k')+C^{(2)}(q,k|q',k')+C^{(3)}(q,k|q'|k')$, is inversely proportional to the surface length, due to the lack of a second delta function. Therefore, in the limit of a long surface or a large illumination area the long-range and infinite-range correlations are small compared to short-range correlation functions and vanish in the limit of an infinitely long surface. Thus, the experimental observation of the $C^{(1.5)}$, $C^{(2)}$ and $C^{(3)}$ correlation functions requires the use of a short segment of random surface and/or the use of a beam of narrow width for the incident field. A detailed discussion of the conditions under which they may be observed will therefore be deferred to a separate paper.

The preceding results are consistent with the usual assumptions and conclusions encountered in conventional speckle theory [23–25]. Thus, when the surface profile function is assumed to be a stationary random process, and the random surface is assumed to be infinitely long, the scattering matrix $S(q|k)$ becomes the sum of a very large number of independent contributions from different points on the surface. On invoking the central limit theorem, it is found that $S(q|k)$ obeys complex Gaussian statistics. In this case equation (2.4) becomes rigorously [23]

$$C(q,k|q',k') = \frac{1}{L_1^2}\frac{\omega^2}{c^2}[|\langle \delta S(q|k)\delta S^*(q'|k')\rangle|^2 + |\langle \delta S(q|k)\delta S(q'|k')\rangle|^2] \quad (2.5)$$

$$= C^{(1)}(q,k|q',k') + C^{(10)}(q,k|q',k'), \quad (2.6)$$

because all cumulant averages of products of more than two Gaussian random processes vanish. The last term on the right-hand side of equation (2.4) therefore gives the correction to the prediction of the central limit theorem due to the finite length of the random surface.

If it is further assumed, as is done in speckle theory, where the disorder is presumed to be strong, that $\delta S(q|k)$ obeys circular complex Gaussian statistics [24, 25], then $\langle \delta S(q|k)\delta S(q'|k')\rangle = 0$ and the expression for $C(q,k|q',k')$ simplifies to

$$C(q,k|q',k') = \frac{1}{L_1^2}\frac{\omega^2}{c^2}|\langle \delta S(q|k)\delta S^*(q'|k')|^2 \quad (2.7)$$

$$= C^{(1)}(q,k|q',k'). \quad (2.8)$$

This approximation is often called the factorization approximation to $C(q,k|q'k')$ [17].

We recall that, if the complex random variables F_1 and F_2 are jointly circular complex Gaussian random variables, then the conditions

$$\langle \mathrm{Re}\, F_1 \mathrm{Re}\, F_2 \rangle = \langle \mathrm{Im}\, F_1 \mathrm{Im}\, F_2 \rangle, \tag{2.9}$$

$$\langle \mathrm{Re}\, F_1 \mathrm{Im}\, F_2 \rangle = -\langle \mathrm{Im}\, F_1 \mathrm{Re}\, F_2 \rangle, \tag{2.10}$$

have to be satisfied. To analyse how the scattering matrix transforms from a complex Gaussian random process into a circular complex Gaussian random process we represent the scattering matrix in the form $\delta S(q|k) = \delta S_1(q|k) + \mathrm{i}\delta S_2(q|k)$. The expressions for the averages of the products of the real and imaginary parts of $\delta S(q|k)$ can be written in terms of $\langle \delta S(q|k)\delta S^*(q'|k')\rangle$ and $\langle \delta S(q|k)\delta S(q'|k')\rangle$:

$$\langle \delta S_1(q|k)\delta S_1(q'|k')\rangle = \tfrac{1}{2}\mathrm{Re}\,[\langle \delta S(q|k)\delta S^*(q'|k')\rangle + \langle \delta S(q|k)\delta S(q'|k')\rangle] \tag{2.11}$$

$$\langle \delta S_2(q|k)\delta S_2(q'|k')\rangle = \tfrac{1}{2}\mathrm{Re}\,[\langle \delta S(q|k)\delta S^*(q'|k')\rangle - \langle \delta S(q|k)\delta S(q'|k')\rangle] \tag{2.12}$$

$$\langle \delta S_1(q|k)\delta S_2(q'|k')\rangle = -\tfrac{1}{2}\mathrm{Im}\,[\langle \delta S(q|k)\delta S^*(q'|k')\rangle - \langle \delta S(q|k)\delta S(q'|k')\rangle] \tag{2.13}$$

$$\langle \delta S_2(q|k)\delta S_1(q'|k')\rangle = \tfrac{1}{2}\mathrm{Im}\,[\langle \delta S(q|k)\delta S^*(q'|k')\rangle + \langle \delta S(q|k)\delta S(q'|k')\rangle]. \tag{2.14}$$

When $q = q'$ and $k = k'$, the average $\langle \delta S(q|k)\delta S(q|k)\rangle$, which is proportional to $2\pi\delta(2q-2k)$ due to the stationarity of the surface profile function, is nonzero only in the specular direction $q = k$. Therefore, if the surface is infinitely long, and if we omit the specular direction from equations (2.9), (2.10) and (2.11)–(2.14) we see that the scattering matrix is a circular complex Gaussian random process. Consequently, apart from the specular direction, the speckle contrast $\rho = \sqrt{[\langle(\delta S(q|k)\delta S^*(q|k))^2\rangle/\langle \delta S(q|k)\delta S^*(q|k)\rangle^2] - 1}$ is unity [24–26]. This result contradicts the well known result of [25] and [26] that the statistics of the diffuse component of the scattered field is highly non-circular when the surface is weakly rough, and only in the limit of very rough surfaces is the circularity of the statistics restored. The contradiction stems from the representation of the amplitude of the scattered field as the convolution of a real-valued amplitude weighting function and a random phase factor in [25, 26]. The assumption of a real-valued amplitude weighting function, which represents the finite width of the aperture, is identical to the assumption of a finite length of the randomly rough surface. As a result, the statistics of the scattering amplitude is nonstationary in [25, 26]. In the present work we are interested only in the case where the statistics of the surface profile function, as well as of the scattering matrix, is stationary.

The set of the scattering matrices $\delta S(q|k)$ is a set of jointly circular complex Gaussian random variables when $\langle \delta S(q|k)\delta S(q'|k')\rangle$ vanishes. But when $\langle \delta S(q|k)\delta S(q'|k')\rangle$ vanishes the correlation function $C^{(10)}$ vanishes since, within a coefficient, $C^{(10)}(q, k|q'k') \sim |\langle \delta S(q|k)\delta S(q'|k')\rangle|^2$.

Thus, calculations and measurements of the correlation function $C(q, k|q', k')$ yield important information about the statistical properties of the amplitude of the scattered field. If the random surface is such that only the $C^{(1)}$ and $C^{(10)}$ correlation functions are observed, then $S(q|k)$ obeys complex Gaussian statistics. If the random surface is such that only $C^{(1)}$ is observed, then $S(q|k)$ obeys circular complex Gaussian statistics. Finally, if the random surface is such that $C^{(1.5)}$, $C^{(2)}$ and $C^{(3)}$ are observed in addition to both $C^{(1)}$ and $C^{(10)}$, then $S(q|k)$ is not a Gaussian random process, but the statistics it obeys in this case are not known at the present time.

To conclude this section we introduce the normalized angular intensity correlation functions of interest to us, which in terms of $\delta S(q|k)$ are defined by

$$\Xi^{(1)}(q, k|q', k') = \frac{|\langle \delta S(q|k)\delta S^*(q'|k')\rangle|^2}{\langle \delta S(q|k)\delta S^*(q|k)\rangle\langle \delta S(q'|k')\delta S^*(q'|k')\rangle}, \tag{2.15}$$

and

$$\Xi^{(10)}(q,k|q',k') = \frac{|\langle \delta S(q|k)\delta S(q'|k')\rangle|^2}{\langle \delta S(q|k)\delta S^*(q|k)\rangle\langle \delta S(q'|k')\delta S^*(q'|k')\rangle}. \tag{2.16}$$

We also introduce the envelopes $C_0^{(1)}$ and $C_0^{(10)}$ of the correlation functions $C^{(1)}$ and $C^{(10)}$, which we define by

$$C^{(1)}(q,k|q',k') = 2\pi\delta(q-k-q'+k')C_0^{(1)}(q,k|q',q'-q+k) \tag{2.17}$$

and

$$C^{(10)}(q,k|q',k') = 2\pi\delta(q-k+q'-k')C_0^{(10)}(q,k|q'q'+q-k). \tag{2.18}$$

3. Light scattering from a perfectly conducting randomly rough surface in the framework of phase perturbation theory

In this section we study the statistical properties of the scattering matrix for the problem of the scattering of a scalar plane wave from a randomly rough infinitely long surface defined by the equation $x_3 = \zeta(x_1)$. The region $x_3 > \zeta(x_1)$ is vacuum, while the region $x_3 < \zeta(x_1)$ is a perfectly conducting medium. It is assumed that the Dirichlet boundary condition is satisfied on the surface $x_3 = \zeta(x_1)$.

The surface profile function $\zeta(x_1)$ is assumed to be a single-valued function of x_1 that is differentiable and constitutes a zero-mean, stationary, Gaussian random process defined by the properties

$$\langle\zeta(x_1)\rangle = 0, \qquad \langle\zeta(x_1)\zeta(x_1')\rangle = \delta^2 W(|x_1 - x_1'|). \tag{3.1}$$

In equations (3.1) the angle brackets denote an average over the ensemble of realizations of $\zeta(x_1)$, $\delta = \langle\zeta^2(x_1)\rangle^{1/2}$ is the rms height of the surface and, $W(|x_1|)$ is the surface height autocorrelation function. In numerical examples we will use the Gaussian form for $W(|x_1|)$:

$$W(|x_1|) = \exp(-x_1^2/a^2), \tag{3.2}$$

where a is the transverse correlation length of the surface roughness.

A reciprocal phase-perturbation theory for the scattering matrix $S(q|k)$ was constructed in [27] and [28]. The term of lowest order in the surface profile function was shown to have the form

$$S(q|k) = \int_{-\infty}^{\infty} \mathrm{d}x_1\, \mathrm{e}^{-\mathrm{i}(q-k)x_1} \mathrm{e}^{-2\mathrm{i}\sqrt{\alpha_0(q)\alpha_0(k)}\zeta(x_1)}. \tag{3.3}$$

Since

$$\langle S(q|k)\rangle = 2\pi\delta(q-k)\mathrm{e}^{-2\delta^2\alpha_0(q)\alpha_0(k)}, \tag{3.4}$$

we can write the expression for $\delta S(q|k)$ as

$$\delta S(q|k) = \int_{-\infty}^{\infty} \mathrm{d}x_1\, \mathrm{e}^{-\mathrm{i}(q-k)x_1}\left[\mathrm{e}^{-2\mathrm{i}\sqrt{\alpha_0(q)\alpha_0(k)}\zeta(x_1)} - \mathrm{e}^{-2\delta^2\alpha_0(q)\alpha_0(k)}\right]. \tag{3.5}$$

We calculate the averages $\langle\delta S(q|k)\delta S^*(q'|k')\rangle$ and $\langle\delta S(q|k)\delta S(q'|k')\rangle$ using the expression (3.5) for the scattering matrix. For $\langle\delta S(q|k)\delta S^*(q'|k')\rangle$ we obtain

$$\begin{aligned}\langle\delta S(q|k)\delta S^*(q'|k')\rangle = &\int_{-\infty}^{\infty}\mathrm{d}x_1\int_{-\infty}^{\infty}\mathrm{d}x_1'\, \mathrm{e}^{-\mathrm{i}(q-k)x_1+\mathrm{i}(q'-k')x_1'} \\ &\times\left\langle\left[\mathrm{e}^{-2\mathrm{i}\sqrt{\alpha_0(q)\alpha_0(k)}\zeta(x_1)} - \mathrm{e}^{-2\delta^2\alpha_0(q)\alpha_0(k)}\right]\right. \\ &\times\left.\left[\mathrm{e}^{2\mathrm{i}\sqrt{\alpha_0(q')\alpha_0(k')}\zeta(x_1')} - \mathrm{e}^{-2\delta^2\alpha_0(q')\alpha_0(k')}\right]\right\rangle\end{aligned} \tag{3.6}$$

$$= \mathrm{e}^{-2\delta^2(\alpha_0(q)\alpha_0(k)+\alpha_0(q')\alpha_0(k'))} \int_{-\infty}^{\infty} \mathrm{d}x_1 \int_{-\infty}^{\infty} \mathrm{d}x_1' \, \mathrm{e}^{-\mathrm{i}(q-k)x_1+\mathrm{i}(q'-k')x_1'}$$
$$\times \left[\mathrm{e}^{4\delta^2\sqrt{\alpha_0(q)\alpha_0(q')\alpha_0(k)\alpha_0(k')}W(|x_1-x_1'|)} - 1\right] \tag{3.7}$$
$$= 2\pi\delta(q-k-q'+k')\mathrm{e}^{-2\delta^2(\alpha_0(q)\alpha_0(k)+\alpha_0(q')\alpha_0(k'))}$$
$$\times \int_{-\infty}^{\infty} \mathrm{d}u\left[\mathrm{e}^{4\delta^2\sqrt{\alpha_0(q)\alpha_0(q')\alpha_0(k)\alpha_0(k')}W(|u|)} - 1\right]\mathrm{e}^{-\mathrm{i}(q'-k')u}, \tag{3.8}$$

while for $\langle \delta S(q|k)\delta S(q'|k')\rangle$ we have

$$\langle \delta S(q|k)\delta S(q'|k')\rangle = \int_{-\infty}^{\infty} \mathrm{d}x_1 \int_{-\infty}^{\infty} \mathrm{d}x_1' \, \mathrm{e}^{-\mathrm{i}(q-k)x_1-\mathrm{i}(q'-k')x_1'}$$
$$\times\left\langle\left[\mathrm{e}^{-2\mathrm{i}\sqrt{\alpha_0(q)\alpha_0(k)}\zeta(x_1)} - \mathrm{e}^{-2\delta^2\alpha_0(q)\alpha_0(k)}\right]\right.$$
$$\left.\times\left[\mathrm{e}^{-2\mathrm{i}\sqrt{\alpha_0(q')\alpha_0(k')}\zeta(x_1')} - \mathrm{e}^{-2\delta^2\alpha_0(q')\alpha_0(k')}\right]\right\rangle \tag{3.9}$$
$$= \mathrm{e}^{-\delta^2(\alpha_0(q)\alpha_0(k)+\alpha_0(q')\alpha_0(k'))/2} \int_{-\infty}^{\infty} \mathrm{d}x_1 \int_{-\infty}^{\infty} \mathrm{d}x_1' \, \mathrm{e}^{-\mathrm{i}(q-k)x_1-\mathrm{i}(q'-k')x_1'}$$
$$\times\left[\mathrm{e}^{-4\delta^2\sqrt{\alpha_0(q)\alpha_0(q')\alpha_0(k)\alpha_0(k')}W(|x_1-x_1'|)} - 1\right] \tag{3.10}$$
$$= 2\pi\delta(q-k+q'-k')\mathrm{e}^{-2\delta^2(\alpha_0(q)\alpha_0(k)+\alpha_0(q')\alpha_0(k'))}$$
$$\times \int_{-\infty}^{\infty} \mathrm{d}u\left[\mathrm{e}^{-4\delta^2\sqrt{\alpha_0(q)\alpha_0(q')\alpha_0(k)\alpha_0(k')}W(|u|)} - 1\right]\mathrm{e}^{-\mathrm{i}(q'-k')u}. \tag{3.11}$$

It is readily seen that, in contrast to $\langle \delta S(q|k)\delta S^*(q'|k')\rangle$, the average $\langle \delta S(q|k)\delta S(q'|k')\rangle$ vanishes with increasing roughness parameters δ and a, due to the negative exponential under the integral sign in the last line of equation (3.11). Plots of the normalized correlation functions $\Xi^{(1)}(q,k|q',k')$ and $\Xi^{(10)}(q,k|q',k')$ as functions of δ for different values of a are presented in figure 1(a), while plots of the envelopes of the correlation functions $C^{(1)}$ and $C^{(10)}$ as functions of δ for different values of a are presented in figure 1(b), for fixed values of q, k and q', while k' is determined by the constraint of the corresponding δ function. When calculating the results presented in figures 1(a) and (b) the value of q' was chosen to produce the same values of $C^{(1)}$ and $C^{(10)}$ in the limit of a weakly rough surface. From the plots presented in figure 1(a) we see that $\Xi^{(10)}(q,k|q',k')$ vanishes even for quite moderately weakly rough surfaces for which $\Xi^{(1)}(q,k|q',k')$ is still about unity. We note that $C^{(1)}$ also decreases with increasing δ (figure 1(b)). Using equations (2.11)–(2.14), (3.8) and (3.11) we obtain the expressions for $\langle(\delta S_1(q|k))^2\rangle$, $\langle(\delta S_2(q|k))^2\rangle$ and $\langle \delta S_1(q|k)\delta S_2(q|k)\rangle$:

$$\langle(\delta S_1(q|k))^2\rangle = \mathrm{e}^{-4\delta^2\alpha_0(q)\alpha_0(k)}\left[\frac{L_1}{2}\int_{-\infty}^{\infty} \mathrm{d}u\cos(q-k)u(\mathrm{e}^{\delta^2\alpha_0(q)\alpha_0(k)W(|u|)} - 1)\right.$$
$$\left. + \tfrac{1}{2}\pi\delta(q-k)\int_{-\infty}^{\infty}\cos(q-k)u(\mathrm{e}^{-\delta^2\alpha_0(q)\alpha_0(k)W(|u|)} - 1)\right], \tag{3.12}$$

and

$$\langle(\delta S_2(q|k))^2\rangle = \mathrm{e}^{-4\delta^2\alpha_0(q)\alpha_0(k)}\left[\frac{L_1}{2}\int_{-\infty}^{\infty} \mathrm{d}u\cos(q-k)u(\mathrm{e}^{\delta^2\alpha_0(q)\alpha_0(k)W(|u|)} - 1)\right.$$
$$\left. - \tfrac{1}{2}\pi\delta(q-k)\int_{-\infty}^{\infty}\cos(q-k)u(\mathrm{e}^{-\delta^2\alpha_0(q)\alpha_0(k)W(|u|)} - 1)\right], \tag{3.13}$$

while

$$\langle \delta S_1(k|k)\delta S_2(k|k)\rangle = 0. \tag{3.14}$$

In figure 2 we present plots of the ratio $\langle(\delta S_2(k|k))^2\rangle/\langle(\delta S_1(k|k))^2\rangle$ as a function of the rms height of the surface roughness δ. Since this ratio is calculated for the specular direction $q = k$,

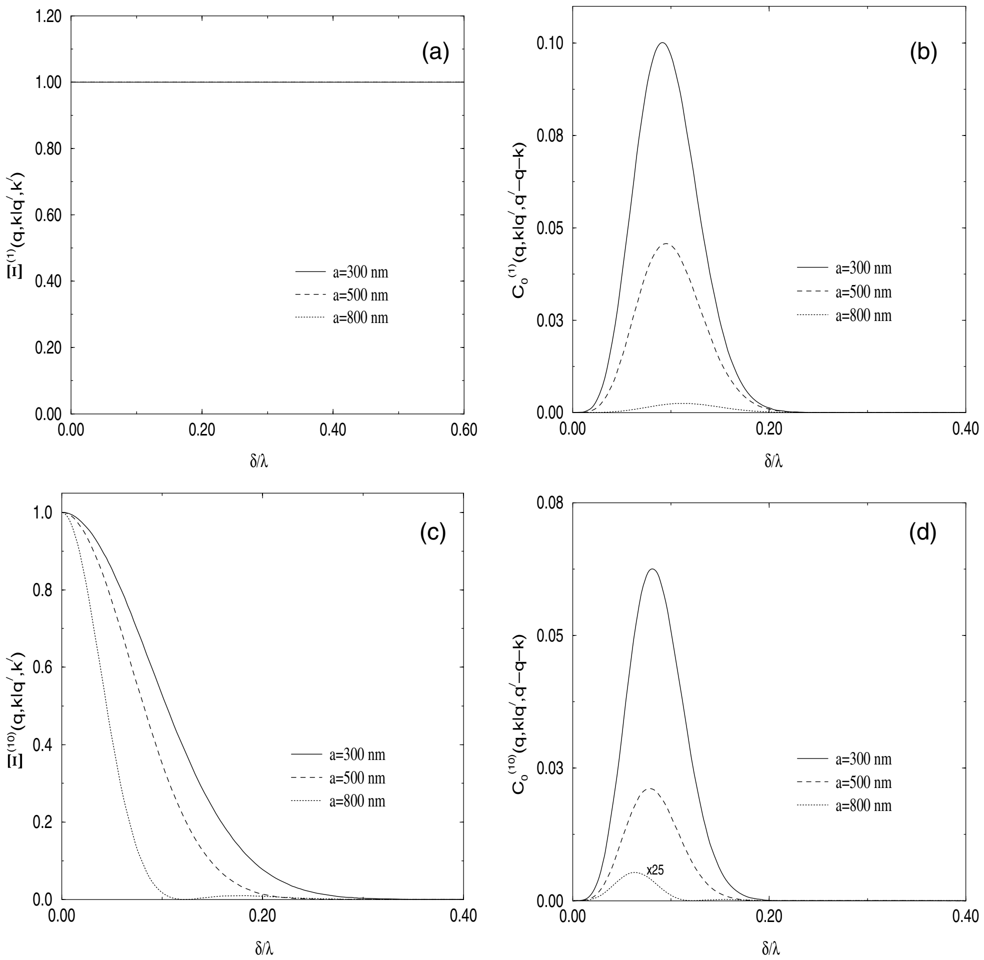

Figure 1. The normalized correlation functions $\Xi^{(1)}$ (a) and $\Xi^{(10)}$ (c) and the envelopes $C_0^{(10)}$ (b) and $C_0^{(10)}$ (d) as functions of δ/λ for values of the transverse correlation length $a = 300$ nm, 500 nm and 800 nm. The incident light was s-polarized and of wavelength 632.8 nm. The scattering medium was a randomly rough perfect conductor. Furthermore $\theta_0 = 30°$, $\theta_s = 0°$ and $\theta_s' = 0°$. In figure 1(a) the results for the different correlation lengths considered could not be distinguished.

it is independent of the transverse correlation length a. From the plot presented it is easily seen that for large values of the rms height, the incoherent part of the scattering matrix, $\delta S(q|k)$, becomes a circular complex Gaussian variable, even in the specular direction.

4. Light scattering from a randomly rough penetrable surface

The results of the preceding section enable us to make several conclusions when studying the scattering of light from a randomly rough surface of a penetrable medium. For simplicity we consider here the scattering of s-polarized light from a randomly rough surface of a medium characterized by a dielectric function $\epsilon(\omega)$. As is well known (see, e.g., [29–31]), if the surface profile function is such that the conditions for the applicability of the Rayleigh hypothesis are

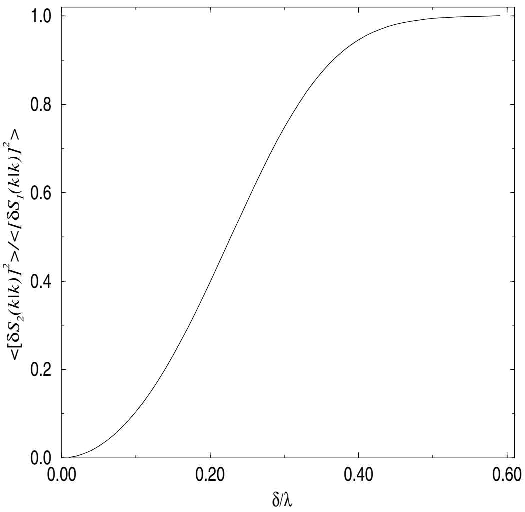

Figure 2. The ratio $\langle(\delta S_2(k|k))^2\rangle/\langle(\delta S_1(k|k))^2\rangle$ as a function of δ/λ.

satisfied the scattering amplitude $R(q|k)$ obeys the reduced Rayleigh equation. Rewritten in terms of the scattering matrix $S(q|k)$ it has the form

$$S(q|k) = 2\pi\delta(q-k)R_0(k) + N(q|k) + \int_{-\infty}^{\infty}\frac{\mathrm{d}p}{2\pi}M(q|p)S(p|k), \tag{4.1}$$

where, for the case of the scattering of s-polarized light,

$$R_0(k) = \frac{\alpha_0(k) - \alpha(k)}{\alpha_0(k) + \alpha(k)}, \tag{4.2}$$

$$\alpha_0(k) = \sqrt{\frac{\omega^2}{c^2} - k^2}, \qquad \alpha(k) = \sqrt{\epsilon(\omega)\frac{\omega^2}{c^2} - k^2}, \tag{4.3}$$

$$N(q|k) = -(\epsilon - 1)\frac{\omega^2/c^2}{\alpha_0(q) + \alpha(q)}\sqrt{\frac{\alpha_0(q)}{\alpha_0(k)}}\frac{J(\alpha(p) + \alpha_0(k)|p-k)}{\alpha(p) + \alpha_0(k)}, \tag{4.4}$$

$$M(q|k) = -(\epsilon - 1))\frac{\omega^2/c^2}{\alpha_0(q) + \alpha(q)}\sqrt{\frac{\alpha_0(q)}{\alpha_0(p)}}\frac{J(\alpha(p) - \alpha_0(k)|p-k)}{\alpha(p) - \alpha_0(k)}, \tag{4.5}$$

and

$$J(\gamma|Q) = \int_{-\infty}^{\infty}\mathrm{d}x_1\,\mathrm{e}^{-\mathrm{i}Qx_1}(\mathrm{e}^{-\mathrm{i}\gamma\zeta(x_1)} - 1). \tag{4.6}$$

We can write the solution of equation (4.1) formally as

$$\begin{aligned} S(q|k) = {} & R_0(k)2\pi\delta(q-k) + F(q|k) + \int_{-\infty}^{\infty}\frac{\mathrm{d}p}{2\pi}M(q|p)F(p|k) \\ & + \int_{-\infty}^{\infty}\frac{\mathrm{d}p}{2\pi}M(q|p)\int_{-\infty}^{\infty}\frac{\mathrm{d}p'}{2\pi}M(p|p')F(p'|k) + \cdots, \end{aligned} \tag{4.7}$$

where

$$F(q|k) = N(q|k) + M(q|k)R_0(k), \tag{4.8}$$

and we keep all terms in the infinite iterative series. Both $N(q|k)$ and $M(q|p)$ contain the surface disorder only in the functions $J(\gamma|Q)$. Therefore, having in hand the recipe for

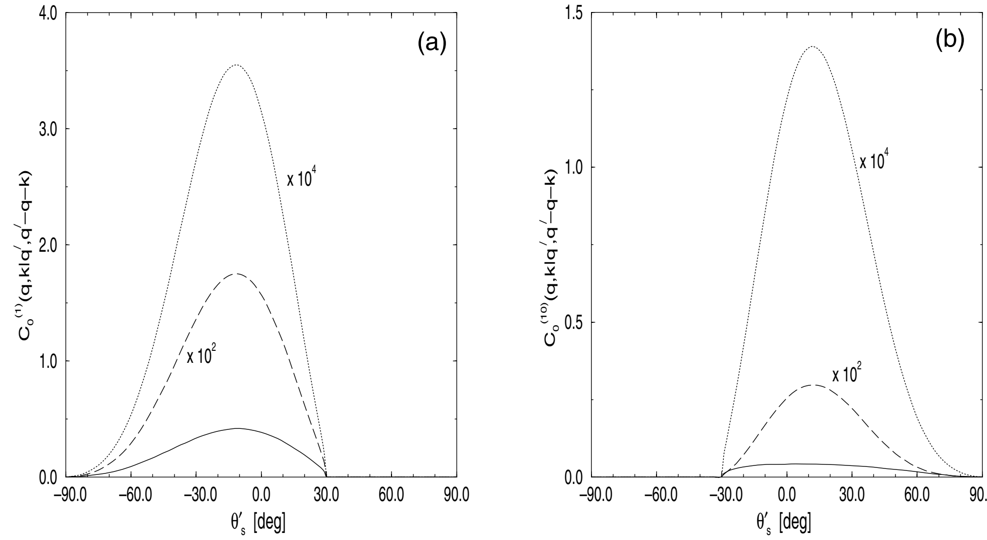

Figure 3. The envelopes of the $C^{(1)}$ (*a*) and $C^{(10)}$ (*b*) correlation functions as functions of θ'_s for $\theta_0 = 30°$ and $\theta_s = 0°$, while θ'_0 is determined by the constraints of the corresponding δ functions for the scattering of s-polarized light from a randomly rough silver surface with $a = 500$ nm and $\delta = 20$ nm (full curves), $\delta = 50$ nm (broken curves) and $\delta = 100$ nm (dotted curves).

calculating the average of the product of any number of functions $J(\gamma|Q)$, we can calculate, in principle, both $\langle\delta S(q|k)\delta S(q'|k')\rangle$ and $\langle\delta S(q|k)\delta S^*(q'|k')\rangle$. The basics of such calculations were described in [32].

To calculate the averages $\langle\delta S(q|k)\delta S(q'|k')\rangle$ and $\langle\delta S(q|k)\delta S^*(q'|k')\rangle$ we multiply the series (4.7) for $S(q|k)$ by the corresponding series for $S(q'|k')$ and average the product term-by-term. From the result we subtract the product $\langle S(q|k)\rangle\langle S(q'|k')\rangle$. In a similar fashion we calculate the average $\langle\delta S(q|k)\delta S^*(q'|k')\rangle$ by multiplying the series (4.7) for $S(q|k)$ by the complex conjugate of the corresponding series for $S(q'|k')$, averaging the product term-by-term, and subtracting the product $\langle S(q|k)\rangle\langle S^*(q'|k')\rangle$ from the result. In the product$\langle\delta S(q|k)\delta S^*(q'|k')\rangle$ the contribution of nth order in the functions $J(\gamma|Q)$ and $J^*(\gamma|Q)$ contains $n-1$ terms of the form

$$\sum_{m=1}^{n-1}\left\{\left\langle\prod_{r=1}^{m} J(\gamma_r|Q_r)\prod_{s=1}^{n-m} J^*(\gamma'_s|Q'_s)\right\rangle - \left\langle\prod_{r=1}^{m} J(\gamma_r|Q_r)\right\rangle\left\langle\prod_{s=1}^{n-m} J^*(\gamma'_s|Q'_s)\right\rangle\right\}. \tag{4.9}$$

To obtain a nonzero contribution, for each value of m at least one $J(\gamma_r|Q_r)$ must be contracted with at least one $J^*(\gamma'_s|Q'_s)$. Therefore each term in this sum contains at least one factor with a positive exponential of the form $\exp\{\delta^2\gamma\gamma' W(|u|)\} - 1$. In contrast, when calculating $\langle\delta S(q|k)\delta S(q'|k')\rangle$ the contribution of the nth order in the functions $J(\gamma|Q)$ contains the sum

$$\sum_{m=1}^{n-1}\left\{\left\langle\prod_{r=1}^{m} J(\gamma_r|Q_r)\prod_{s=1}^{n-m} J(\gamma'_s|Q'_s)\right\rangle - \left\langle\prod_{r=1}^{m} J(\gamma_r|Q_r)\right\rangle\left\langle\prod_{s=1}^{n-m} J(\gamma'_s|Q'_s)\right\rangle\right\}. \tag{4.10}$$

In this case, to obtain a nonzero contribution, for each value of m at least one $J(\gamma_r|Q_s)$ must be contracted with at least one $J(\gamma'_s|Q'_s)$. Therefore, each term in this sum contains only negative exponentials of the form $\exp\{-\delta^2\gamma\gamma' W(|u|)\} - 1$. Owing to this lack of the positive exponential, $\langle\delta S(q|k)\delta S(q'|k')\rangle$ vanishes when the roughness parameters increase.

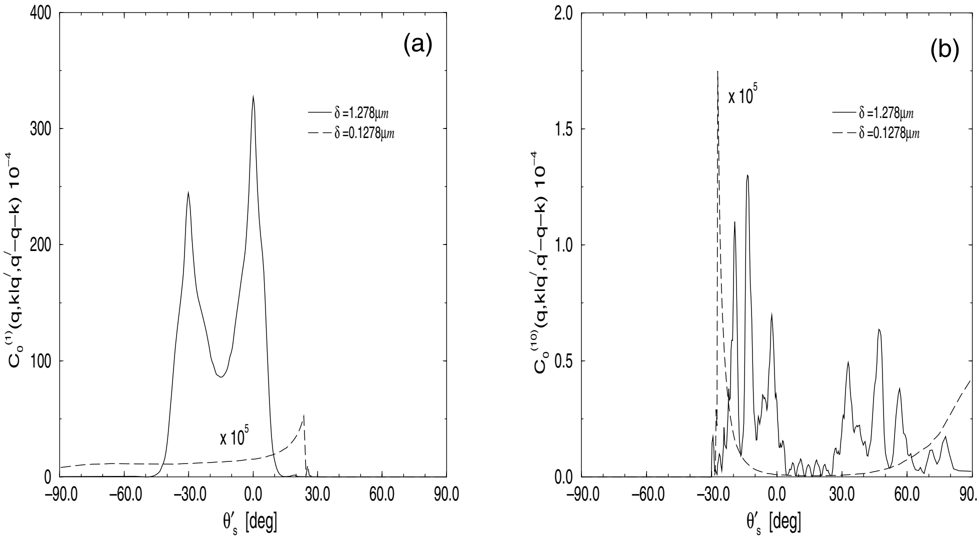

Figure 4. The same as in figure 3, but for $a = 3.85$ μm and $\delta = 1.278$ μm (full curves) and $\delta = 0.1278$ μm (broken curves).

In figure 3 we present plots of the envelopes $C_0^{(1)}$ and $C_0^{(10)}$ of the correlation functions $C^{(1)}$ (figure 3(a)) and $C^{(10)}$ (figure 3(b)) as functions of θ_s' for fixed values of θ_0 and θ_s, while θ_0' is determined by the constraints of the corresponding δ functions. The calculations were carried out for the scattering of s-polarized light, of 612.7 nm wavelength, from a weakly rough random surface of silver characterized by the complex dielectric constant $\epsilon = -17.2 + \mathrm{i}0.479$ for different values of the roughness parameters δ and a. In calculating the results presented in figure 3 we kept all terms in the infinite iterative series equation (4.7) which would give contributions to the averages we calculate through terms of $\mathrm{O}(\delta^8)$ if they were to be expanded in powers of the small parameter $(\omega/c)\delta$.

In figure 4 we present rigorous numerical simulation calculation results [33] for the envelopes of the correlation functions $C^{(1)}$ (figure 4(a)) and $C^{(10)}$ (figure 4(b)). The surface parameters used here were the same as those used in obtaining figure 3, except that the roughness now was $\delta = 1.278$ μm (full curves) and $\delta = 0.1278$ μm (broken curves). It should be pointed out that for the scattering of s-polarized light from a weakly rough random metal surface, there should be no memory or reciprocal memory effect present in $C_0^{(1)}$. This is indeed confirmed by our numerical calculations where the $C_0^{(1)}$ for $\delta = 0.1278$ μm (figure 4(a), broken curve) is a smooth function of its argument, as well as by the results presented in figure 3(a). In particular, there are no peaks at angles $\theta = 0^\circ$ and 30°, which are the positions of the memory and reciprocal memory effects. As the roughness is increased to $\delta = 1.278$ μm one sees from figure 4(a) (full curve) that the overall amplitude of the envelope $C_0^{(1)}$ is increased and, more importantly, that two peaks have developed at the aforementioned angles. These peaks are due, in the large roughness limit, to volume waves scattered multiply at the rough surface. In figure 4(b) the corresponding results for the $C_0^{(10)}$ envelopes are presented. It is observed that in the low roughness limit this envelope is structureless, and that $C_0^{(1)}$ and $C_0^{(10)}$ are roughly of the same order of magnitude. However, as δ is increased, the scattering matrix $S(q|k)$ starts to obey circular complex Gaussian statistics and thus, as discussed earlier, the envelope $C_0^{(10)}$ should in principle vanish. From our numerical results for $\delta = 1.278$ μm (full curve) we indeed see that $C_0^{(10)}$ is much smaller then the corresponding $C_0^{(1)}$ shown in figure 4(a). In fact, $C_0^{(10)}$ is just noise, consistent with this function vanishing in the large roughness limit.

5. Conclusions

In this paper we calculated the angular intensity correlation functions $C(q, k|q', k')$ by means of an approach that explicitly separates out different contributions to it. We have shown that calculations and measurements of the correlation function $C(q, k|q', k')$ yield important information about the statistical properties of the amplitude of the scattered field. In particular, we have shown that the short-range correlation function $C^{(10)}$ is, in a sense, a measure of the noncircularity of the complex Gaussian statistics of the scattering matrix. Thus, if the random surface is such that only the $C^{(1)}$ and $C^{(10)}$ correlation functions are observed, then $S(q|k)$ obeys complex Gaussian statistics. If the random surface is such that only $C^{(1)}$ is observed, then $S(q|k)$ obeys circular complex Gaussian statistics. Finally, if the random surface is such that $C^{(1.5)}$, $C^{(2)}$ and $C^{(3)}$ are observed, in addition to both $C^{(1)}$ and $C^{(10)}$, then $S(q|k)$ is not a Gaussian random process. In addition, we can conclude that if a surface is sufficiently weakly rough and long enough, its speckle pattern should display the memory and reciprocal memory effects when the angle of incidence is changed and, for a fixed angle of incidence, should be symmetric about the specular direction. However, if the roughness of the surface is sufficiently great and the surface is long enough, its speckle pattern should display only the memory and reciprocal memory effects, when the angle of incidence is changed.

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References

[1] Michel T R and O'Donnell K A 1992 *J. Opt. Soc. Am.* A **9** 1374
[2] Knotts M E, Michel T R and O'Donnell K A 1992 *J. Opt. Soc. Am.* A **9** 1822
[3] Nieto-Vesperinas M and Sánchez-Gil J A 1992 *Phys. Rev.* B **46** 3112
[4] Nieto-Vesperinas M and Sánchez-Gil J A 1993 *J. Opt. Soc. Am.* A **10** 150
[5] Arsenieva A D and Feng S 1993 *Phys. Rev.* B **47** 13 047
[6] Nieto-Vesperinas M and Sánchez-Gil J A 1993 *Phys. Rev.* B **48** 4132
[7] Kuga Y, Le C T C, Ishimaru A and Ailes-Sengers L 1996 *IEEE Trans. Geosci. Remote Sens.* **34** 1300
[8] Le C T C, Kuga Y and Ishimaru A 1996 *J. Opt. Soc. Am.* A **13** 1057
[9] Malyshkin V, McGurn A R, Leskova T A, Maradudin A A, and Nieto–Vesperinas M 1997 *Opt. Lett.* **22** 946
[10] Malyshkin V, McGurn A R, Leskova T A, Maradudin A A, and Nieto–Vesperinas M 1997 *Waves Random Media* **7** 479
[11] Nieto-Vesperinas M, Maradudin A A, Shchegrov A V and McGurn A R 1997 *Opt. Commun.* **142** 1
[12] West C S and O'Donnell K A 1999 *Phys. Rev.* B **59** 2393
[13] Lu J Q and Gu Z-H 1997 *Appl. Opt.* **36** 4562
Gu Z-H and Lu J Q 1997 *SPIE* **3141** 269
[14] McGurn A R, Maradudin A A and Celli V 1985 *Phys. Rev.* B **31** 4866
[15] McGurn A R and Maradudin A A 1989 *Opt. Commun.* **72** 279
[16] Feng S, Kane C, Lee P A and Stone A D 1988 *Phys. Rev. Lett.* **61** 834
[17] Shapiro B 1986 *Phys. Rev. Lett.* **57** 2168
[18] Garcia N and Genack A Z 1989 *Phys. Rev. Lett.* **63** 1678
[19] van Albada M P, de Boer J F and Lagendijk A 1990 *Phys. Rev. Lett.* **64** 2787

[20] Freund I and Rosenbluh M 1991 *Opt. Commun.* **82** 362
[21] Kubo R 1962 *J. Phys. Soc. Japan* **17** 1100
[22] Stuart A and Keith Ord J 1987 *Kendall's Advanced Theory of Statistics* vol 1, 5th edn (London: Charles Griffin) p 84ff
[23] Stoffregen B 1979 *Optik* **52** 385
[24] Goodman J W 1985 *Statistical Optics* (New York: Wiley) ch 2
[25] Goodman J W 1975 *Opt. Commun.* **14** 324
[26] Goodman J W 1984 *Laser Speckle and Related Phenomena* ed J C Dainty (Berlin: Springer) ch 2
[27] Shen J and Maradudin A A 1980 *Phys. Rev.* B **22** 4234–40
[28] Fitzgerald R M and Maradudin A A 1994 *Waves Random Media* **4** 275
[29] Lord Rayleigh 1985 *The Theory of Sound* vol 2 (London: Macmillan) pp 89, 297
[30] Petit R and Cadilhac M 1966 *C. R. Acad. Sci., Paris* B **262** 468
[31] Hill N R and Celli V 1978 *Phys. Rev.* B **17** 2478
[32] Leskova T A, Maradudin A A and Novikov I 2000 *J. Opt. Soc. Am.* A **17** 1288
[33] Maradudin A A, Michel T, McGurn A R and Méndez E R 1990 *Ann. Phys., NY* **203** 255