3. STOCHASTIC EQUATION OF MOTION

* Time evolution in <u>variable space</u>

- Langevin equations

* Time evolution in <u>probability space</u>

- Master equations (including Fokker-Planck eq.)

3.1 Langevin Equations

We consider a particle (the system) moving around in a thermal bath (the surroundings).

In one-dimension, the (linear) Langevin eq. is

$$\boxed{m\ddot{x} = \underbrace{-\partial_x V(x)}_{①} \underbrace{- m\gamma\dot{x}}_{②} + \underbrace{R(t)}_{③}}$$

① An external potential

② Energy dissipation

To see this set $R(t)=0$, and multiply LE by $\dot{x}$

$$m\dot{x}\ddot{x} + \dot{x}\,\partial_x V(x) = \frac{d}{dt}(K+V) = -2\gamma K$$

③ The stochastic force caused by the surroundings. It's properties need to be specified.

— " —

Stochastic properties of $R(t)$

1| Symmetry argument for stationary system (ensamble or time average)

$$\langle R(t) \rangle = 0$$

2| $R(t)$ is related to the temperature (T) of the bath (since $R(t) \to 0$ when $T \to 0$)

3| Relaxation processes (thermal) of the bath is much faster than that of the system (thermal relaxation times given by γ^{-1})

In this case:

$$\langle R(t+t') R(t') \rangle \simeq C\,\delta(t)$$

and the system is Markovian.

On the other hand if

$$\langle R(t+t')R(t')\rangle \propto \underset{\uparrow}{Z}(t) \neq \delta(t)$$

Memory kernal (correlation function)

the system is non-Markovian.

<u>Note</u> : Both γ and C come from the system-bath coupling. They should therefore be related.

3.I.1 Position independent potential $V(x) = V_0$

The LE becomes (assuming one-dimension)

$$m\ddot{x} = -m\gamma\dot{x} + R(t)$$

or $(v = \dot{x})$

$$\dot{v} = -\gamma v + \frac{1}{m}R(t)$$

This eq. is simple to solve as it is a first-order inhomogeneous diff. eq.
(solution is the sum of the general homogeneous and a particular solution)

$$v(t) = v(t{=}0)\, e^{-\gamma t} + \frac{1}{m} \int_0^t dt'\, e^{-\gamma(t-t')} R(t')$$

[Need to average over all possible realizations of $R(t)$]

For long times, i.e. $t \to \infty$, only the last term remains, and thus (in thermal equilibrium)

$$\langle v(t) \rangle = 0 \qquad (t \to \infty)$$

On the other hand $(t \to \infty)$:

$$\langle v^2 \rangle = \frac{1}{m^2} \int_0^t dt' \int_0^t dt''\, e^{-\gamma(t-t')}\, e^{-\gamma(t-t'')} \underbrace{\langle R(t')\, R(t'') \rangle}_{C\, \delta(t'-t'')}$$

$$= \frac{1}{m^2} \int_0^t dt'\, e^{-2\gamma(t-t')}\, C$$

$$= \frac{C}{2m^2\gamma}$$

When $t \to \infty$ the system should be in thermal equilibrium so that (one-dim.):

$$\langle K \rangle = \frac{1}{2} m \langle v^2 \rangle = \frac{1}{2} k_B T$$

$$\Rightarrow \langle v^2 \rangle = \frac{k_B T}{m}$$

This implies that (one-dim.)

$$C = 2m\gamma k_B T \qquad [= 6m\gamma k_B T \text{ in } 3D]$$

and

$$\langle R(t+t')R(t')\rangle = 2m\gamma k_B T\,\delta(t)$$

— " —

Notice : The relation (one-dim)

$$C = 2m\gamma k_B T$$

relates the fluctuations (T) and dissipations (γ) in the system.

This is an example of the fluctuation-dissipation theorem.

It essentially says that

$$\langle R(t+t')R(t')\rangle = 2m\gamma k_B T\,\delta(t)$$

is equivalent to detailed balance (condition imposed on the transition rates of the master eq.), in order to satisfy the requirment that thermal equilibrium is reached at lang times.

Note : The evolution of the stochastic state variable $x(t)$ is determined by the state of the system and the bath at the same time. t.

Note : Only when the timescales of relaxation and thermal enviornment are short relative to the characteristic system time, is the system Markovian.

If this is not the case, the so-called generalized Langevin eq. result ;

$$m\ddot{x} = -\partial_x V(x) \underbrace{- m\int_0^t d\tau\, Z(t-\tau)\dot{x}(\tau)}_{\text{Non-Markovian friction term.}} + R(t)$$

Where :

$$\langle R(t) \rangle = 0$$

$$\langle R(t+t')R(t') \rangle = m k_B T \underbrace{Z(t)}_{\text{Memory kernel}}$$

Some comments:

* It is (partly) due to the Fluctuation-Dissipation theorem that relates the size of the fluctuations to the (dissipative) damping that the Langevin approach is so successful.

* The noise is fully expressed in terms of the macroscopic damping constant together with the temperature.

* Like with Einsteins relation, one obtains an identity because one knows beforehand the size of the equilibrium fluctuations from ordinary equilibrium stat. mech.

* The physical picture is that the random "kicks" tend to spread out $v(t)$, while the damping term tries to bring v back to zero. The balance between these two opposing tendencies is the equilibrium distribution.

$$x(t) = x_0 - \frac{v_0}{\gamma}\left(e^{-\gamma t} - 1\right) + \frac{1}{m}\int_0^t dt' \int_0^{t'} dt'' \, e^{-\gamma(t'-t'')} R(t'')$$

$$\int_0^t dt' \int_0^{t'} dt'' \, e^{-\gamma(t'-t'')} R(t'') \qquad \int_0^t d\tau' \int_0^{\tau'} d\tau'' \, e^{-\gamma(\tau'-\tau'')} R(\tau'')$$

$$\Rightarrow \int_0^t dt' \int_0^{t'} dt'' \int_0^t d\tau' \int_0^{\tau'} d\tau'' \, e^{-\gamma(t'-t'')} e^{-\gamma(\tau'-\tau'')} \underbrace{\langle R(t'') R(\tau'') \rangle}_{\delta(t''-\tau'')}$$

$\tau'' - t''$

$$\int_0^t dt' \int_0^t d\tau' \cdot \underbrace{\int_0^{t'} dt'' \int_0^{\tau'} d\tau''}\, e^{-\gamma(t'-t'')} e^{-\gamma(\tau'-\tau'')} \langle R(t'') R(\tau'') \rangle$$

Do the integration over the variable corresponding to max (t', τ') for the upper integration limit.

$$\int_0^t dt' \int_0^t d\tau' \int_0^{\min(t',\tau')} dt'' \int_0^{\max(t',\tau')} d\tau'' \; \ldots$$

$$v = v_0 e^{-\gamma t} + \frac{1}{m}\int_0^t dt' e^{-\gamma(t-t')} R(t')$$

Integrerer: fra $t=0$ til $t=t$

$$x = x_0 - \frac{v_0}{\gamma}(e^{-\gamma t} - 1) + \frac{1}{m}\int_0^t dt' \int_0^{t'} dt'' e^{-\gamma(t'-t'')} R(t'')$$

$$\langle (x-x_0)^2 \rangle = \frac{v_0^2}{\gamma^2}(1-e^{-\gamma t})^2 + 2\frac{v_0}{\gamma}(e^{-\gamma t}-1)\frac{1}{m}\underbrace{\int_0^t dt' \int_0^{t'} dt'' e^{-\gamma(t'-t'')} \langle R(t'') \rangle}_{=0}$$

min(t', τ')

$$+ \frac{1}{m^2}\int_0^t dt' \int_0^t d\tau' \int_0^{t'} dt'' \int_0^{\tau'} d\tau'' e^{-\gamma(t'-t''+\tau'-\tau'')} \langle R(t'')R(\tau'') \rangle \quad (= C\delta(t))$$

$$= \frac{v_0^2}{\gamma^2}(1-e^{-\gamma t})^2 + \frac{C}{m^2}\int_0^t dt' \int_0^t d\tau' \int_0^{t'} dt'' e^{-\gamma(t'+\tau'-2t'')}$$

$$= \frac{v_0^2}{\gamma^2}(1-e^{-\gamma t})^2 + \frac{\gamma k_B T}{m}\int_0^t dt' \int_0^t d\tau' \frac{1}{2\gamma}\left[e^{-\gamma(\tau'-t')} - e^{-\gamma(t'+\tau')}\right]$$

$$= \text{---}\ \text{''}\ \text{---} + \frac{k_B T}{2m}\int_0^t d\tau' \left[\frac{1}{\gamma}(e^{-\gamma(\tau'-t)} - e^{-\gamma\tau'}) - \left(-\frac{1}{\gamma}\right)(e^{-\gamma(t+\tau')} - e^{-\gamma\tau'})\right]$$

$$= \text{---}\ \text{''}\ \text{---} + \frac{k_B T}{2m\gamma}\int_0^t d\tau' \left(e^{-\gamma(\tau'-t)} - 2e^{-\gamma\tau'} + e^{-\gamma(t+\tau')}\right)$$

$$= \text{---}\ \text{''}\ \text{---} + \frac{k_B T}{2m\gamma^2}\left(-(e^0 - e^{\gamma t}) + 2(e^{-\gamma t} - 1) - (e^{-2\gamma t} - e^{-\gamma t})\right)$$

$$= \text{---}\ \text{''}\ \text{---} + \frac{k_B T}{2m\gamma^2}\left(-1 + e^{\gamma t} + 2e^{-\gamma t} - 2 - e^{-2\gamma t} + e^{-\gamma t}\right)$$

$$= \text{---}\ \text{''}\ \text{---} + \frac{k_B T}{2m\gamma^2}\left(3e^{-\gamma t} - 3 + e^{\gamma t} - e^{-2\gamma t}\right)$$

Integrasjonsgrenser feil ved/etter $t'' = \tau''$?

③ Fra ②: $\frac{m}{2}\frac{d^2\langle x^2\rangle}{dt^2} - \overbrace{m\langle v^2\rangle}^{k_BT} = -\frac{m\gamma}{2}\frac{d\langle x^2\rangle}{dt}$

$$\langle x^2\rangle = C_1 e^{-\gamma t} + C_2 t^2 + C_3 t + C_4$$

$$\frac{m}{2}\left(\gamma^2 C_1 e^{-\gamma t} + \cancel{2C_2}\right) + \frac{m\gamma}{2}\cdot\left(-\gamma C_1 e^{-\gamma t} + 2C_2 t + C_3\right) = k_BT$$

$$C_2 = 0 \quad , \quad C_3 = \frac{2k_BT}{m\gamma} = 2D$$

$$\langle x^2(0)\rangle = C_1 + C_4 = 0 \Rightarrow C_4 = -C_1$$

$$\langle x^2\rangle = C_1(e^{-\gamma t} - 1) + 2Dt$$

$$\langle x^2\rangle = \frac{2D}{\gamma}(e^{-\gamma t} - 1) + 2Dt$$

$t \gg \frac{1}{\gamma}$: $\langle x^2\rangle = \frac{2D}{\gamma}((\approx 0) - 1) + 2Dt = 2Dt - \frac{2D}{\gamma}$

$t \ll \frac{1}{\gamma}$: $\langle x^2\rangle = \frac{2D}{\gamma}\left(\cancel{1} - \cancel{\gamma t} + \frac{(\gamma t)^2}{2} - \cancel{1}\right) + \cancel{2Dt}$

$$= \frac{2D\gamma t^2}{2} = D\gamma t^2 = \frac{k_BT}{\cancel{\gamma} m}\cancel{\gamma} t^2 = \frac{m\langle v^2\rangle}{m} t^2 = \underline{\underline{\langle v^2\rangle t^2}}$$

④ Fra ③: $\langle x^2\rangle \approx 2Dt$ for store t.

$$t \to \Delta t \Rightarrow \langle x^2\rangle = 2D\Delta t$$

$$\underline{\underline{\sigma_{\Delta t} = \sqrt{\langle x^2\rangle} = \sqrt{\lambda}\cdot\sigma_{\delta t}}}$$

⑤ $m\ddot{x} = -\partial_x V(x) - m\gamma\dot{x} + R(t)$ $\quad F = \frac{R}{m}$

$\langle F(\tau)F(\tau')\rangle = C\delta(t$

$\langle F(\tau)\rangle = 0$

Med $V(x) = V_0$: $m\ddot{x} = -m\gamma\dot{x} + R(t)$

Løsning: $v(t) = v_0 e^{-\gamma t} + \int^t d\tau\, e^{-\gamma(t-\tau)} F(\tau)$

$$v(t)^2 = v_0^2 e^{-2\gamma t} + 2v_0\int_0^t d\tau\, e^{-\gamma(2t-\tau)}F(\tau) + \int_0^t d\tau'\int_0^t d\tau\, e^{-\gamma(2t-\tau-\tau')}F(\tau)F(\tau')$$

Ensemble-middel: $\langle v(t)^2\rangle = v_0^2 e^{-2\gamma t} + \underbrace{2v_0\int_0^t d\tau\, e^{-\gamma(2t-\tau)}\langle F(\tau)\rangle}_{0} + \int_0^t d\tau\cdot C\,\underbrace{e^{-\gamma(2t-2\tau)}}_{e^{-2\gamma t}\cdot e^{2\gamma\tau}}$

$$= v_0^2 e^{-2\gamma t} + \frac{C}{2\gamma}e^{-2\gamma t}(e^{2\gamma t} - 1) = v_0^2 e^{-2\gamma t} + \frac{C}{2\gamma}(1 - e^{-2\gamma t})$$

Vi vet at $\frac{1}{2}m\langle v^2\rangle \to \frac{3}{2}k_BT$ når $t \to \infty$: $\frac{3k_BT}{m} = \frac{C}{2\gamma}$ $\quad C = \frac{6\gamma k_BT}{}$

We will now focus on the position of the particle:

Going one step back, and recalling that

$$v(t) = \underbrace{v(t=0)}_{v_0} e^{-\gamma t} + \frac{1}{m}\int_0^t dt' e^{-\gamma(t-t')} R(t')$$

one obtains by direct integration that:

$$x(t) = \underbrace{x(t=0)}_{x_0} + v(t=0)\frac{1}{\gamma}(1-e^{-\gamma t}) + \frac{1}{m}\int_0^t dt' \int_0^{t'} dt'' \, e^{-\gamma(t'-t'')} R(t'').$$

From this expression, one can calculate the mean square displacement $\langle (x(t)-x_0)^2\rangle$ of the particle, and the result is ($v_0 = v(t=0)$)

$$\langle (x(t)-x_0)^2\rangle = \frac{v_0^2}{\gamma^2}(1-e^{-\gamma t})^2 + \frac{k_B T}{m\gamma^2}(2\gamma t - 3 + 4e^{-\gamma t} - e^{-2\gamma t})$$

Show this!

For large times ($\gamma t \gg 1$) one gets

$$\langle (x(t)-x_0)^2\rangle \simeq 2\frac{k_B T}{m\gamma} t \quad [= 2Dt]$$

according to Einstein

$$\Rightarrow \quad D = \frac{k_B T}{m\gamma} \qquad \text{Einstein relation}$$

The LE can also be rigorously derived from microscopic models with Hamiltonian.

$$H = \frac{p^2}{2m} + V(x) + H_{bath} + H_{int}$$

We will not follow this route here.

3.2 Differential form of the Chapman-Kolmogorov Equation

From the CK-eq. it can be shown that (see notes handed out in the lectures)

$$
\begin{aligned}
\partial_t\, p(x,t\,|\,x_0,t_0) &\\
= & \left.\begin{array}{l} -\,\partial_x\left[A(x,t)\, p(x,t\,|\,x_0,t_0)\right] \\ +\,\frac{1}{2}\,\partial_x^2\left[B(x,t)\, p(x,t\,|\,x_0,t_0)\right] \end{array}\right\}\ \text{Fokker-Planck} \\
& \left.\begin{array}{l} +\displaystyle\int dx_1 \Big\{ W(x\,|\,x_1;t)\, p(x_1,t\,|\,x_0,t_0) \\ \qquad\qquad -\, W(x_1\,|\,x;t)\, p(x_1,t\,|\,x_0,t_0)\Big\} \end{array}\right\}\ \text{Master Eq.}
\end{aligned}
$$

From this equation two special limits can be taken:

1) The Master eq.: $A = B = 0$

2) The Fokker-Planck eq.: $W = 0$

We will now discuss them in tern.

General Comments

The Fokker-Planck eq.

- applies for continuous Markov proc. i.e. diffusion like processes
- the FP eq. is the differential form of the Chapman-Kolmogorov eq for continuous MPs
- the FP eq. is a partial diff. eq. for $p(x,t)$

The Master Equation

- applies to non-continuous MPs i.e. jump processes
- the ME is the diff. form of th CK eq for non-continuous MPs
- the ME is an ordinary differential eq. for the probabilities $p(x,t)$.

3.3 The Fokker-Planck Equation

When the transition prob. $W=0$, the diff. form of the Chapman-Kolmogorov eq. reads:

$$\partial_t p(x,t|x_0,t_0)$$

$$= -\partial_x [A(x,t) \cdot p(x,t|x_0,t_0)] + \frac{1}{2}\partial_x^2 [B(x,t)\, p(x,t|x_0,t_0)] \quad (*)$$

$$B(x,t) > 0$$

This eq. for the probabilities is known as the Fokker-Planck Equation.

The range of x is continuous and normally supposed to be $(-\infty, +\infty)$

The first right-hand side term is known as the "transport term", the "convection term", or the "drift term".

The second term is known as:
the "diffusion term" or the "fluctuation term".

Notice that the FP-eq. is a partial diff. eq. for the (conditional) probability density.

When both A and B are time-independent the resulting FP eq. is also known as the Smoluchowski eq., the second Kolmogorov eq. or the generalized diffusion eq. if we are in one-dimension.

<u>Note</u> : Some authors distinguish between the FP eq. and master eq. (like us) and reserve the latter for jump processes.

Others (like van Kampen) say that the FP eq. is a special case of the master eq. which in this context is the diff. form of the Chapman-Kolmogorov eq.

Note: One may write the FP eq as

$$\partial_t p(x,t|x_0,t_0) + \partial_x J_x = 0$$

$$J_x = A(x,t)\, p(x,t|x_0,t_0) - \frac{1}{2}\partial_x[B(x,t)\, p(x,t|x_0,t_0)]$$

Here $J(x,t)$ is the probability current.

Historic note:

Planck derived the Fokker-Planck eq. as an approximation to the Master eq.

The elegant mathematical properties of the FP eq. should not obscure the fact that its applications in physical situations requires a physical justification. (which is not always obvious, in particular not in non-linear systems)

The FP eq. is an approximate description for <u>any</u> Markov process $x(t)$ whose individual jumps are <u>small</u>.

A word on terminology:

The FP eq. is, by definition, always a linear PDE in p.

However, the term non-linear FP eq. is used. in the literature.
If A is a linear function of x, and B a constant, the resulting FP eq. is said to be linear. In all other cases, the FP eq. is non-linear

3.4 The Master Equation

The Master eq. reads:

① $$\partial_t p(x,t|x_0,t_0)$$

$$= \int dx_1 \left[W(x|x_1;t)\, p(x_1,t|x_0,t_0) - W(x_1|x;t)\, p(x,t|x_0,t_0) \right]$$

(for continuous variables) or

②

$$\partial_t p_n(t) = \sum_{n'} \Big[\underbrace{W_{nn'} p_{n'}(t)}_{\text{gain}} - \underbrace{W_{n'n} p_n(t)}_{\text{loss}} \Big]$$

for discrete variables ($W_{nn'} \geq 0$ $W_{nn} = 0$).

This latter form shows that the ME is a gain-loss eq. of probabilities of the separate states n.

[$W_{nn'}$ is the transition rate from state n' to state n. (and not opposite)]

Terminology

The name Master Eq. first appeared in a paper in which it actually had the role of a general eq from which all other results were derived
[Ref. Physica 7, 344 (1940)]

This name got stuck onto a special type of eq.

"Loose definition"

The Master Equation is a (set) of <u>ordinary</u> first-order (in time) differential eq. describing the time-develop. of the probability of the system.

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The Master eq. is, in particular, used for discrete systemes, i.e. a system of discrete # states:

Note that for such system the ME can be written in vector notation as:

$$\partial_t \vec{p}(t) = \mathbb{W} \vec{p}(t)$$

where

$$\mathbb{W}_{nn'} = W_{nn'} - \delta_{nn'} \sum_m W_{mn}$$

Formally, the solution of this eq. is

$$\vec{p}(t) = e^{\mathbb{W} t} \vec{p}(0)$$

but this form is not so helpful to explicitly finding $\vec{p}(t)$. [It is hard to calculate the exponential function of a matrix in general].

Typically exp of matrices is calculated by first calculating the eigenvectors and eigenvalues of the matrix. However, in our case the matices involved are not neccessarily symmetric.

However, typically one has

1] $\mathbb{W}_{nn'} \geqslant 0 \qquad n \neq n'$

2] $\sum_n \mathbb{W}_{nn'} = 0 \qquad \forall n'$ (column sum is zero)

2] => $\mathbb{W}$ has a ~~constant~~ left egenvector where all elements are equal corresponding to the eigenvalue zero.

Thus also a right eigenvector exist with the zero eigenvalue $\mathbb{W} \phi = 0$

This solution, when propperly normalized, represents a stationary solution of the Master Equation, i.e. the solution for which $\partial_t \vec{p} = 0$.

In the long time limit, $t \to \infty$, the ME will bring the system to this stationary (equilibrium) solution

$$\lim_{t \to \infty} \vec{p}(t) = \vec{p}_\infty \qquad \partial_t \vec{p}_\infty = 0$$

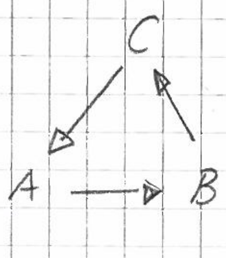
In this case :

$$\sum_{n'} W_{nn'} \, p_{n'}^{\infty} = \sum_{n'} W_{n'n} \, p_{n}^{\infty}$$

This relation is known as detailed balance.

Note : $\vec{p}_\infty$ is known from (equilibrium) statistical mechanics, and the detailed balance is a relation ~~between~~ among the transition probabilities.

Mot ex

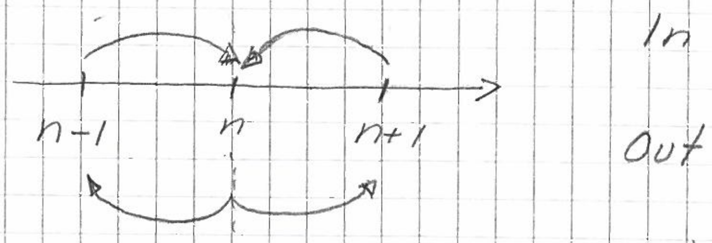


Likevekt kan oppnås via "loop-transport"

I ett-trinns prosesser er alltid DB oppfylt da loops ikke er mulig.

Ex : The Master Eq. for a Random Walk

$$\partial_t P_n(t) = \frac{1}{2} P_{n-1}(t) + \frac{1}{2} P_{n+1}(t) - P_n(t)$$



$$P_0(t=0) = 1$$

— 〃 —

3.4.1 The Fokker-Planck Eq. as a Continuous Limit of the Master Eq.

Let us consider the ordinary random walk, as given by the Master Eq. of the above ex.

$$\partial_t P_n(t) = \frac{1}{2} P_{n-1}(t) + \frac{1}{2} P_{n+1}(t) - P_n(t)$$

We will use the notation: $P_n(t) \equiv P(n,t)$

Hence

$$\partial_t P(n,t) = -\frac{1}{2}\left[P(n,t) - P(n-1,t)\right] - \frac{1}{2}\left[P(n,t) - P(n+1,t)\right]$$

* Planck derived the FP eq. as an approximation to the Master eq. (see Van kampen p. 197)

If we now can treat n as a continuous variable one gets after a Taylor expansion:

$$\partial_t p(n,t) = -\frac{1}{2}\Big[p(n,t) - p(n,t) + \partial_n p(n,t) - \frac{1}{2}\partial_n^2 p(n,t) + \dots\Big]$$
$$- \frac{1}{2}\Big[p(n,t) - p(n,t) - \partial_n p(n,t) - \frac{1}{2}\partial_n^2 p(n,t) + \dots\Big]$$

$$\simeq \frac{1}{2}\partial_n^2 p(n,t) + \frac{1}{4!}\partial_n^4 p(n,t) + \dots$$

Hence, to lowest order we have obtained that the Fokker-Planck equation for a (symmetric) random walk is the diffusion eq.

$$\partial_t p(x,t|x_0,t_0) = D\,\partial_x^2 p(x,t|x_0,t_0), \quad D = 1/2$$

$$p(x,t_0|x_0,t_0) = \delta(x-x_0)$$

This means:

NB The Fokker-Planck equation for the Wiener process is the diffusion eq.

A Wiener-process can therefore be defined as a Markov process where the conditional probability (transition prob.) satisfies the diffusion eq (with $D = 1/2$).

Note : The above derivation is by no means rigorous. For instance, that we can ~~stop the~~ terminate the expansion after the lowest order term has to be justified.

Technical note :

$$p(n \pm 1, t) = e^{\pm \partial_n} p(n,t)$$

Note : If we have considered an asymmetric random walk, the corresponding FP eq. would also have contained a drift-term.

Technical note :

The expansion we just did can more generally be written like (our calc. is a spesial case)

$$\partial_t p(x,t) = \sum_n \frac{1}{n!} (-\partial_x)^n \{ \alpha_n(x) p(x,t) \} .$$

This is known as the Kramers-Moyal expansion

3.5 Important Markov Processes Revised

We will look at the two important Markov processes:

1) Wiener process
2) the Ornstein-Uhlenbeck process

3.5.1 The Wiener process

The FP-eq.

$$\partial_t p(x,t|x_0,t_0) = D\partial_x^2 p(x,t|x_0,t_0)$$

$$p(x,t_0|x_0,t_0) = \delta(x-x_0)$$

Solution (one-dimension)

$$p(x,t|x_0,t_0) = \frac{1}{[4\pi D(t-t_0)]^{1/2}} \exp\left\{-\frac{(x-x_0)^2}{4D(t-t_0)}\right\}$$

3.5.2 The Orenstein-Uhlenbeck process

The corresponding Fokker-Planck eq. reads

OBS

$$\partial_t p(x,t|x_0,t_0) = \gamma \partial_x [x\, p(x,t|x_0,t_0)] + D \partial_x^2 p(x,t|x_0,t_0)$$

$$p(x,t_0|x_0,t_0) = \delta(x-x_0)$$

$$\gamma > 0$$

Solution (one-dimension):

$$p(x,t|x_0,t_0) = \sqrt{\frac{\gamma}{2\pi D(1-\Gamma^2)}} \exp\left\{-\frac{\gamma(x-\Gamma x_0)^2}{2D(1-\Gamma^2)}\right\}$$

$$\Gamma = e^{-\gamma(t-t_0)}$$

Note: In the limit $\gamma \to 0$ the OU-process reduces to the Wiener process.

— " —

General comments:

The FP eq. given above ~~can be used~~ for the transition prob. can be used as definitions of the Wiener and OU Markov processes.

Note: The stationary Markov process determined by the linear FP eq. is the Ornstein-Uhlenbeck process.

3.6 Summary

Langevin eq.
- stochastic diff. Eq.

Master Eq.
- Differentio-integral eq for p
- Deterministic

Fokker-Planck Eq
- Partial diff. eq. (PDE) for $p(x,t|x_0,t_0)$
- Deterministic