4. ANOMALOUS DIFFUSION

4.1 Introduction

We have in previous chapters seen that the hallmark of diffusion processes, $x(t)$, is the <u>linear</u> time dependence of the mean square displacement

$$\langle x^2(t) \rangle \sim Dt \qquad (*)$$

This eq. signals that the underlying transport mechanism is described by Fick's second law.

The scaling relation $(*)$ is a direct concequence of:

1) The central limit theorem
2) The Markovian nature of $x(t)$

Lifting any of these two properties will make $(*)$ invalid. That is, anomalous diffusion will result.

Def: Anomalous diffusion

A stochastic process, $x(t)$, where the mean square displacement no longer is linear in time

$$\langle x^2(t) \rangle \neq \text{lin}(t)$$

— " —

Several functional forms are popular for $\langle x^2(t) \rangle$ in the anomalous case, in particular the power-law

$$\langle x^2(t) \rangle \sim D_\alpha t^\alpha \qquad \alpha \neq 1 \quad (**)$$

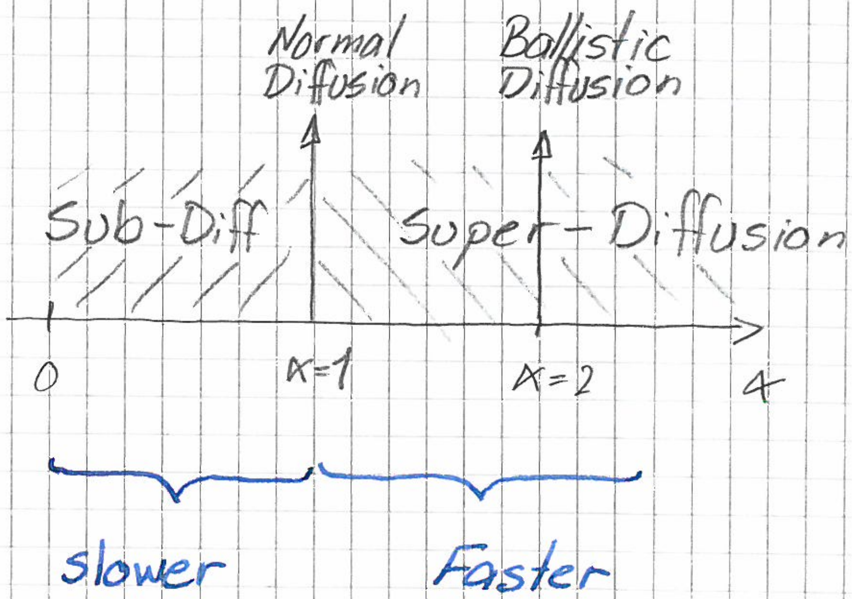
(or the logarithmic).

We will follow common practice and focus on the power-law form.

Note that D_α, the anomalous diff. const., now has dimension

$$[D_\alpha] = \frac{m^2}{s^\alpha}$$

The exponent α is known as the anomalous diffusion exponent.



Note:

Anomalous diffusion

$\Rightarrow$ Non-Fickian transport
"anomalous transport"

The anomalous diffusion behaviour manifested by

$$\langle x^2(t) \rangle \sim D_\alpha t^\alpha$$

is intimately connected with the breakdown of the cental limit theorem caused by either

1) broad distributions ($\sigma \neq \infty$)
2) long-range correlations

Anomalous diffusion rests on the validity of the so-called Lévy-Gnedenko generalized central limit theorem for such situations where not all moments of the underlying elementary transport events exist.

The broad spatial jump or waiting time distributions lead to non-Gaussian propagators (fundamental solutions) and a possibly non-Markovian time evolution (manifestation of non-local temporal phenomena).

4.2 Levy Flights

We will now give a first example of anomalous diffusion — the so-called Lévy Flights.

LFs are random walks where the increments are drawn from a "fat-tailed" distribution, i.e. a distribution of diverging moments (2nd or 1st.).

$$x(t) = \sum_{i=1}^{t} \xi_i \quad \leftarrow \text{IID}$$

One familiar example of such a distribution is the Cauchy-Lorentz distr., but there are many others as we will see.

Show some plots.

4.2.1 The Lévy Distribution

The mathematicians call it the α-stable distribution.

Consider

$$x = \xi_1 + \xi_2$$

where ξ_i are IID random variables.
If $\xi_i = N(0,1)$ we know that x also will be Gaussian (Normally) distributed.

One says that the Normal dist. is stable under addition.

French math. P. Lévy asked ~~the question of~~ what is the largest class of distributions having this property.

This class is today known as the Lévy distribution (physics) or α-stable dist. (math.).

We will also see that this dist. is intimately related with the generalized central limit theorem.

The Levy Distribution is defined by a characteristic function

NB
$$\hat{L}_\alpha(k;a) = \exp(-a|k|^\alpha)$$

i.e. a stretched exponential.

Hence, we recall, that the distribution is given by the Fourier transform:

$$L_\alpha(x;a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\; e^{ikx - a|k|^\alpha}$$

(because of symmetry)

$$= \frac{1}{\pi}\int_0^{\infty} dk\; \cos(kx)\; e^{-ak^\alpha}$$

In general this integral can not be evaluated analytically. However, two noticeable exceptions do exist:

$\alpha = 2$: The Gaussian distr.

$\alpha = 1$: Cauchy-Lorentz ; $\frac{1}{\pi} \frac{a}{x^2+a^2}$

Technical comment:

Above we have only given the symmetric case. Strictly speaking the full class of Lévy dist. do also include non-symmetric distributions. One writes

$$L_{\alpha\beta}(x;a)$$

Where β is related to the asymmetry (actually the skewness). The symmetric case corresponds to $\beta = 0$.

Some plots; e.g. Wikipedia.

* Large x expansion of $L_\alpha(x;a)$

$$L_\alpha(x;a) = \frac{1}{\pi}\int_0^\infty dk\, \underbrace{\cos kx}_{u'}\, \underbrace{e^{-ak^\alpha}}_{v}$$

(Integration by part) $\quad \frac{d}{dk} e^{-ak^\alpha} = -e^{-ak^\alpha} a\alpha k^{\alpha-1}$

$$= \frac{a\alpha}{\pi}\int_0^\infty dk\, \frac{\sin k|x|}{|x|}\, e^{-ak^\alpha} k^{\alpha-1}$$

$$\xi = k|x| \qquad d\xi = dk\,|x|$$

$$= \frac{a\alpha}{\pi |x|^{1+\alpha}} \int_0^\infty d\xi\, \xi^{\alpha-1} \sin\xi\, \exp\left\{-\frac{a\xi^\alpha}{|x|^\alpha}\right\}$$

$$\underset{x\to\infty}{\sim} \frac{a\alpha}{\pi |x|^{1+\alpha}} \int_0^\infty d\xi\, \xi^{\alpha-1}\sin\xi$$

$$\sim \frac{a\alpha\, \Gamma(\alpha) \sin\left(\frac{\pi\alpha}{2}\right)}{\pi\, |x|^{1+\alpha}}$$

∴ A inverse power-law tail

$$L_\alpha(x;a) \underset{x\to\pm\infty}{\sim} \frac{A_\pm}{|x|^{\alpha+1}}$$

— // —

$$\Gamma(x) = \int_0^\infty dt\, t^{x-1} e^{-t} \quad \text{(generalized } n!\text{)}$$

$$\Gamma(n+1) = n!$$

* Small x expansion of $L_\alpha(x;a)$

$$L_\alpha(x;a) = \frac{1}{\pi}\int_0^\infty dk \; \cos kx \; e^{-ak^\alpha}$$

Taylor expansion

$$= \frac{1}{\pi}\int_0^\infty dk \sum_{n=0}^\infty (-1)^n \frac{(kx)^{2n}}{(2n)!} \; e^{-ak^\alpha}$$

$$w = ak^\alpha \Rightarrow dw = \alpha a k^{\alpha-1} dk$$
$$= \alpha a \left(\frac{w}{a}\right)^{\frac{\alpha-1}{\alpha}} dk$$

$$= \frac{1}{\pi}\sum_{n=0}^\infty (-1)^n \int_0^\infty \frac{dw}{\alpha a \left(\frac{w}{a}\right)^{\frac{\alpha-1}{\alpha}}} \frac{x^{2n}}{(2n)!} \left(\frac{w}{a}\right)^{\frac{2n}{\alpha}} e^{-w}$$

$$= \frac{1}{\pi}\sum_{n=0}^\infty (-1)^n \frac{x^{2n}}{(2n)!} \int_0^\infty dw \, \frac{1}{\alpha a} \underbrace{\left(\frac{w}{a}\right)^{\frac{2n}{\alpha}-\frac{\alpha-1}{\alpha}} e^{-w}}_{\frac{e^{-w}}{\left(\frac{w}{a}\right)^{\frac{\alpha-1-2n}{\alpha}}}}$$

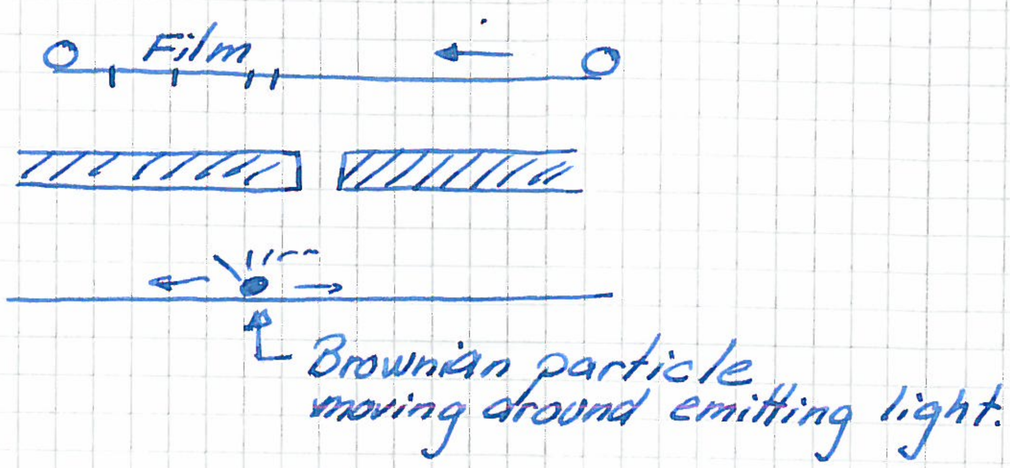
$x \to 0$ is the interesting limit

$$= \frac{1}{\pi}\sum_{n=0}^\infty \frac{(-1)^n}{(2n)!} \frac{x^{2n}}{\alpha a^{\frac{2n+1}{\alpha}}} \Gamma\left(\frac{2n+1}{\alpha}\right)$$

$$\sim \frac{1}{\pi} \frac{\Gamma\left(\frac{1}{\alpha}\right)}{\alpha \, a^{1/\alpha}}$$

Increases as $\alpha \to 0$

Ex : Roaming Brownian Particle



Each time the roaming particle passes the slit a mark is put on the film.

The distribution of time intervals between consecutive marks is given by the one-sided distribution :

$$L_{1/2;-1/2}(x) = \begin{cases} \dfrac{1}{2\sqrt{\pi}}\, x^{-3/2}\, e^{-\frac{1}{4x}} & x > 0 \\ 0 & x < 0 \end{cases}$$

Application : Single molecule spectroscopy.

Ex: The Holtsmark distribution

$L_{\alpha\beta}(x)$ with $\alpha = 3/2$ and $\beta = 0$

This symmetric dist. is of some use in cosmology.

Holtsmark was Prof. of physics at NTH (and UiO).

4.2.2 Lévy-Gnedenko Generalized Central Limit Theorem

Theorem

Define

$$X_N = \sum_{i=1}^{N} \xi_i$$

where $\{\xi_i\}$ are IID random variables.
If the distribution of X_N, with an appropriate normalization, converges to some distribution $P(X_N)$ in the limit $N \to \infty$, $P(X_N)$ is stable.
In particular, if its variance is finite, $P(X_N)$ is Gaussian and obeys the Central Limit Theorem.

— // —

Q : Given a distribution for $\{\xi_i\}$, which one of the stable distributions $L_{\alpha\beta}(x;a)$ does ~~the~~ the distribution of X_N satisfy?

A : If the distribution $p(\xi)$ satisfies

$$p(\xi) \sim \frac{A_\pm}{|\xi|^{\alpha+1}} \qquad \xi \to \pm\infty$$

$$\beta = \frac{A_+ - A_-}{A_+ + A_-}$$

Then $p(X_N)$ when $N \to \infty$ be $L_{\alpha\beta}(x;a)$

The parameter a_N is like the σ_N in the Gaussian case; a measure of the spread.

Note: If the variance of ξ, $\sigma_\xi^2 < \infty$, then the limiting dist. is the Gaussian.

4.2.3 Scaling and Superdiffusion

We once more consider the random walk where

$$X_N = \sum_{n=1}^{N} \xi_n$$

where $\{\xi_n\}$ are IID random variables.

If the second moment of the distribution of ξ, $p(\xi)$, is finite ($\sigma_\xi < \infty$) then we know that $p(X_N)$ is Gaussian (for sufficiently large N) and that the <u>width</u>

$$\sigma_N = \langle X_N^2 \rangle^{1/2} \sim N^{1/2}$$

How is the corresponding scaling for Lévy Flights?

Hence, consider $p(\xi) = L_{\alpha,0}(\xi, a)$.

Previously, we showed (Chapter 1) that the

$$P_N(x_N) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \; \underbrace{\hat{P}_N(k)}_{\text{characteristic func.}} \; e^{ikx_N}$$

where

$$\hat{P}_N(k) = \left[\exp(-a|k|^\alpha)\right]^N$$
$$= e^{-aN|k|^\alpha}$$

Therefore, one gets

$$P_N(x_N) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \; e^{-aN|k|^\alpha} e^{ikx_N}$$

$\cancel{w^\alpha = Nk^\alpha}$

$\Rightarrow k = N^{-1/\alpha} w \Rightarrow dk = +N^{-1/\alpha} dw$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{dw}{N^{1/\alpha}} \; e^{-a|w|^\alpha} \; e^{iw\left(\frac{x_N}{N^{1/\alpha}}\right)}$$

$$= \frac{1}{N^{1/\alpha}} \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} dw \; e^{-a|w|^\alpha} e^{iw\left(\frac{x_N}{N^{1/\alpha}}\right)}}_{L_\alpha\left(\frac{x_N}{N^{1/\alpha}};\, a\right)}$$

$$P_N(x_N) = \frac{1}{N^{1/\alpha}} L_\alpha\left(\frac{x_N}{N^{1/\alpha}}; a\right)$$

Thus, the width scales as $N^{1/\alpha}$.

4.3 Continuous Time Random Walks (CTRW)

This model was introduced by Weiss and Montroll (1965).

The main difference from the classic Random Walk model is that the time-interv. between jumps now also is a stochastic variable

The CTRW model introduces the jump pdf $\psi(x,t)$ (joint prob. dist.) for jumps of size x and waiting time t between consecutive jumps.

From the joint probability $\psi(x,t)$, the marginal distributions for jump size and waiting time may be defined.

They read

$$\lambda(x) = \int_0^\infty dt \; \psi(x,t)$$

for the jump size (or lenght) pdf, and

$$w(t) = \int_{-\infty}^{\infty} dx \; \psi(x,t)$$

for the waiting time pdf.

If the jump size and waiting time are independent

$$\psi(x,t) = \lambda(x)\,\omega(t)$$

Otherwize they are coupled

$$\psi(x,t) = p(x|t)\,\omega(t) = \underbrace{p(t|x)\,\lambda(x)}$$

jumps of a certain size involves a time cost.

Different CTRW processes can be categorized by

$$T = \langle t \rangle = \int_0^\infty dt\; t\,\omega(t)$$

the characteristic waiting time, and

$$\Sigma^2 = \langle x^2 \rangle = \int_{-\infty}^{\infty} dx\; x^2 \lambda(x)$$

the characteristic jump size variance.

We will see what matters is if these variabeles, T and Σ^2, are finite or not.

Ultimately, we want to know the fundamental solution — the propagator $W(x,t)$ which satisfies the init. cond. $W(x,t_0) \equiv W_0(x)$.

Note when $W_0(x) = \delta(x)$, one has that $W(x,t) \equiv P(x,t|x_0,t_0)$.

— " —

It can be shown that in ~~the~~ Fourier-Laplace space the propagator can be expresse in terms of $\psi(x,t)$ like (Phys. Rev. A 35, 3081 (1987))

$$\hat{\tilde{W}}(k,u) = \frac{1-\hat{\omega}(u)}{u} \frac{\tilde{W}_0(k)}{1-\hat{\tilde{\psi}}(k,u)} \qquad (*)$$

In the following we will consider a decoupled joint prob.

$$\hat{\tilde{\psi}}(k,u) = \tilde{\lambda}(k)\,\hat{\omega}(u)$$

4.3.1 Brownian Motion : $T < \infty$, $\Sigma^2 < \infty$

We will now consider the case where both $T = \langle t \rangle < \infty$ and $\Sigma^2 = \langle x^2 \rangle < \infty$ are finite. This means that both $\omega(t)$ and $\lambda(x)$ will have to decay sufficiently fast, but otherwise be arbitrary.

This decay implies the following behaviour in Laplace and Fourier space

$$\left.\begin{aligned} \hat{\omega}(u) &\sim 1 - \tau u + O(u^2) \\ &\text{and} \\ \tilde{\lambda}(k) &\sim 1 - \sigma^2 k^2 + O(k^4) \end{aligned}\right\} \text{Universal behavior when } T \text{ and } \Sigma^2 \text{ are finite}$$

when $(k, u) \to (0,0)$ [corresponding to $(|x|, t) \to (\infty, \infty)$].

Here we have that

$$T \propto \tau$$

and

$$\Sigma^2 \propto \sigma^2$$

The prefactors depend on the dist.

Installing these results into (*) gives

$$\hat{\tilde{W}}(k,u) = \frac{\tau u + \dots}{u} \frac{\tilde{W}_0(k)}{1-(1-\sigma^2 k^2 + \dots)(1-\tau u + \dots)}$$

$$= \tau \frac{\tilde{W}_0(k)}{1-(1-\tau u - \sigma^2 k^2 + \dots)}$$

$$= \frac{\tilde{W}_0(k)}{u + \frac{\sigma^2}{\tau} k^2} + \dots$$

$$\equiv \frac{\tilde{W}_0(k)}{u + K_1 k^2} \qquad K_1 = \frac{\sigma^2}{\tau}$$

We will now go back to (x,t)-coordinates

$$(u + K_1 k^2)\, \hat{\tilde{W}}(k,u) = \tilde{W}_0(k)$$

$$\underbrace{u\, \hat{\tilde{W}}(k,u) - \tilde{W}_0(k)}_{\mathcal{L}[\partial_t \tilde{W}(k,t)]} + K_1 \underbrace{k^2 \hat{\tilde{W}}(k,u)}_{\mathcal{F}[-\partial_x^2 \hat{W}(x,u)]} = 0$$

Hence, the classic DE

$$\partial_t W(x,t) = K_1 \partial_x^2 W(x,t)$$

results after applying $\mathcal{L}^{-1} \mathcal{F}^{-1}$.

Notice that we did not specify $\Psi(x,t)$ in order to obtain this result, only did we require that $\langle t \rangle$ and $\langle x^2 \rangle$ were finite.

Hence, the results are generally valid, and any choice of $w(t)$ and $\lambda(x)$ will result in Brownian Motion (i.e. classic Diffusion) if the first and second moments, respect., are finite.

Note: The Diffusion Constant K_1 defined as

$$K_1 = \frac{\sigma^2}{\tau}$$

is coming from the ratio of the prefactors of the $(k,u) \to (0,0)$ expansion prop. to u and k^2

This result is rather general we will see.

4.3.2 Sub-Diffusion; "Long Rests" $T=\infty, \Sigma^2<\infty$

We will now consider the situation where the waiting time distribution $w(t)$ is so "fat-tailed" that $\langle t\rangle = \infty$, and at the same time the jump size dist. gives $\langle x^2\rangle < \infty$.

In this case we have $[w(t) \sim A_\alpha (\tau/t)^{\alpha+1}]$

$$\hat{w}(u) \sim 1 - (\tau u)^\alpha + \dots$$

and as before

$$\tilde{\lambda}(k) \sim 1 - \sigma^2 k^2 + \dots$$

Then from $(*)$ it follows that:

$$\hat{\tilde{W}}(k,u) = \frac{\tilde{W}_0(k)}{u + K_\alpha u^{1-\alpha} k^2} ; \quad K_\alpha = \frac{\sigma^2}{\tau^\alpha}$$

[in the limit $(k,u) \to (0,0)$]

Show this!

As in the BM case, we would like to make the inversion to (x,t)-space.

However, this is not so easy due to the $u^{1-\alpha}$-term.

To this end, we will use that

$$\mathcal{L}\left[{}_0D_t^{-p} W(x,t)\right] = u^{-p}\,\hat{W}(x,u) \qquad p \geq 0$$

where ${}_0D_t^{-p}$ is related to the Riemann-Liouville operator

$${}_0D_t^{1-p} = \frac{\partial}{\partial t}\,{}_0D_t^{-p}\ , \quad 0 < p < 1$$

that can be defined as

$${}_0D_t^{1-\alpha} W(x,t) = \frac{1}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t dt'\,\frac{W(x,t')}{(t-t')^{1-\alpha}}.$$

It has the property

$$ {}_0D_t^q \, t^p = \frac{\Gamma(1+p)}{\Gamma(1+p-q)} t^{p-q} $$

valid for any real q.

Notice in particular the special case $p = 0$:

$$ {}_0D_t^q \, 1 = \frac{1}{\Gamma(1-q)} t^{-q} $$

Compare this to $\partial^n 1$ where n is an integer. (the special case $q = n$ is included in the above expressions for ${}_0D_t^q 1$ via the poles of the Γ-functions. ($q = 1, 2, \ldots$).

— // —

So back to the expression for the propagator (in the Fourier-Laplace domain)

$$\hat{\tilde{W}}(k,u) = \frac{\tilde{W}_0(k)}{u + K_\alpha u^{1-\alpha} k^2} \quad ; \quad K_\alpha = \frac{\sigma^2}{\tau^\alpha}$$

Proceeding like before:

$$(u + K_\alpha u^{1-\alpha} k^2)\, \hat{\tilde{W}}(k,u) = \tilde{W}_0(k)$$

$$\underbrace{u\,\hat{\tilde{W}}(k,u) - \tilde{W}_0(k)}_{\mathcal{L}[\partial_t \tilde{W}(k,t)]} + K_\alpha \underbrace{u^{1-\alpha} k^2\, \hat{\tilde{W}}(k,u)}_{k^2 \mathcal{L}[{}_0D_t^{1-\alpha} \tilde{W}(k,t)]} = 0$$

Inv. Laplace transform:

$$\partial_t \tilde{W}(k,t) + K_\alpha k^2\, {}_0D_t^{1-\alpha} \tilde{W}(k,t) = 0$$

Now performing the inverse Fourier transf. gives the final result:

$$\partial_t W(x,t) = {}_0D_t^{1-\alpha} K_\alpha \partial_x^2 W(x,t)$$

This is the <u>Fractional Diffusion Eq</u> for sub-Diffusion

Note that the long-range kernel $M(t) \propto 1/t^{1-\alpha}$ of the Riemann-Liouville fractional operator is responcable for the non-Markovian nature of sub-diffusion ("fat-tailed" waiting time dist.)

How to calculate the mean square displacement $\langle x^2(t) \rangle$?

1] Solving the FDE and calculating the 2nd moment. Difficult. (Fox functions)

2] Infering it from the $(k,u) \to (0,0)$ limit of the propagator $\hat{\tilde{W}}(k,u)$ (in Fourier-Laplace space)

This can be achieved from

$$\langle x^2(u) \rangle = \lim_{k \to 0} (-\partial_k^2)\ \hat{\tilde{W}}(k,u)$$

and then making the inverse Laplace transform.

The result is :

$$\langle x^2(t) \rangle = \frac{2K_\alpha}{\Gamma(1+\alpha)}\, t^\alpha$$

3) The last option (that we will present) is to integrate FDE over

$$\int_{-\infty}^{\infty} dx \; x^2$$

resulting in

$$\partial_t \langle x^2(t) \rangle = {}_0D_t^{1-\alpha} \, 2K_\alpha = \frac{2K_\alpha \, t^{\alpha-1}}{\Gamma(\alpha)}$$

The solution of this eq. is as given before.

Note: In the limit $\alpha \to 1$ the FDE reduces to Ficks second law (ordinary DE), as it should.

4.3.3 Super-Diffusion; "Long Jumps" Lévy Flights; $T < \infty$ $\Sigma^2 = \infty$

In this case one has:

$$\hat{w}(u) \sim 1 - u\tau + O(u^2)$$

and (Lévy jump size)

$$\tilde{\lambda}(k) \sim 1 - \sigma^\mu |k|^\mu, \qquad 0 < \mu < 2$$

From (*) it follows that

$$\hat{\tilde{W}}(k,u) = \frac{\tilde{W}_0(k)}{u + K^\mu |k|^\mu}$$

Inverting this propagator to (x,t)-space gives:

$$\partial_t W(x,t) = K^\mu \, {}_{-\infty}D_x^\mu \, W(x,t)$$

Where ${}_{-\infty}D_x^\mu$ is another fractional derivative, known as the Weyl operator. It is defined via the property

$$\mathcal{F}\left[{}_{-\infty}D_x^\mu f(x) \right] = -|k|^\mu \tilde{f}(k).$$

One may also check that in the limit $\mu \to 2$ the classic DE results.

Moreover, from the general solution of the above FDE, — the Fox functions — one obtains that

$$W(x,t) \sim \frac{K^\mu t}{|x|^{1+\mu}}, \qquad \mu < 2$$

(typical for Lévy distributions).

Due to this property, it follows that

$$\langle x^2(t) \rangle \to \infty$$

"Instead" one has a finite moment of the form

$$\langle |x|^\delta \rangle \propto t^{\delta/\mu}, \qquad 0 < \delta < \mu < 2.$$

Notice: The above process is called Lévy Flights.

Due to the finite value of T the process is Markovian.

Notice since the jumps can be quite large, and the waiting time is finite, that the speed can be arbitrarily large. (and ~~bic~~ higher than speed of light)

This has caused the introduction of a related process — the <u>Lévy walk</u> — where the speed is constant. (or at least smaller than c). This implies that long jumps have a "time-cost."

This cost can be introduced via a spatiotemoral coupling, usually throug the δ-coupling

$$\psi(x,t) = \frac{1}{2} w(t)\, \delta(|x| - vt)$$

where v is the propagation speed of the walker.

4.3.4. Competition between "Long Rest" and "Long Jumps"; $T=\infty$ $\Sigma^2=\infty$

In this case both the waiting time PDF and the jump size PDF have fat-tails.

In this case the process can be both sub- and super-diffusive.

The relevant Fractional DE is

$$\partial_t w(x,t) = {}_0D_t^{1-\alpha} K_\alpha^\mu \, {}_{-\infty}D_x^\mu W(x,t)$$

$$K_\alpha^\mu = \frac{\sigma^\mu}{\tau^\alpha}$$

$$\hat{w}(u) \sim 1-(u\tau)^\alpha$$

$$\tilde{\lambda}(k) \sim 1-\sigma^\mu |k|^\mu$$

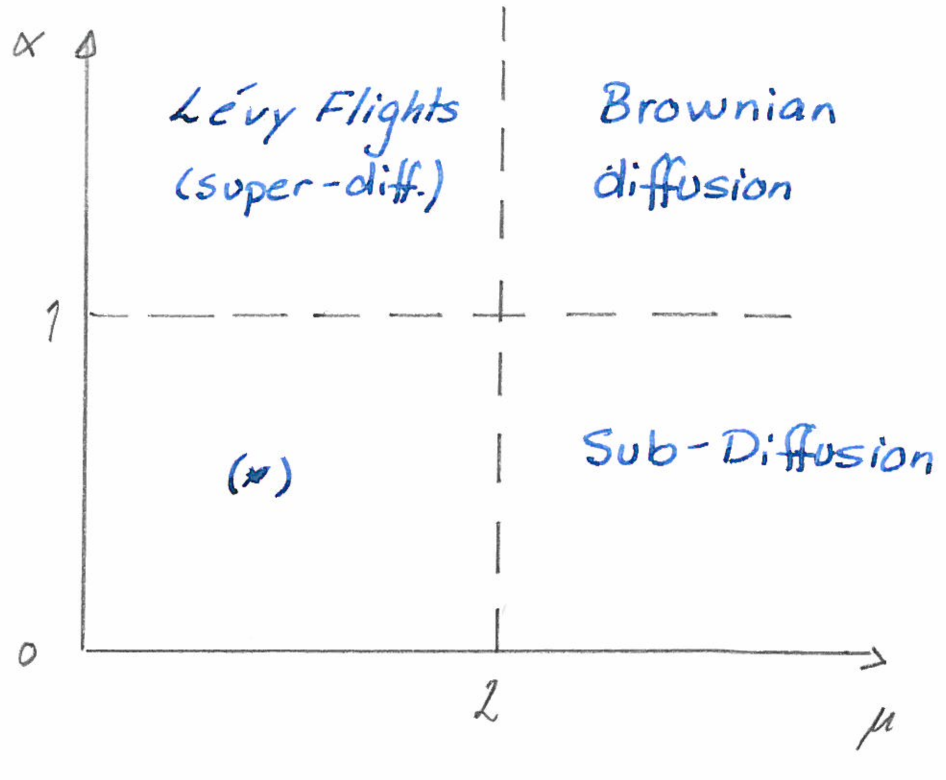
4.3.5 Summary Anomalous Diffusion

Assume, that asymptotically one has

$$\hat{w}(u) \sim 1 - (u\tau)^{\alpha}$$

$$\tilde{\lambda}(k) \sim 1 - \sigma^{\mu} |k|^{\mu}$$

Then one has



(*) A non-Markovian process where the single modes decay like a stretched exponential (Mittag-Leffler function)