① SANNSYNLIGHETS TEORI

1.1 Def : Stochastic Variables

1) Range "set of states"
2) Sannsynlighets tetthet, def. over dette settet
function (pdf)

Egenskaper

1) diskret, kontinuerlig, blanding av de to
Gi exempler

2) Ikke-negativ : $p(x) \geq 0 \quad \forall x$
Normalisering : $\int dx\, p(x) = 1$

Tolkning av $p(x)$:

$p(x)dx$ = sannsynligheten for at X har en verdi mellom x og $x+dx$

$$\mathcal{P}(a < X < b) = \int_a^b dx\, p(x)$$

$\mathcal{P}(\cdot)$ = sanns. for noe

Merk : Fysikkere liker ofte å visualisere en pdf via et ensamble.

Isteden for å tenke på en stokastisk variabel X og dens pdf $p(x)$, tenker man seg et stort antall realiseringer N ax X slik at antall med verdi mellom x og $x+dx$ er lik $N\,p(x)dx$

1.2 Def. : kumulativ fordeling funksjon (CDF)

$$\mathcal{P}(x) = \int_{-\infty}^{x} dx' \, p(x') \qquad [\mathcal{P}(x) = \mathcal{P}(X < x)]$$

— ∽ —

Egenskaper:

$$\mathcal{P}(-\infty) = 0$$
$$\mathcal{P}(+\infty) = 1$$

$$1 - \mathcal{P}(x) = \int_{x}^{\infty} dx' \, p(x') \qquad [= \mathcal{P}_{>}(x)]$$

Tolkning : $\mathcal{P}(x)$ er sannsynligheten for at $X \leq x$.

Merk : Mye brukt notasjon er at
X : betegner den stok. var.
x : er en av de mulige verdiene av X.

OBS : I fysikk synder man ofte mot denne konvensjonen og bruker x om begge. Det er klart fra sammenhengen hvilken man snakker om.

PDF og CDF, $p(x)$ og $\underline{P}(x)$, inneholder den samme informasjonen!

Hvorfor skille mellom de to?

- Ofte fordelaktig å job med $\underline{P}(x)$ når man studerer empiriske data
  1. Trenger ikke å "binne" dataene
  2. Lettere å studere halen til fordelingen (so-called "tails" svarende til "rare events")

$\Rightarrow$ Se Bouchaud/Potters p. 58/59

1.3 Def : Middelverdier

$$\langle f(x) \rangle = \int dx\, f(x)\, p(x)$$

— ∽ —

[Lineær operat.
$\langle f+g \rangle = \langle f \rangle + \langle g \rangle$]

Q : Hvorfor opptrer x på venstre siden i denne likningen når vi har integrert den ut?

A : Slurvete notasjon. Skulle ha vært $\langle f(\overline{X}) \rangle$

Momenter av fordelingen

Spesielt $f(x) = x^m$

$\langle x^m \rangle \equiv \mu_m$ m-te moment

- $m=1$: $\langle x \rangle$ Middelverdien
- $m=2$: $\langle x^2 \rangle$

$$\sigma^2 = \langle (x - \langle x \rangle)^2 \rangle$$
$$= \langle x^2 \rangle - \langle x \rangle^2 \quad \text{Variansen}$$

$$\sigma = \sqrt{\sigma^2} \quad \text{Standard avvik.}$$

Merk : Ikke alle $p(x)$ gir endelig σ

[Ex. Cauchy dist.

$$p(x) = \frac{1}{\pi} \frac{\gamma}{(x-a)^2 + \gamma^2}$$]

instead of the random variable ξ, the probability density $W_\eta(y)$ of the random variable η is, according to (2.7, 10), given by

$$\begin{aligned} W_\eta(y) &= \langle \delta(y-\eta) \rangle = \langle \delta(y - g(\xi)) \rangle \\ &= \int \delta(y - g(x))\, W_\xi(x)\, \mathrm{d}x\,. \end{aligned} \tag{2.13}$$

The last integral is easily evaluated. If $g_n^{-1}(y)$ is the nth simple root of $g(x) - y = 0$, then

$$W_\eta(y) = \sum_n W_\xi(g_n^{-1}(y))\, [\, |\mathrm{d}g(x)/\mathrm{d}x|]^{-1}|_{x = x_n = g_n^{-1}(y)}\,, \tag{2.14}$$

from a well-known expression for the δ function, see e.g. [2.7]. (Another possibility to obtain (2.14) is by transformation of the differentials.)

As an example, we calculate from the one-dimensional Maxwell distribution

$$W(v) = \sqrt{\frac{m}{2\pi kT}} \exp\left(-\frac{mv^2}{2kT}\right) \tag{2.15}$$

the probability density of the energy

$$E = \tfrac{1}{2} m v^2 = g(v)\,. \tag{2.16}$$

Here we have

$$\begin{aligned} v_{\substack{1\\2}} &= g_{\substack{1\\2}}^{-1}(E) = \pm\sqrt{2E/m} \\ \left|\frac{\mathrm{d}g}{\mathrm{d}v}\right|_{v = g_{\substack{1\\2}}^{-1}(E)} &= |m v_{\substack{1\\2}}| = \sqrt{2mE} \end{aligned} \tag{2.17}$$

$$\begin{aligned} W(E) &= \sqrt{\frac{m}{2\pi kT}} \exp\left(-\frac{E}{kT}\right) \frac{1}{\sqrt{2mE}} + \sqrt{\frac{m}{2\pi kT}} \exp\left(-\frac{E}{kT}\right) \frac{1}{\sqrt{2mE}} \\ &= \frac{1}{\sqrt{\pi kTE}} \exp\left(-\frac{E}{kT}\right). \end{aligned} \tag{2.18}$$

2.2 Characteristic Function and Cumulants

The characteristic function $C_\xi(u)$ is the average

$$C_\xi(u) = \langle \mathrm{e}^{\mathrm{i}u\xi} \rangle = \int \mathrm{e}^{\mathrm{i}ux}\, W_\xi(x)\, \mathrm{d}x\,. \tag{2.19}$$

$$\begin{aligned} K_3 &= M_3 - 3M_1M_2 + 2M_1^3 \\ K_4 &= M_4 - 3M_2^2 - 4M_1M_3 + 12M_1^2M_2 - 6M_1^4 \end{aligned} \tag{2.26}$$

$$\begin{aligned} M_1 &= K_1 \\ M_2 &= K_2 + K_1^2 \\ M_3 &= K_3 + 3K_2K_1 + K_1^3 \\ M_4 &= K_4 + 4K_3K_1 + 3K_2^2 + 6K_2K_1^2 + K_1^4 . \end{aligned} \tag{2.27}$$

General expressions for the connection between cumulants and moments may be found in [Ref. 2.4, p. 165]. A very convenient form in terms of determinants reads

$$K_n = (-1)^{n-1} \begin{vmatrix} M_1 & 1 & 0 & 0 & 0 & \dots \\ M_2 & M_1 & 1 & 0 & 0 & \dots \\ M_3 & M_2 & \binom{2}{1}M_1 & 1 & 0 & \dots \\ M_4 & M_3 & \binom{3}{1}M_2 & \binom{3}{2}M_1 & 1 & \dots \\ M_5 & M_4 & \binom{4}{1}M_3 & \binom{4}{2}M_2 & \binom{4}{3}M_1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}_n \tag{2.28}$$

$$M_n = \begin{vmatrix} K_1 & -1 & 0 & 0 & 0 & \dots \\ K_2 & K_1 & -1 & 0 & 0 & \dots \\ K_3 & \binom{2}{1}K_2 & K_1 & -1 & 0 & \dots \\ K_4 & \binom{3}{1}K_3 & \binom{3}{2}K_2 & K_1 & -1 & \dots \\ K_5 & \binom{4}{1}K_4 & \binom{4}{2}K_3 & \binom{4}{3}K_2 & K_1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix}_n \tag{2.29}$$

where the determinants $||_n$ contain n rows and n columns and where $\binom{n}{m} = \dfrac{n!}{(n-m)!\,m!}$ are the binomial coefficients. These connections between the expansion coefficients of (2.24 and 25) are found in [2.8]. [To obtain (2.28) the ith row of the corresponding determinant in [2.8] has to be multiplied with $(i-1)!$ and the jth column with the exception of the first column has to be divided by $(j-2)!$. For the derivation of (2.29) the ith row of the corresponding determinant has to be multiplied with $(i-1)!$ and the jth column has to be divided by $(j-1)!$].

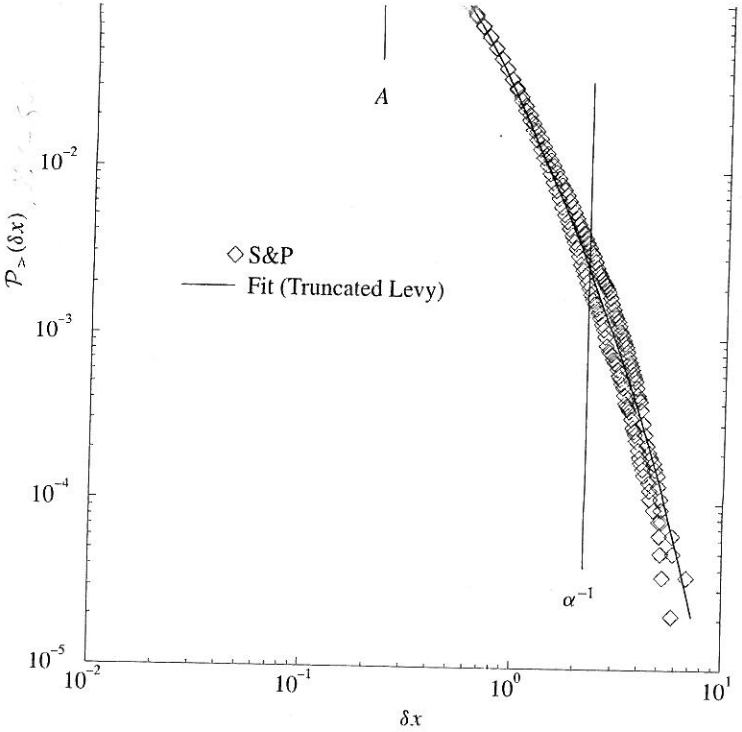


Fig. 2.8. Elementary cumulative distribution $\mathcal{P}_{1>}(\delta x)$ (for $\delta x > 0$) and $\mathcal{P}_{1<}(\delta x)$ (for $\delta x < 0$), for the S&P 500, with $\tau = 15$ min. The thick line corresponds to the best fit using a symmetric TLD $L_\mu^{(t)}$, of index $\mu = \frac{3}{2}$. We have also shown on the same graph the values of the parameters A and α^{-1} as obtained by the fit.

- The particular value $\mu = \frac{3}{2}$ has a simple theoretical interpretation, which we shall briefly present in Section 2.8.

In order to characterize a probability distribution using empirical data, it is always better to work with the cumulative distribution function rather than with the distribution density. To obtain the latter, one indeed has to choose a certain width for the bins in order to construct frequency histograms, or to smooth the data using, for example, a Gaussian with a certain width. Even when this width is carefully chosen, part of the information is lost. It is furthermore difficult to characterize the tails of the distribution, corresponding to rare events, since most bins in this region are empty. On the other hand, the construction of the cumulative distribution does not require to choose a bin width. The trick is to order the observed data according to their rank, for example in decreasing order. The value x_k of the

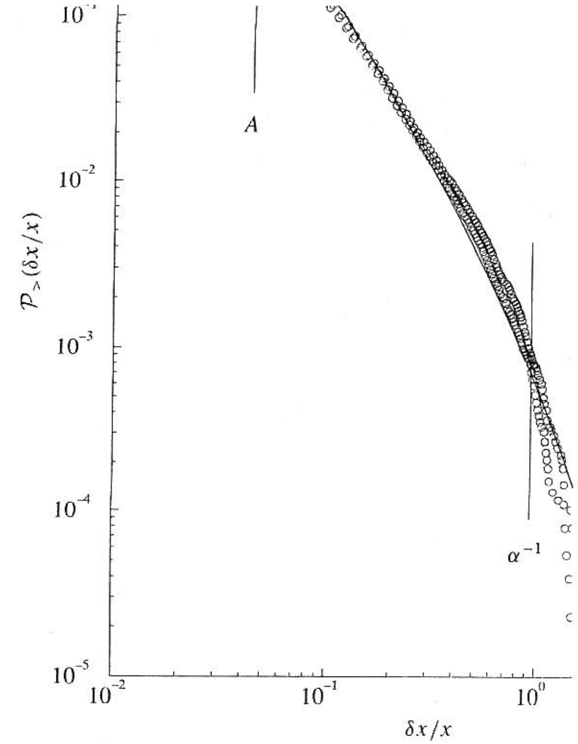
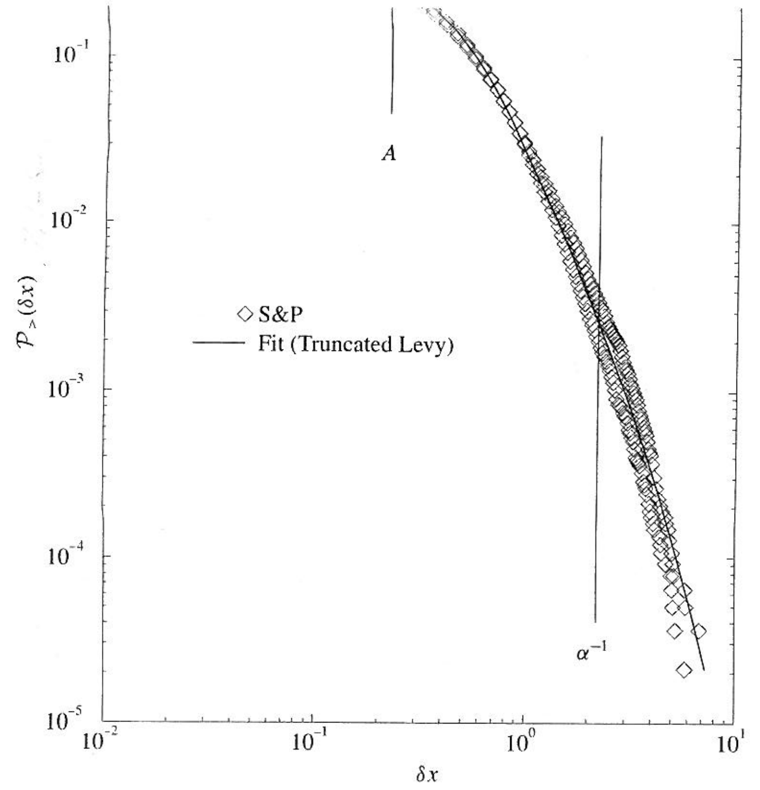


Fig. 2.9. Elementary cumulative distribution for the DEM/$, 1
using a symmetric TLD $L_\mu^{(t)}$, of index $\mu = \frac{3}{2}$. In this case, it
been considered. The fit is not very good, and would have beer
of $\mu \sim 1.2$. This increases the weight of very small variations.

kth variable (out of N) is then such that:

$$\mathcal{P}_>(x_k) = \frac{k}{N+1}.$$

This result comes from the following observation: if one dr
variable from the same distribution, there is an a priori equa
it falls within any of the $N+1$ intervals defined by the previ
probability that it falls above the kth one, x_k is therefore equa
beyond x_k, which is equal to k, times $1/(N+1)$. This is also equ
(See also the discussion in Section 1.4, and Eq. (1.45)). Since
distribution is a particular interest, it is convenient to choose
probabilities. Furthermore, in order to check visually the s



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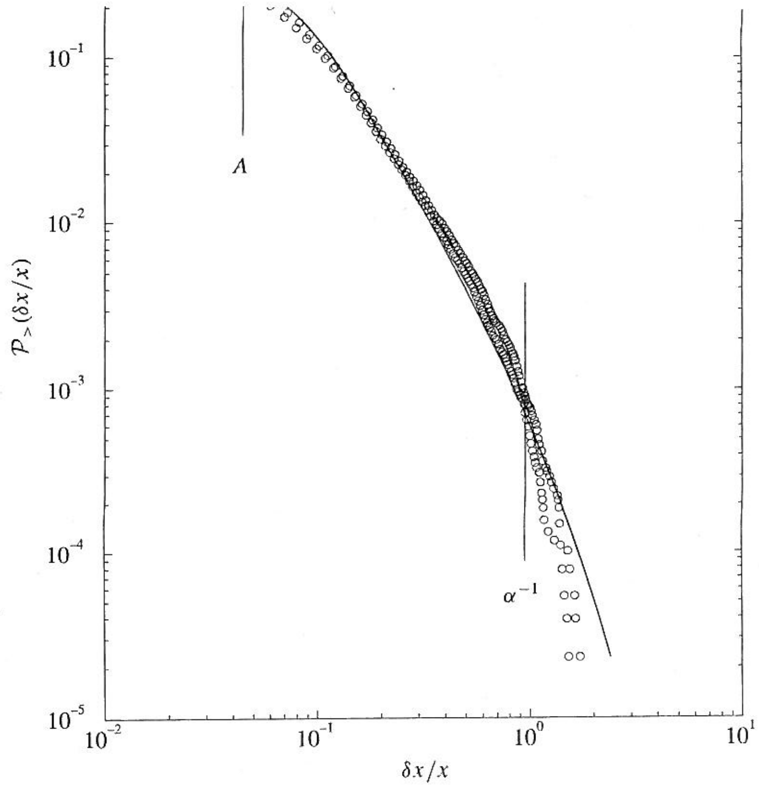


Fig. 2.9. Elementary cumulative distribution for the DEM/\$, for $\tau = 15$ min, and best fit using a symmetric TLD $L_\mu^{(t)}$, of index $\mu = \frac{3}{2}$. In this case, it is rather $100\delta x/x$ that has been considered. The fit is not very good, and would have been better with a smaller value of $\mu \sim 1.2$. This increases the weight of very small variations.

kth variable (out of N) is then such that:

$$\mathcal{P}_>(x_k) = \frac{k}{N+1}. \tag{2.7}$$

This result comes from the following observation: if one draws an $(N+1)$th random variable from the same distribution, there is an a priori equal probability $1/N+1$ that it falls within any of the $N+1$ intervals defined by the previously drawn variables. The probability that it falls above the kth one, x_k is therefore equal to the number of intervals beyond x_k, which is equal to k, times $1/N+1$. This is also equal, by definition, to $\mathcal{P}_>(x_k)$. (See also the discussion in Section 1.4, and Eq. (1.45)). Since the rare events part of the distribution is a particular interest, it is convenient to choose a logarithmic scale for the probabilities. Furthermore, in order to check visually the symmetry of the probability

1.4 Karakteristisk funksjon (av stok. var. X)

<u>Def</u>: $(k \in \mathbb{R})$ [Noen forf. bruker not $\hat{p}(k)$]

$$G(k) = \underset{\text{stok. var.}}{\langle e^{ikX} \rangle} = \int_\Omega dx \, e^{ikx} p(x)$$

—∽—

Egenskaper:

$$G(0) = 1 \qquad |G(k)| \leq 1$$

1.4.1 Moment generating function

$$G(k) = \int_\Omega dx \underbrace{\sum_{m=0}^{\infty} \frac{(ikx)^m}{m!}}_{e^{ikx}} p(x)$$

$$= \sum_{m=0}^{\infty} \frac{(ik)^m}{m!} \int_\Omega dx \, x^m p(x)$$

$$= \sum_{m=0}^{\infty} \frac{(ik)^m}{m!} \langle x^m \rangle \qquad \langle x^m \rangle = \mu_m$$

- $G(k)\big|_{k=0} = 1$
- $\partial_k G(k)\big|_{k=0} = i \langle x \rangle$
- $\partial_k^2 G(k)\big|_{k=0} = - \langle x^2 \rangle$

$\vdots$

- $\partial_k^m G(k)\big|_{k=0} = i^m \langle x^m \rangle$

$$\boxed{\langle x^m \rangle = (-i)^m \partial_k^m G(k)\big|_{k=0}}$$

1.4.2 Kumulanter

[se Risken p. 18]

$$\ln G(k) = \ln\Big[1 + \underbrace{\sum_{m=1}^{\infty} \frac{(ik)^m}{m!}\langle x^m\rangle}\Big]$$

Motivation

$\langle e^{x} \rangle$

$= \exp\left[\varkappa_1 + \frac{\varkappa_2}{2!} + \frac{\varkappa_3}{3!} + \dots\right]$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

$$= \sum_{m=1}^{\infty} \frac{(ik)^m}{m!} \varkappa_m$$

$\varkappa_m$ ← kumulant

De første kumulanter blir:

$$\varkappa_1 = \langle x \rangle = \mu_1$$

$$\varkappa_2 = \langle x^2 \rangle - \langle x \rangle^2 = \sigma^2 = \mu_2 - \mu_1^2$$

$$\varkappa_3 = \langle x^3 \rangle - 3\langle x^2 \rangle \langle x \rangle + 2\langle x \rangle^3$$

$$= \mu_3 - 3\mu_2\mu_1 + 2\mu_1^3$$

$$\varkappa_4 = \mu_4 - 4\mu_3\mu_1 - 3\mu_2^2 + 12\mu_2\mu_1^2 - 6\mu_1^4$$

Merk: Kumulanten $\varkappa_m$ er et polynom i momentene $\langle x^p \rangle$ med $p \leq m$.

Ofte brukes også normaliserte kumulanter

$$\bar{\varkappa}_n = \frac{\varkappa_n}{\sigma^n}$$

$$\bar{\varkappa}_3 = \frac{\langle (x - \langle x \rangle)^3 \rangle}{\sigma^3} \qquad \text{skewness}$$

$$\bar{\varkappa}_4 = \frac{\langle (x - \langle x \rangle)^4 \rangle}{\sigma^4} - 3 \qquad \text{kurtosis*}$$

[*annen def: $\bar{\varkappa}_4 + 3$]

- skewness : mål på symmetrien til ford.
- kurtosis : mål på avviket av $p(x)$ fra en Gaussian
  [$\bar{\varkappa}_4 > 0$: leptokurtic. "fat tails"]

Def. av $\varkappa_m$ kan virke noe tilfeldig. Dette er også riktig, men $\varkappa_m$ har ganske mange fine egenskaper

- For en Gauss fordeling er $\varkappa_m = 0$, $m > 2$
- $\varkappa_m$ summeres når man summerer uavh. random var [Bourchad/Potters 1.5]

Merk : På samme måte som $\partial_k^n G(k)|_{k=0}$ genererer $\langle x^n \rangle$, genererer $\partial_k^n \ln G(k)|_{k=0}$ kumulantene

Altså

$\ln G(k)$: kumulant genererende funk.

$$\varkappa_m = (-i)^m \partial_k^m \ln G(k)\Big|_{k=0}$$

sml.

$$\langle x^m \rangle = (-i)^m \partial_k^m G(k)\Big|_{k=0}$$

Disse to uttrykkene kan alternativt tas som def. av de moment/kurtosis genererende funksjoner.

Comment on notation

$\varkappa_2 = \langle x^2 \rangle_c$

$= \langle\langle x^2 \rangle\rangle$

1.4.3 Relasjonen mellom G(k) og Fourier Transform

Tidligere definerte vi

$$G(k) = \langle e^{ikx} \rangle = \int_{\Omega} dx \, e^{ikx} \, p(x)$$

Definer

$$\bar{p}(x) = \begin{cases} p(x) & x \in \Omega \\ 0 & x \notin \Omega \end{cases}$$

Kan da skrive:

$$G(k) = \int_{-\infty}^{\infty} dx \, e^{ikx} \, \bar{p}(x) = \mathcal{F}[\bar{p}](k)$$

Fourier transform.

Dvs.

NB

Den karakteristiske funksjonen G(k) er essensielt den Fourier transformerte av den tilhørende sannsynlighets tettheten p(x).

1.5 Multivariate fordelinger

1.5.1 Joint prob. distribusjon

La $\vec{x} = (x_1, x_2, \ldots, x_r)$ være en stok. (random) variabel med r komponenter.
Dens PDF blir da

$$p(\vec{x}) = p(x_1, x_2, \ldots, x_r)$$

og kalles "the joint prob. dist.".

Ex : Dart kasting

1.5.2 Marginal distribution

Anta for enkelhets skyld $r=2$; $\vec{x} = (x_1, x_2)$

Sans. for å finne $\bar{x}_1 = x_1$ uavh. av $\bar{x}_2$ er

$$p_1(x_1) = \int dx_2 \, p(x_1, x_2)$$

Marginale fordelingen

Mer generelt:

$$p_s(x_1, x_2, \ldots, x_s) = \int dx_{s+1} \cdots dx_r \, p(x_1, \ldots, x_r)$$

$\ldots, x_s, x_{s+1}, \ldots$

1.6 SUM AV RANDOM VARIABLE

La X_1 og X_2 være to random variable med gitt joint prob. $p(x_1, x_2)$.

Problem: Hva er fordelingen av y gitt ved

$$y = x_1 + x_2$$

— " —

Sanns. for å finne y mellom y og $y+dy$:

$$P_y(y)\,dy = \iint\limits_{y < x_1 + x_2 < y+dy} dx_1\,dx_2 \quad p(x_1, x_2)$$

$$\int_a^{a+\Delta} dx\, f(x) \simeq \Delta f(a)$$

— ⟲ —

$$P_y(y)\,dy = dy \int_{-\infty}^{\infty} dx_1\,dx_2 \quad \delta(x_1 + x_2 - y)\; p(x_1, x_2)$$

$$= dy \int_{-\infty}^{\infty} dx_1 \quad p(x_1, y - x_1)$$

$$\Rightarrow \boxed{P_y(y) = \int_{-\infty}^{\infty} dx_1 \; p(x_1, y - x_1)}$$

Change notation to

$x_i \to \xi_i$

Spesielt : x_1 og x_2 er stat. uavh.

$$p(x_1, x_2) = p_1(x_1)\, p_2(x_2)$$

$$P_Y(y) = \underbrace{\int_{-\infty}^{\infty} dx_1\, p_1(x_1)\, p_2(y - x_1)}_{\text{convolution}}$$

$$\boxed{P_Y(y) = (p_1 * p_2)(y)}$$

Merk!
Krever stat. uavh.

— // —

Man kan lett utlede

- $\langle y \rangle = \langle x_1 \rangle + \langle x_2 \rangle$
- $\sigma_Y^2 = \sigma_{x_1}^2 + \sigma_{x_2}^2$ (iff x_1 og x_2 er ukorrelert)

1.6.1 Karakteristisk funksjon for addisjon av random variable

- Generelt $G_Y(k) = G_{x_1, x_2}(k, k)$
- Statistisk uavh. x_1 og x_2

$$\boxed{G_Y(k) = G_{x_1}(k)\, G_{x_2}(k)}$$

— // —

Merk: Dette ses lett når man minnes sammenhengen mellom $G(k)$ og den Fourier transformerte av $p(x)$

Tar man FT av begge sider av

$$p_y(y) = (p_1 * p_2)(y)$$

følger

$$G_y(k) = G_{x_1}(k)\, G_{x_2}(k)$$

Ex: Man vet at Gauss-fordelingen er stabil under addisjon (om ikke annet skal vi diskutere dette senere)

Det følger da at convolution av to Gauss-funksjoner er en Gauss-funksj.

Vis dette!

1.6.2 Ex: Diskrete Random Walk.

En "fyllik" tar ett steg til høyre eller venstre med lik sanns i hvert tidssteg.

ξ_i : Steg-variabelen ($i = 0, 1, 2, \ldots$ tid)
Utfallsrommet = ± 1

Problem: Hva er posisjonen for "vår venn" ved tiden $i = t$ (antar start fra 0)

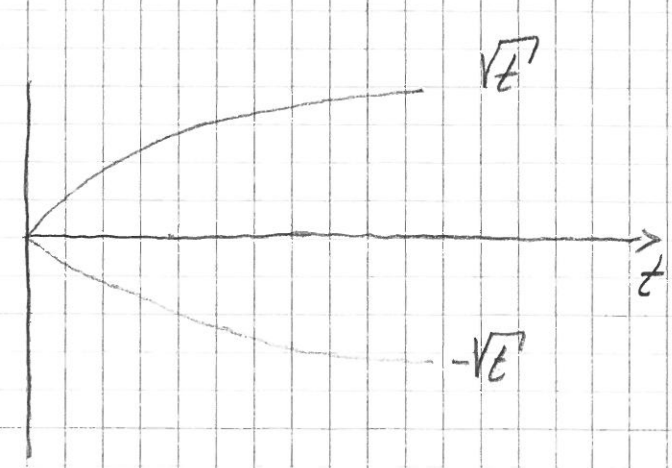
$$x_t = \xi_1 + \xi_2 + \xi_3 + \cdots \xi_t = \sum_{i=1}^{t} \xi_i$$

Finner umiddelbart at

- $\langle x_t \rangle = 0$
- $\langle x_t^2 \rangle = \sum_{i=1}^{t} \langle \xi_i^2 \rangle = t \langle \xi^2 \rangle = t$

Dvs

$$\boxed{\sigma_x = \sqrt{t}}$$



Karakteristisk funksjon for ξ ($p_{\pm 1} = 1/2$)

$$G_\xi(k) = \langle e^{ik\xi} \rangle = \frac{1}{2}\left[e^{ik} + e^{-ik}\right]$$

Vis vha fourier transforms.

Karakteristisk funksjon for X_t:

Da alle ξ_i er stat. uavh. følger:

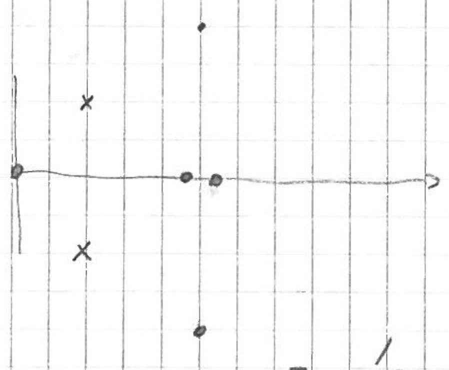
$$G_x(k;t) \equiv G_x(k) = \left[G_\xi(k)\right]^t$$

$$= \left[\frac{1}{2}e^{ik} + \frac{1}{2}e^{-ik}\right]^t$$

$$= \frac{1}{2^t}\sum_{n=0}^{t}\binom{t}{n} e^{ikn} e^{-ik(t-n)}$$

$$\binom{N}{k} = \frac{N!}{k!\,(N-k)!}$$

$$(a+b)^N = \sum_{k=0}^{N}\binom{N}{k} a^k b^{N-k}$$



$$= \frac{1}{2^t}\sum_{n=0}^{t}\binom{t}{n} e^{\underbrace{2ikn - ikt}}$$

$$ik(2n-t) = ikm$$

$$\Rightarrow m = \cancel{2n-t} \; -t+2n$$

$$= \frac{1}{2^t}\sum_{m=-t,\,-t+2}^{t}{}' \binom{t}{\frac{1}{2}(m+t)} e^{ikm}$$

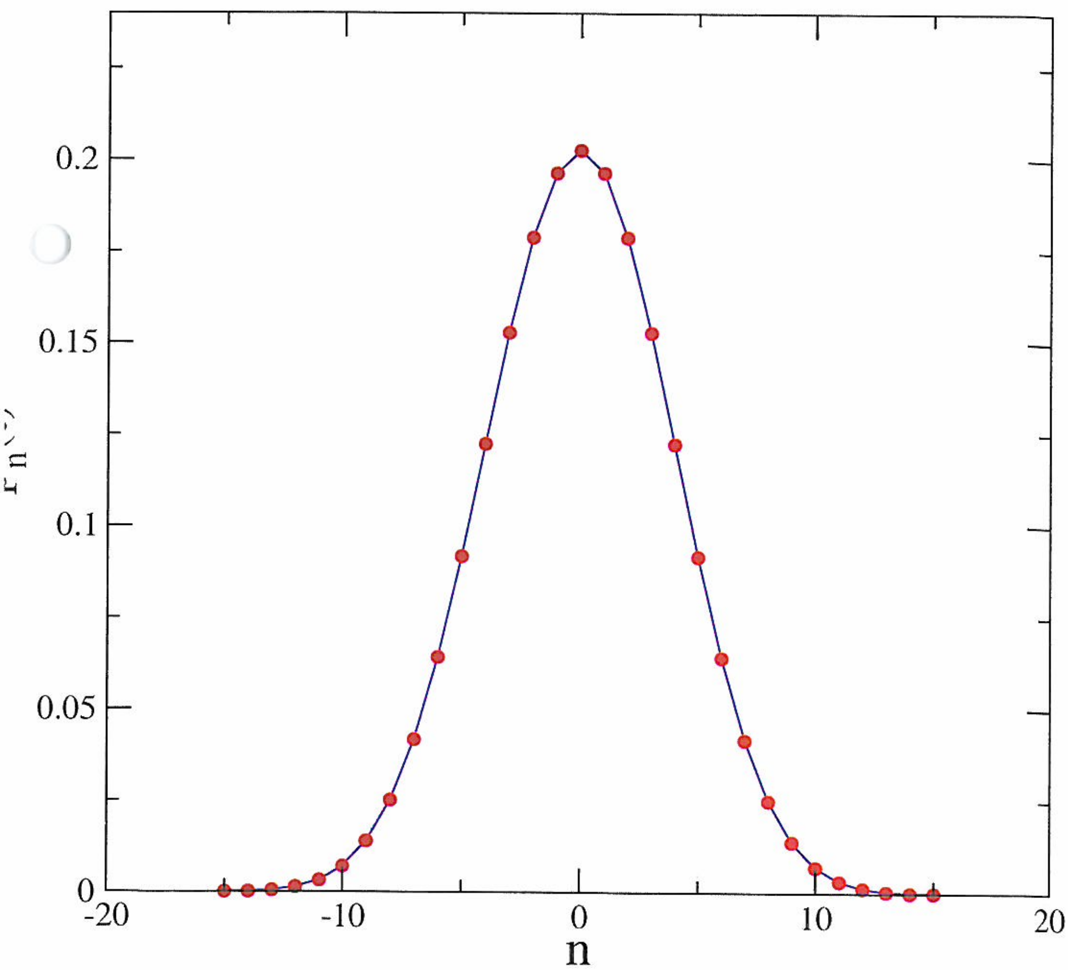
Tilsv. steg lengde 2

Feil fortegn? van K p.16

$$G_x(k) = \langle e^{ikx} \rangle = \sum_{n=-\infty}^{\infty} p_n e^{ikn}; \; x \in N$$

Discrete Random Walk

t=15;   Stepsize +/- 1



As Fourier transforms

$$
\begin{aligned}
G_\xi(k) &= \mathcal{F}[p_\xi](k) \\
&= \int dk\; p_\xi(k)\, e^{ik\cancel{\xi}} \\
&= \frac{1}{2} \int dk \left[\delta(r+1/2) + \delta(r-1/2) \right] e^{ikr} \\
&= \frac{1}{2} \left[e^{ik} + e^{-ik} \right]
\end{aligned}
$$

$$\Rightarrow \quad p_n(t) = \frac{1}{2^t}\binom{t}{\frac{1}{2}(n+t)}$$

Merk: Alternativt kan man skrive

$$p_{\xi}(r) = \frac{1}{2}\,\delta(r+\underbrace{1/2}_{1}) + \frac{1}{2}\,\delta(r-\underbrace{1/2}_{+1})$$

NB Observer hvor elegant (enkelt) vi kom frem til dette resultatet vha den karakter. funks.

Å bruke sanns. tettheten direkte, og suksesivt beregnet convolutions, er mye mer komplisert og tidkrevende.

Sett som en kontinuerlig fordeling:

$$p(x,t) = \mathcal{F}^{-1} G_x(k)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \; e^{-ikx} \; G_x(k)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \; e^{-ikx} \; \frac{1}{2^t} \sum_{n=-t}^{t}{}' \binom{t}{\frac{1}{2}(n+t)} e^{ikn}$$

$$= \frac{1}{2^t} \sum_{n=-t}^{t}{}' \binom{t}{\frac{1}{2}(n+t)} \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} dk \; e^{-ik(x-n)}}_{\delta(x-n)}$$

$$= \frac{1}{2^t} \sum_{n=-t}^{t}{}' \binom{t}{1/2(n+t)} \delta(x-n)$$

Plot the results for the discrete RW

Octave:

```
            -t:2:t
t = 15;
n = -t:t;     pn = bincoeff(t, 0.5*(n+t))/2^t
plot(n,pn)
```

Show an example graph!

1.7 Transformasjon av stokastiske variable

Gitt en stokastisk var x og PDF $p(x)$.

Problem: Hva blir da fordelingen av

$$y = f(x)$$

— " —

Som tidligere [sanns. for y mellom y og $y+dy$]

$$P_y(y)\,dy = \int_{y<f(x)<y+dy} dx\; p(x)$$

$$\simeq dy \int_{-\infty}^{\infty} dx\; \delta(y - f(x))\; p(x)$$

$$\underline{\underline{P_y(y) = \int_{-\infty}^{\infty} dx\; \delta(y - f(x))\; p(x)}}$$

[Mer elegant: $P_Y(y) = \langle \delta(y - \tilde{Y}) \rangle = \langle \delta(y - f(x)) \rangle =$

1.7.1 Karakteristisk funksjon

$$G_y(k) = \langle e^{iky} \rangle$$

$$= \int_{-\infty}^{\infty} dy\; e^{iky} \overbrace{\int_{-\infty}^{\infty} dx\; \delta(f(x)-y)\, p(x)}^{P_y(y)}$$

$$= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\; e^{iky}\, \delta(f(x)-y)\, p(x)$$

$$= \int_{-\infty}^{\infty} dx \; e^{ikf(x)} \, p(x)$$

$$\equiv \langle e^{ikf(x)} \rangle \leftarrow \text{Middel mhp } x.$$

Dvs, resultat som forventet.

<u>Oppsummering</u>:

$$y = f(x) \qquad [p(x) \text{ gitt}]$$

$$P_y(y) = \int_{-\infty}^{\infty} dx \; \delta(y - f(x)) \, p(x) = \langle \delta(y - f(x)) \rangle$$

$$G_y(k) = \langle e^{ikf(x)} \rangle$$

Merk: De ovenstående resultatene generalis. enkelt til høyere dimensjoner. Dim av x og y trenger ikke å være tilsvarende.

Ex: Maxwells hast. dist gitt

$$P_M(\vec{v}) \rightarrow p(E) \qquad E = \frac{1}{2} m \vec{v}^{\,2}$$

Ex : Energi fordelingen av fri punkt-partikkel

Ifølge Maxwell-Boltzmanns lov har man

$$P_M(\vec{v}) = \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{mv^2}{2kT}} \qquad \left[e^{-E/kT}\right]$$

For en punkt-partikkel har man

$$E = +\frac{1}{2}m\vec{v}^2 \quad \left[\equiv f(\vec{v})\right]$$

Da følger at:

$$= \langle \delta(E - \tfrac{1}{2}m\vec{v}^2)\rangle_{\vec{v}}$$

$$p(E) = \int d^3v \; \delta(\tfrac{1}{2}m\vec{v}^2 - E)\, P_M(\vec{v})$$

$$= \frac{2}{\sqrt{\pi}} (kT)^{-3/2} E^{1/2} e^{-E/kT} \qquad \chi^2\text{-dist} \quad \Gamma\text{-dist.}$$

Vis dette!

Kopi H. Risken: The Fokker-Planck Eq. p. 16.
Van Kampen p. 18

1.8 Betinget sannsynlighet

Betrakt to stokastiske variable x_A og x_B.

Da gjelder (Bayes' regel)

$$\boxed{p(x_A, x_B) = p_{A|B}(x_A|x_B) \cdot p_B(x_B)}$$

Dersom $p_{A|B}(x_A|x_B) = p_A(x_A)$ da er x_A og x_B statistisk uavh.
[Sml: $p(x_A, x_B) = p_A(x_A) p_B(x_B)$]

Ofte brukt notasjon:

$$p_{m|n}(x_1, \dots, x_m | x_{m+1}, \dots, x_{m+n})$$

m : variable
n : betingelser

1.8 Sentral Grense Teoremet

1.8.1 Gauss Fordelingen / Normal Fordelingen

* PDF

$$P_G(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

σ : standard avvik

μ : middel

* Karakteristisk funksjon :

$$G_G(k) = \exp\left[i\mu k - \frac{\sigma^2 k^2}{2}\right]$$ [Vis dette!]

Fra dette følger at: ~~$\chi_n = 0 \quad n \geq 3$~~

$$\chi_n \propto \partial_k^n \ln G_G(k)\Big|_{k=0} = 0 \quad n \geq 3$$

Kumulanter av orden 3 og større er null for Gauss-fordelingen.

* Standard Normal fordeling

$$\mu = 0 \quad \sigma = 1.$$

La ξ være en vilkårlig Gauss-fordelt stok. var. Da vil

$$\frac{\xi - \langle \xi \rangle}{\sigma}$$

være standard Normal fordeling, og man skriver $N(\mu,\sigma) = N(0,1)$.

1.8.2 CLT and its derivation

Teorem:

La $\{\xi_i\}$ være sekvensen av uavhengige random var. med $\langle \xi_i \rangle = 0$ og std. $\sigma_i < \infty$

Da vil fordelingen av summen

$$S_n = (\xi_1 + \xi_2 + \dots + \xi_n)/\sqrt{n}$$

gå mot Normal fordelingen $N(0, \sigma_n)$ hvor $\sigma_n^2 = \frac{1}{n}\sum_{i=1}^{n} \sigma_i^2$ når $n \to \infty$

— " —

Merk.

* Teoremet krever at ξ_i er uavh.
* Teoremet krever at std er endelig.
* Teoremet krever ikke at ξ_i har samme fordeling. (PDF'ene er vilkårlig)

Skisse av utledningen: ξ_i er iid (Antatt)

Gi antagelser

$$G_\xi(k) = \langle e^{ik\xi} \rangle = 1 - \frac{\sigma^2 k^2}{2} + \dots \quad \text{pga } \langle \xi_i \rangle = 0$$

$$G_S(k) = \left[\langle e^{ik\xi/\sqrt{n}} \rangle\right]^n$$

$$= \left[G_\xi\left(\frac{k}{\sqrt{n}}\right)\right]^n$$

$$= \left[1 - \frac{\sigma^2 k^2}{2n} + \dots\right]^n$$

$$\simeq 1 - \frac{\sigma^2 k^2}{2} \longrightarrow e^{-\frac{\sigma^2 k^2}{2}}$$

Skisse av utledning

For enkelhets skyld, anta at ξ_i er i.i.d

Da følger:

$$\sigma_\xi = \sigma$$

$$\sigma_S = \left[\frac{1}{n}\sum_{l=1}^{n}\sigma_\xi^2\right]^{1/2} = \sigma_\xi = \sigma$$

$$G_\xi(k) = \langle e^{ik\xi}\rangle = \sum_{m=0}^{\infty}\frac{(ik)^m}{m!}\langle\xi^m\rangle$$

$$= 1 - \frac{\sigma^2 k^2}{2} + \dots \qquad \mu_1 = 0$$

$$= \langle e^{ikS_n}\rangle$$

$$G_{S_n}(k) = \left[\langle e^{ik\xi/\sqrt{n}}\rangle\right]^n = \prod_{m=1}^{n}\langle e^{ik\xi/\sqrt{n}}\rangle$$

$$= \left[G_\xi\left(\frac{k}{\sqrt{n}}\right)\right]^n$$

$$O(n^{-3/2})$$

$$= \left[1 - \frac{\sigma_\xi^2 k^2}{2n} + \dots\right]^n$$

$$\cancel{\simeq 1 - \frac{\sigma^2 k^2}{2} + \dots \qquad [(1+x)^n \simeq 1+nx} \quad x \ll 1]$$

$$\simeq \left[e^{-\frac{\sigma^2 k^2}{2n}}\right]^n \qquad \cancel{n \gg 1} \quad n \to \infty$$

$$= e^{-\frac{\sigma^2 k^2}{2}}$$

Dette er den karakteristiske funksj. for en Gussisk var. med middel null og std σ.

For $\xi^r = (\xi_1 c_1 + \ldots + \xi_n c_n)^r$, the multiplier of $c_1 c_2 \ldots c_n$ in $\langle \xi^r \rangle$ is $\delta_{n,r} n! \langle \xi_1 \xi_2 \ldots \xi_n \rangle$, while the multiplier of $c_1 c_2 \ldots c_{2s}$ in $\langle \xi^2 \rangle^s$ is

$$2^s s! \sum_{\substack{i_1 < i_2 < \ldots < i_{2s} \\ i_1 < j_1, \ldots, i_{2s} < j_{2s}}} \left\langle \xi_{i_1} \xi_{j_1} \right\rangle \ldots \left\langle \xi_{i_{2s}} \xi_{j_{2s}} \right\rangle.$$

Thus, on comparing the coefficients of $c_1 c_2 \ldots c_n$ on both sides of Eq. (1.6.3.4), we have Isserlis's theorem. The second Eq. (1.6.3.3) may also be written in the form

$$\langle \xi_1 \xi_2 \ldots \xi_n \rangle = \sum \prod_{i_r > i_s} \left\langle \xi_{i_r} \xi_{i_s} \right\rangle,$$

the summation being over all products of mean values of different pairs with decreasing suffixes.

1.6.4 *Central Limit Theorem*

The properties of characteristic functions are of considerable use in developing a fundamental statistical theorem known [101] as the *Central Limit Theorem*, which may be stated as follows.

Let $\{\xi_i\}$ be a sequence of *independent* random variables each having *arbitrary* distributions, then the *sums*,

$$\xi = (\xi_1 + \xi_2 + \xi_3 \ldots + \xi_n)/\sqrt{n},$$

approach a *normally* distributed random variable as n approaches infinity. Further, if ξ_i has mean zero and variance $\langle \xi_i^2 \rangle = \sigma_i^2 < \infty$ then ξ has mean zero and variance σ^2, where

$$\sigma^2 = \frac{1}{n} \sum_{i=1}^{n} \sigma_i^2 .$$

The theorem may be proved heuristically as follows. Let

$$\left\langle \xi_i^3 \right\rangle = \tau_i^3, \left\langle \xi_i^4 \right\rangle = v_i^4, i = 1, 2, \ldots n$$

(the exact values of these higher order moments will be of no concern to us in the present investigation as long as they are uniformly bounded). The characteristic function $\phi_\xi(u)$ may be written as

$$\phi_\xi(u) = \left\langle e^{iu\xi} \right\rangle = \prod_{k=1}^{n} \left\langle e^{iu\xi_k/\sqrt{n}} \right\rangle.$$

Taking the logarithm of this gives the logarithmic characteristic function

$$\ln \phi_\xi(u) = \sum_{k=1}^{n} \ln\left(1 - \frac{u^2 \sigma_k^2}{2n} - \frac{i\tau_k^3 u^3}{6n\sqrt{n}} + \frac{u^4 v_k^4}{24n^2} \ldots\right).$$

For $n \to \infty$, the term on the right-hand side under the summation sign may be approximated by $-u^2 \sigma_k^2 / (2n)$. Hence,

$$\lim_{n \to \infty} \left[\ln \phi_\xi(u) \right] = -\frac{1}{2} u^2 \sigma^2$$

and thus

$$\phi_\xi(u) = e^{-u^2 \sigma^2 / 2},$$

where

$$\sigma^2 = \lim_{n \to \infty} \left(\frac{1}{n} \sum_{k=1}^{n} \sigma_k^2 \right).$$

The probability density function, $f_\xi(x)$, of ξ is then

$$f_\xi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_\xi(u) e^{-iux} du = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/(2\sigma^2)}, \qquad (1.6.4.1)$$

which proves the theorem. It should be noted that a rigorous proof of the theorem requires justification of the various limiting processes involved in going to Eq. (1.6.4.1). This can be done by appealing to Lebesgue's dominated convergence theorem [29].

The most important concept of probability theory in relation to Brownian motion is the notion of a *random process*, which we now outline.

1.6.5 *Random processes*

Consider a random variable ξ which depends on the time t, i.e., $\xi = \xi(t)$. A *random process* (also known as a *stochastic process*) is [11] a family of random variables $\{\xi(t), t \in T\}$, where t is some parameter, generally the time, defined on a set T. $\xi(t)$ does not depend in a completely definite way on the independent variable t [12]. Instead one gets different functions $y(t)$ in different observations. To describe the random process completely [11] we decompose the set T into instants $t_1 < t_2 < \ldots < t_n < T$ and then approximate the family of random variables $\xi(t)$ by $\xi(t_1), \xi(t_2), \ldots, \xi(t_n)$. We may then use the following set of *joint* distribution functions: $P_1(y_1, t_1)\, dy_1$ is the probability of finding $\xi(t_1)$ in the interval $(y_1, y_1 + dy_1)$. $P_2(y_1, t_1; y_2, t_2)\, dy_1\, dy_2$ is the joint probability of finding $\xi(t_1)$ in the interval $(y_1, y_1 + dy_1)$ and $\xi(t_2)$ in the interval $(y_2, y_2 + dy_2)$. $P_3(y_1, t_1; y_2, t_2; y_3, t_3) dy_1 dy_2 dy_3$ is the joint probability of finding $\xi(t_1)$ in the interval $(y_1, y_1 + dy_1)$, $\xi(t_2)$ in the interval $(y_2, y_2 + dy_2)$ and $\xi(t_3)$ in the interval $(y_3, y_3 + dy_3)$. This may be continued up to $P_n(y_1, t_1; y_2, t_2; \ldots; y_n, t_n)$.

Merk!

* Vi antok ikke noe spesiell form av $G_\xi(k)$, bare at $\langle \xi \rangle = 0$.

*

Q : - Hva skjer dersom (minst en) $\sigma_\xi = \infty$?

- Hvor bryter da ovenstående argument sammen?

- Hva vil $p(S_n)$ være dersom minst en $\sigma_\xi = \infty$?

$\Longrightarrow$ Levy fordeling.

1.9 Stokastisk Prosess – time-dependent rnd var.

Def. : En stokastisk prosess Y er en funksjon av en stokastisk variabel X <u>OG</u> tiden t;

$$Y_x(t) = f(\underbrace{X, t}_{\text{to variable!}})$$

Spesielt : kan ha $Y_x = X$

Kommentar :
- Som tidligere nevn skiller man ikke alltid mellom prosessen $Y_x = f(X,t)$ og en realisering $y_x = f(x,t)$ (gjelder i fysikk)
- Ofte gir man ikke eksplisitt subsk. x; dvs man skriver bare $Y(t)$

<u>Momenter</u>

$$\langle Y(t) \rangle = \int Y(t)\, p(x)\, dx$$

$$\langle Y(t_1)\, Y(t_2) \dots Y(t_n) \rangle = \int Y(t_1)\, Y(t_2) \dots Y(t_n)\, p(x)\, dx$$

— // —

Merk! Multi-dimensjonal stokastisk prosess:

$$\vec{Y}(t) = f(\vec{X}, t)$$

1.9.1 Hirarki av fordelings funksjoner

Sanns. tetthets funksjonen for at $\bar{Y}_x(t)$ skal ha verdien y ved tiden t $[Y_x(t) = f(\bar{x},t)]$

$$P_1(y,t) = \int \delta(y - Y_x(t))\, p_x(x)\, dx$$
$$= \langle \delta(Y - f(x,t)) \rangle$$

[sml: transf. av variable]

Tilsvarende:

$$P_n(y_1,t_1; y_2,t_2; \ldots; y_n,t_n)$$
$$= \int \delta(y_1 - \bar{Y}(t_1)) \ldots \delta(y_n - \bar{Y}(t_n))\, p_x(x)\, dx$$
$$= \langle \delta(Y_1 - \bar{Y}(t_1))\, \delta(Y_2 - \bar{Y}(t_2)) \ldots \rangle$$

$$t_n \geq t_{n-1} \geq \ldots t_2 \geq t_1$$

NB: Hirarkiet av joint-pdf $\{p_n\}$ bestemmer entydig den stok. pros.

Opposite: The whole hierarchy is needed

Kommentar: – Dog er det ikke enkelt å ha kjenskap til $\{p_n\}$.

– Fra $\{p_n\}$ kan f.eks. alle momenter av $Y(t)$ bestemmes.

1.9.2 Momenter av $Y(t)$

Fra tidligere:

$$\langle Y_x(t) \rangle \overset{\text{Def.}}{=} \int y\, P_1(y,t)\, dy$$

$$= \int y \left[\int \delta(y - Y_x(t))\, p_x(x)\, dx \right] dy$$

$$= \int dx\, p_x(x) \underbrace{\int dy\, \delta(y - Y_x(t))\, y(t)}_{Y_x(t)}$$

$$= \int dx\, \underset{f(x,t)}{Y_x(t)}\, p_x(x)$$

$$\boxed{\langle Y(t) \rangle = \int dx\, \underset{f(x,t)}{Y_x(t)}\, p_x(x)}$$

Tilsvarende finner man:

$$\langle Y_x(t_1) \dots Y_x(t_n) \rangle = \int dx\, Y_x(t_1) \dots Y_x(t_n)\, p_x(x)$$

$$\left[\text{Def.}\quad \langle \quad \rangle = \int y_1 y_2 \dots y_n\, P_n(y_1, t_1; y_2, t_2 \dots y_n, t_n)\, dy_1 \dots dy_n \right]$$

På samme måte som momenter av en stok. var. kan beregnes fra $G(k)$, kan man definere en <u>karakteristisk</u> <u>funksjonal</u> (moment. gen. funksjonal)

$$\underbrace{G[k]}_{\text{funksjonal}} = \left\langle \exp\left[i \int_{-\infty}^{\infty} dt\ k(t)\ Y(t) \right] \right\rangle$$

$k(t)$: vilkårlig (test) hjelpe funksjon (auxillary function)

Momentene av $Y(t)$ følger som suksesive funksjonal deriverte i mhp $k(t)$
[dvs. koef. i utv. i $k(t_1)k(t_2)\ldots$] [expand exp()]

Tilsvarende har man en kumulant genererende funksj.

$$\ln G[k]$$

Merk: Vet man ikke hva en funksjonal derivert er, er dette ikke viktig.

1.9.3 Stasjonære Stokastiske Prosesser

Def. : En stok. pros. $Y_x(t) = f(x,t)$ sies å være stasjonær dersom alle P_n (i hirarkiet) oppfyller (τ: vilkårlig)

hierarchy

$$P_n(y_1, t_1+\tau; y_2, t_2+\tau; \dots, y_n, t_n+\tau) \equiv P_n(y_1, t_1; y_2, t_2; \dots; y_n, t_n)$$

for vilkårlig τ.

— " —

NB Dvs., alle P_n er tidstranslasjons invariant [og det samme gjelder for momentene]

Kommentar : * I praksis er det ofte ikke så enkelt, spesielt for numeriske data, å teste for stasjonaritet.

* Def. gir at bare tids diff. betyr noe, Dvs. ingen absolutt tid!

Ex: for en stasjonær stok. pros. vil

* $P_1(y,t)$ være tidsuavhengig
* $P_2(y_1, t_1; y_2, t_2)$ avh. av $t_2 - t_1$

Q : Er en (vanlig) random walk stasjonær?

A : ~~Ja~~ Nei

1.9.4 Ergodisitet Ergodicity

Vi skal nå definere begrepet ergodisitet, som gjelder for stasjonære stok. pros.

Motivasjon: Anta at man ønsker å studere den midlere posisjonen $\langle \overset{Y}{\cancel{X}} \rangle$ for en random walk prosess.

To muligheter:

* Studere <u>en</u> walker over lang tid og beregne middelverdien

$$\lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} y(t)\, dt$$

* Studere et ensambel av walkers og beregne middelverdien

$$\int dx\, f(x,t)\, P_1(x)$$

Q: Vil disse to middel-verdiene ha noe med hverandre å gjøre?

A: Ja, dersom processen er ergodisk.

Def. : En stasjonær stokastisk prosess $Y_x(t) = f(x,t)$ sies å være ergodisk dersom

$$\langle Y \rangle = \underbrace{\int dx\, f(x,t)\, p(x)}_{\text{ensamble middel}} = \underbrace{\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt\, y(t)}_{\text{Tids-middel}}$$

— " —

$$\boxed{\text{ensamble midling} = \text{Tids midling}}$$

Prosesser som bryter ergositet ("ergodicity breaking") er i dag et "hot" forskningsfelt.

1.9.5 Korrelasjons Funksjoner

Gitt to stokastiske prosesser $y_1(t)$ og $y_2(t)$. Da definerer man:

$$W_{ij}(t_1, t_2) = \langle [Y_i(t_1) - \langle Y_i \rangle][Y_j(t_2) - \langle Y_j \rangle] \rangle .$$

$$= \langle Y_i(t) Y_j(t) \rangle - \langle Y_i \rangle \langle Y_j \rangle \quad i,j = 1,2$$

Når $= \langle \delta Y_i(t) \, \delta Y_j(t) \rangle \qquad \delta y_i(t) = Y_i(t) - \langle Y_i \rangle$

*

*

[Kom

Merk

Correlation functions:

In general the n'th point correlation function is related to

$$C(t_n, \dots, t_1) = \int d\xi_n \dots d\xi_1 \; \xi_n \dots \xi_1 \; P_n(\xi_n, t_n; \dots; \xi_1, t_1)$$

$$= \langle \xi_n \dots \xi_1 \rangle$$

For n=2, one does often only say correlation function.

ert til

i=j, og

enkelte forfattere skriver

$$W_{ij}(t_1, t_2) = \left. \begin{array}{l} \langle Y_i(t_1) Y_j(t_2) \rangle_c \\ \langle\langle Y_i(t) Y_j(t_2) \rangle\rangle \end{array} \right\} \text{forskj. notasjon}$$

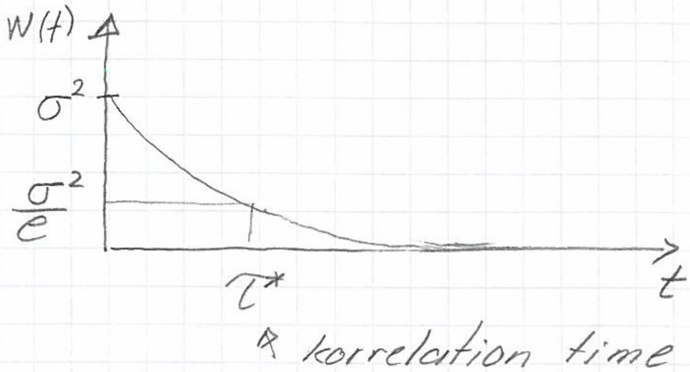
* Matematikkeren kaller $W_{ij}(t_1, t_2)$ for covariansen. Man må dividere med $[\langle Y_i^2 \rangle_c \langle Y_j^2 \rangle_c]^{1/2}$ for å få korr. funksjonen.

I fysikk skiller man ikke strengt mellom de to. Man taler heller om normerte og unormerte korr. fonk.

Spesielt for stasjonære prosesser vil $W_{ij}(t_1, t_2)$ kun avhenge av $|t_1 - t_2| = \tau$ og man skriver

$$W_{ij}(\tau) = \langle Y_1(t)\, Y_2(t+\tau) \rangle_c = \langle Y_1(0)\, Y_2(\tau) \rangle_c$$

Typisk auto-korrelasjons funksjon



Man sier at de stok. pros. er ukorrelert dersom:

$$\cancel{W(t_1, t_2) \propto \delta(t_1 - t_2)}$$

[og dersom $\cancel{W_{ij}(t_1, t_2)} = 0$ når i

$$\boxed{W_{ij}(t_1, t_2) \propto \delta_{ij}\, \delta(t_1 - t_2)}$$

— // —

For komplexe prosesser:

$$W_{ij}(\tau) = \langle Y_i^*(t)\, Y_j(t+\tau) \rangle_c$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} dt\; Y_i^*(t)\, Y_j(t+\tau)$$

1.9.6 Gaussiske Stokastiske Prosesser

Def. : En stokastisk prosess sies å være Gaussisk dersom alle $\{P_n\}$ er (multivariate) Gaussiske fordelinger

— " —

Gaussiske stok. prosesser opptrer ofte i naturen (om ikke annet så approximativt) og de har derfor blitt mye studert.

At en stok. pros. er Gaussisk forenkler ofte "livet" betraktelig da alle kumulanter større enn $m=2$ forsvinner. Vis dette!

Dvs, en Gaussisk stokastisk prosess er fullstendig bestemt av

* $\langle Y(t)\rangle$: Middelverdien
* $\langle Y(t_1) Y(t_2)\rangle_c$: Korrelasjons funksjonen

[cv. $\langle Y(t_1) Y(t_2)\rangle$ 2. momentet]

Mer generelt (Karakteristisk funksjonal)

$$G[k] = \left\langle \exp\left\{ i \int dt\; k(t)\, Y(t) \right\} \right\rangle$$

$$= \exp\left\{ i \int dt_1\; k(t_1) \langle Y(t_1)\rangle - \frac{1}{2}\int dt_1 dt_2\; k(t_1)\, k(t_2) \langle Y(t_1)\, Y(t_2)\rangle_c \right\}$$

OBS

1.9.7 Wiener-Khintchine Teoremet [se Risken p. 30]

Betrakt den stokastiske prosessen $\xi(t)$, som vi vil anta er <u>stasjonær</u>.

Den Fourier transformerte av $\xi(t)$ er da også en stok. pros.

$$\hat{\xi}(\omega) = \int_{-\infty}^{\infty} dt\ \xi(t)\, e^{-i\omega t} = \mathcal{F}[\xi](\omega)$$

Betrakter nå:

$$\langle \hat{\xi}(\omega)\, \hat{\xi}^*(\omega') \rangle$$

$$= \left\langle \int_{-\infty}^{\infty} dt\ \xi(t)\, e^{-i\omega t} \int_{-\infty}^{\infty} dt'\ \xi^*(t')\, e^{i\omega' t'} \right\rangle$$

$$= \int dt\, dt'\ \underbrace{\langle \xi(t)\, \xi^*(t') \rangle}_{\langle \xi(t-t')\, \xi^*(0) \rangle \text{ pga } \xi(t) \text{ er stasjonær}}\ e^{-i\omega t + i\omega' t'}$$

Innfører: $\tau = t - t' \Rightarrow t' = t - \tau$

$$= \int_{-\infty}^{\infty} dt \int_{+\infty}^{-\infty} (-d\tau)\ \langle \xi(\tau)\, \xi^*(0) \rangle\, e^{-i\omega t + i\omega'(t-\tau)}$$

$$= \int_{-\infty}^{\infty} dt\ e^{-i(\omega-\omega')t} \int_{-\infty}^{\infty} d\tau\ \langle \xi(\tau)\, \xi^*(0) \rangle\, e^{-i\omega'\tau}$$

$$= 2\pi\,\delta(\omega-\omega')\int_{-\infty}^{\infty} d\tau\,\underbrace{\langle \xi(\tau)\,\xi^*(0)\rangle}_{W(\tau)}\,e^{-i\omega\tau}$$

$$= 2\pi\,\delta(\omega-\omega')\,\mathcal{F}[W](\omega)$$

$$\equiv \pi\,\delta(\omega-\omega')\,P(\omega)$$

$$\Rightarrow \quad P(\omega) = 2\int_{-\infty}^{\infty} d\tau\,\langle \xi(\tau)\,\xi^*(0)\rangle\,e^{-i\omega\tau}$$

$$= 2\int_{-\infty}^{\infty} d\tau\,W(\tau)\,e^{-i\omega\tau}$$

$P(\omega)$ is called the "power spectral density" (psd)
[The factor of two depends on if one uses the one or two-sided psd. (for two-sided, no 2)]

The Wiener-Khintchine theorem states that the power spectral density of a <u>stationary</u> process is the Fourier transform of its correlation function

$$P(\omega) = 2\,\mathcal{F}[W](\omega)$$

Note: In the time-domain, the (time) correlation function is defined as

$$W(t) = \int_{-\infty}^{\infty} d\tau \; \xi(\tau + t)\, \xi(\tau)$$

[t is often called the lag]

A transformation pair for the Fourier transf. is (correlation theorem where $A = B$ in $\langle AB \rangle$)

$$W(t) \iff |\hat{\xi}(\omega)|^2 \propto P(\omega)$$

$$[\, \mathrm{Corr}(A,B) \iff \hat{A}(\omega)\, \hat{B}^*(\omega) \,]$$

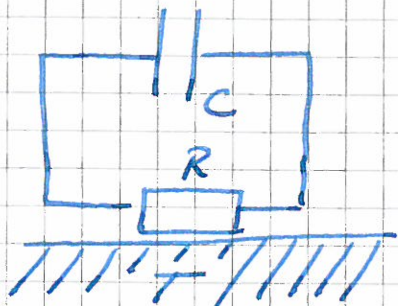
$W(t)$ as defined above is obtained if one assumes $\xi(t)$ to also be ergodic!

9

1.9.10 Stationary Markov Processes

Stochastic proc. that are <u>both</u> stationary and Markovian, are of special interest in physics, in particular for describing equilibrium systems/fluctuations.

Ex. : RC - circuit with fixed temp.



The fluctuating voltage, $V(t)$, over C is Markovian to a good appr.
If the temp. of R is constant, the proc. is also stationary

$$P_1(V_1) = \left(\frac{C}{2\pi kT}\right)^{1/2} \exp\left[-\frac{CV_1^2}{2kT}\right]$$

— " —

The transition prob. does not for a stationary Markov proc. depend on two times (only diff.)

$$\boxed{P_{1|1}(y_2, t_2 \mid y_1, t_1) = T_\tau(y_2 \mid y_1) \quad \tau = t_2 - t_1}$$

$$= P_{1|1}(y_2, t_2 - t_1 \mid y_1, 0)$$

~~1.9.9~~ 1.9.10 Markov Chains Endre rekkefølgen

An especially simple class of Markov proc. are the Markov Chains.

They are defined by the following properties:

i) The range of y is a discrete set of states (discrete state space)

ii) The time variable is descrete and takes only integer values

iii) The process is stationary (or homogen.) so that the trans. prob. depends only on the time difference.

A finite Markov chain is one whose state space is finite (finite No. of states).

$P_1(y,t) \longrightarrow$ N-comp. vector : $p_n(t)$

~~$P_{1|1}(y_2|y_1)$~~ $\longrightarrow$ $N \times N$ matrix : T_{nm}

$P_{1|1}(Y_2,t_2 | Y_1,t_1) = T_\tau(Y_2|Y_1)$

1.9.11 The Chapman – Kolmogorov Equation

Let $y(t)$ be a Markovian Stochastic Process.

Then it follows that $(t_3 > t_2 > t_1)$

$$P_3(y_3, t_3; y_2, t_2; y_1, t_1)$$

$$= P_{1|1}(y_3, t_3 | y_2, t_2)\, P_{1|1}(y_2, t_2 | y_1, t_1)\, P_1(y_1, t_1)$$

Integrating both sides of this eq. with respect to y_2 gives

$$P_2(y_3, t_3; y_1, t_1)$$

$$= \int dy_2\, P_{1|1}(y_3, t_3 | y_2, t_2)\, P_{1|1}(y_2, t_2 | y_1, t_1)\, P_1(y_1, t_1)$$

and after dividing both sides by $P_1(y_1, t_1)$
$[P_2 = P_{1|1} P_1]$

$$\boxed{P_{1|1}(y_3, t_3 | y_1, t_1) = \int dy_2\, P_{1|1}(y_3, t_3 | y_2, t_2)\, P_{1|1}(y_2, t_2 | y_1, t_1)}$$

This is the Chapman – Kolmogorov eq.

It is an identity that must be fullfilled by the transition prob. of <u>any</u> Markov proc.

Note:

The time ordering is essential
$t_3 > t_2 > t_1$

Interpretation of the ck-eq.

The transition prob. $p_{1|1}(x_3,t_3|x_1,t_1)$ is independent of if one goes directly from x_1,t_1 to x_3,t_3, or via all possible intermediate states x_2,t_2.

A Markov process is determined by
$P_{1|1}(y_2, t_2 | y_1, t_1)$ [transition prob.] and $P_1(y_1, t_1)$

However, they cannot be chosen arbitarily but must satisfy:

NB

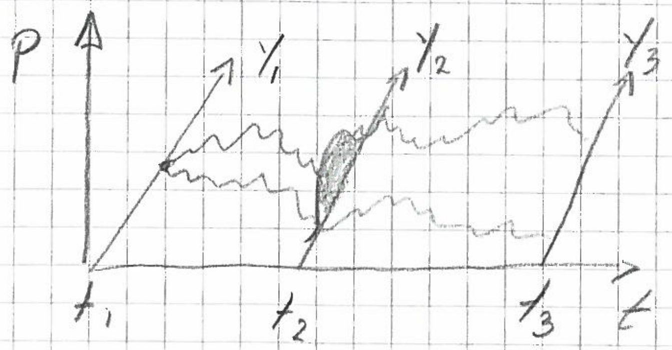
① The CK-eq.

②
$$P_1(y_2, t_2) = \int dy_1 \; P_{1|1}(y_2, t_2 | y_1, t_1) \, P_1(y_1, t_1)$$
$$= \int dy_1 \, P_2(y_2, t_2; y_1, t_1)$$

Vice versa, any two non-negative functions P_1 and $P_{1|1}$ that obey ① and ② ~~dese~~ define uniquely a Markov process.

— // —

Interpretation of the CK-equation



Note that for a stationary Markov proc.

$[P_{1|1}(y_2,t_2|y_1,t_1) = P_{1|1}(y_2,t_2-t_1|y_1,0) = T_{\tau_{21}}(y_2|y_1)]$

the CK-eq. becomes

$$P_{1|1}(y_3,t_3-t_1|y_1,0) = \int dy_2\, P_{1|1}(y_3,t_3-t_2|y_2,0)\, P_{1|1}(y_2,t_2-t_1|y_1,0)$$

$$T_{\tau_{31}}(y_3|y_1) = \int dy_2\, T_{\tau_{32}}(y_3|y_2)\, T_{\tau_{21}}(y_2|y_1)$$

$$\tau_{31} = \tau_{32} + \tau_{21} \qquad \tau_{ij} = t_i - t_j$$

Hence for a finite Markov chain, this becomes

$$T_{\tau+\tau'} = T_{\tau}\, T_{\tau'} \quad \text{(matrix multiplication)}$$

$$p(t) = (T)^t\, p(0)$$

(Studies by Perron / Frobenius)

one eigen-value $\lambda = 1$ that is not degenerate)

[See Van Kampen p. 90]

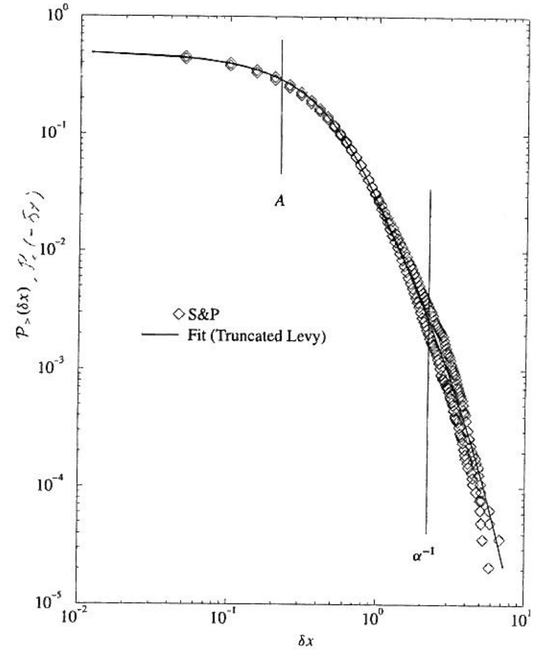


Fig. 2.8. Elementary cumulative distribution $\mathcal{P}_{1>}(\delta x)$ (for $\delta x > 0$) and $\mathcal{P}_{1<}(\delta x)$ (for $\delta x < 0$), for the S&P 500, with $\tau = 15$ min. The thick line corresponds to the best fit using a symmetric TLD $L_\mu^{(t)}$, of index $\mu = \frac{3}{2}$. We have also shown on the same graph the values of the parameters A and α^{-1} as obtained by the fit.

- The particular value $\mu = \frac{3}{2}$ has a simple theoretical interpretation, which we shall briefly present in Section 2.8.

In order to characterize a probability distribution using empirical data, it is always better to work with the cumulative distribution function rather than with the distribution density. To obtain the latter, one indeed has to choose a certain width for the bins in order to construct frequency histograms, or to smooth the data using, for example, a Gaussian with a certain width. Even when this width is carefully chosen, part of the information is lost. It is furthermore difficult to characterize the tails of the distribution, corresponding to rare events, since most bins in this region are empty. On the other hand, the construction of the cumulative distribution does not require to choose a bin width. The trick is to order the observed data according to their rank, for example in decreasing order. The value x_k of the

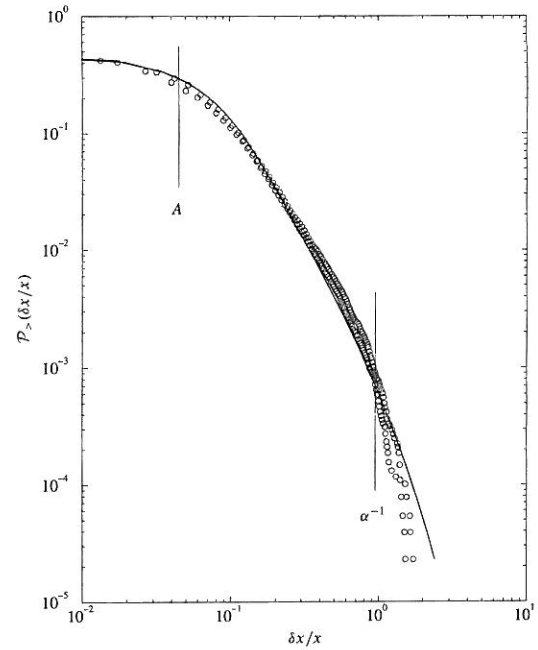


Fig. 2.9. Elementary cumulative distribution for the DEM/\$, for $\tau = 15$ min, and best fit using a symmetric TLD $L_\mu^{(t)}$, of index $\mu = \frac{3}{2}$. In this case, it is rather $100\delta x/x$ that has been considered. The fit is not very good, and would have been better with a smaller value of $\mu \sim 1.2$. This increases the weight of very small variations.

kth variable (out of N) is then such that:

$$\mathcal{P}_>(x_k) = \frac{k}{N+1}. \tag{2.7}$$

This result comes from the following observation: if one draws an $(N+1)$th random variable from the same distribution, there is an a priori equal probability $1/(N+1)$ that it falls within any of the $N+1$ intervals defined by the previously drawn variables. The probability that it falls above the kth one, x_k is therefore equal to the number of intervals beyond x_k, which is equal to k, times $1/(N+1)$. This is also equal, by definition, to $\mathcal{P}_>(x_k)$. (See also the discussion in Section 1.4, and Eq. (1.45)). Since the rare events part of the distribution is a particular interest, it is convenient to choose a logarithmic scale for the probabilities. Furthermore, in order to check visually the symmetry of the probability

N+1 intervals : value can also fall outside the max and min value (=> 2 "extra" intervals)

1.9.12 Important Stochastic Processes

We will now discuss a few stochastic proc. that are of particular importance in physics.

They are:

1) Noise
→ 2) The Wiener process
→ 3) The Ornstein-Uhlenbeck proc.
4) Cauchy process
→ 5) Poisson process

① Noise

Noise is strictly speaking not a well-defined term, but we will discuss how it is normaly used in physics.

Noise is a stationary process which satisfies

$$\langle \xi(t) \rangle = 0$$
$$\langle \xi(t)\, \xi(t') \rangle = W(t-t')$$

If the correlation function is prop. to a δ-function

$$W(t-t') = q\, \delta(t-t'),$$

then the noise is termed white noise (q is the strength of the noise.)

According to the Wiener-Khintchin teoremet its power spectrum reads

$$P(\omega) = \int_{-\infty}^{\infty} dt \;\; w(t)\, e^{i\omega t} \;\; \propto \;\; \omega^{0}$$

This is the reason for the noise being called white; it contains equal amounts of all frequecies.

One also talks about <u>colored noise</u> depending on the scaling with ω of the power spectrum.

Common terms are :

White noise	: $P(\omega) \propto \omega^{0}$		
Pink — " —	: $P(\omega) \propto \omega^{-1}$	($1/f$-noise)	
Bron — " —	: $P(\omega) \propto \omega^{-2}$		
Black — " —	: $P(\omega) \propto \omega^{-\gamma}$	$\gamma > 2$	

* Note that one in practise cannot have white noise since this would imply infinite power. It applies only over ~~an~~ a limited freq. interval.

* If $P(\omega) \propto \omega^{-\gamma}$ with $\gamma \neq 0$ it implies that the noise is correlated. Colored noise is therefore correlated.

* Noise is always stationary, (the way the term is used commonly)

② The Wiener Process

[VK p. 79]

It is named after the am. mathematition Norbert Wiener. It is also known as the Wiener-Levy process.

The Wiener process was introduced in order to describe the position of a Brownian particle.

The Wiener process is a Markov process defined by the conditional prob. (transition prob.) ($D>0$ constant)

$$P_{1|1}(x_2,t_2|x_1,t_1) = \frac{1}{\sqrt{4\pi D(t_2-t_1)}} \exp\left\{-\frac{(x_2-x_1)^2}{4D(t_2-t_1)}\right\}$$

$t_2 > t_1$

and initial condition ($t=0$):

$$P_1(x_1,t_1=0) = \delta(x_1).$$

— " —

Note that $P_{1|1}$ is a Gaussian with std.

$$\sigma = \sqrt{2D(t_2-t_1)}$$

(have you seen this before?)

In transport theory WP is the prototype of a stochastic process.

Exercise: Show that a WP is Markovian!

Q: Is the WP stationary?

A: No, it is not.

This can be shown this way:

$$p_1(y,t) = \int dy_0 \, P_{1|1}(y,t \mid y_0,t_0) \underbrace{p(y_0,t_0)}_{\delta(y_0)} \qquad t_0=0$$

$$= P_{1|1}(y,t \mid 0,0)$$

$$= \frac{1}{\sqrt{4\pi D t}} \exp\left\{ -\frac{y^2}{4Dt} \right\}$$

Since $p_1(y,t)$ is not time-translation invariant, the WP process is non-stationary. This is so because of the boundary condition.

Note though, that the step-size dist. $P_{1|1}$ is stationary.

Summary: WP is a non-stationary Markov process!

Ref: N. Wiener, J. Math. Phys. 2, 131 (1923).

③ The Orenstein-Uhlenbeck Process [VK p. 83]

Ref.: Uhlenbeck and Ornstein, Phys. Rev. 36, 823 (1930)

This process was introduced to describe the stochastic behavior of the velocity of a Brownian particle.

The OU-process is a stationary, Gaussian and Markovian process defined by

* $$P_1(y_1,t_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y_1^2}{2}}$$

* $$P_{1|1}(y_2,\underbrace{t_2-t_1}_{\tau};y_1,0) = P_{1|1}(y_2,t_2|y_1,t_1)$$

$$= \frac{1}{\sqrt{2\pi(1-e^{-2\tau})}} \exp\left[-\frac{(y_2-y_1e^{-\tau})^2}{2(1-e^{-2\tau})}\right]$$

— " —

It can be shown to have the properties ($\xi(t)$ a OU process)

$$\langle \xi(t) \rangle = 0$$

$$\langle \xi(\tau)\xi(0) \rangle \propto e^{-\gamma\tau}$$

Two important theorems

Doob's theorem

"The OU process is essensially the only (non-trivial) process that is stationary, Gaussian, and Markovian." (3 proppertics)

— " —

Essential (non-trivial) means one must allow for linear transformations of y and t.

Trivial conter ex: the fully random process where $p_n(\ldots)$ factorizes.

Theorem

If $\xi(t)$ is stationary, Gaussian and has an exponential auto-correlation function $W(\tau) = W(0)\, e^{-\gamma\tau}$, then $\xi(t)$ is the Ornstein-Uhlenbeck process.

Proof: Assume zero mean and unit variance. Then the generating functional is

$$G[k] = \exp\left\{ -\frac{1}{2}\int dt_2\, dt_1\, k(t_2)k(t_1) \underbrace{e^{-\gamma|t_2-t_1|}}_{\propto W(t)} \right\}$$

which is also the GF of the OU-process.

④ Cauchy Process

CP is a Markov process for which

$$T_\tau(y_2|y_1) = \frac{1}{\pi} \frac{\tau}{(y_2-y_1)^2 + \tau^2} \qquad \begin{matrix} \tau > 0 \\ -\infty < y < \infty \end{matrix}$$

⑤ Poisson Process

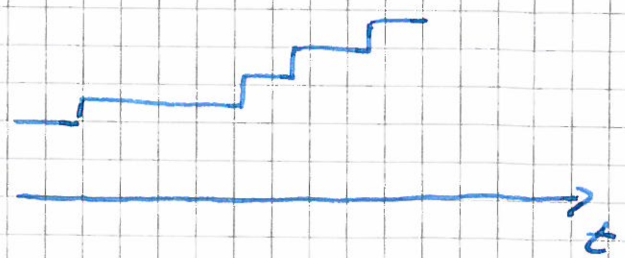
PP is a Markov process.

$Y(t)$ takes on values $0, 1, 2, \ldots$ (integer) and $t \geq 0$. Then the PP is defined by

$$P_1(n, 0) = \delta_{n0}$$

$$P_{1|1}(n_2, t_2 | n_1, t_1) = \frac{(t_2 - t_1)^{n_2 - n_1}}{(n_2 - n_1)!} e^{-(t_2 - t_1)}$$

It is understood that $P_{1|1} = 0$ if $n_2 < n_1$



steps of unit length, at random times

PP is uniquely determined by the time points at which the steps are taken.

The number of time points between t_1 and t_2 is Poisson distributed. ("shot-noise")