The scattering of light from two-dimensionally, randomly rough surfaces

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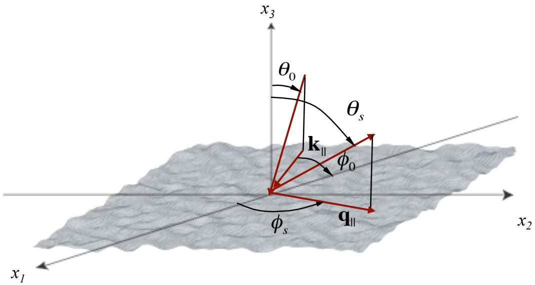
Introduction



- Vacuum : $x_3 > \zeta(\mathbf{x}_\parallel)$
- Scattering medium : $x_3 < \zeta(\mathbf{x}_\parallel)$

The surface profile function is a single-valued function of $\mathbf{x}_\parallel = (x_1, x_2, 0)$ that is at least twice differentiable with respect to x_1 and x_2, and constitutes a stationary, zero-mean Gaussian random process defined by

$$\left\langle \zeta(\mathbf{x}_\parallel)\zeta(\mathbf{x}'_\parallel) \right\rangle = \delta^2 W(\mathbf{x}_\parallel - \mathbf{x}'_\parallel).$$

The angle brackets denote an average over the ensemble of realizations of the surface profile function.

The root-mean-square height of the surface is given by

$$\delta = \left\langle \zeta^2(\mathbf{x}_\parallel) \right\rangle^{\frac{1}{2}} .$$

The power spectrum of the of the surface roughness is defined by

$$g(k_\parallel) = \int \mathrm{d}^2 x_\parallel \, W(\mathbf{x}_\parallel) \exp\left(\mathrm{i}\mathbf{k}_\parallel \cdot \mathbf{x}_\parallel\right) .$$

Outline

A Perfectly Conducting Surface

Stratton-Chu formula for the magnetic Field in vacuum:

$$\theta(x_3 - \zeta(\mathbf{x}_\parallel))\, \mathbf{H}^>(\mathbf{x}|\omega) = \mathbf{H}(\mathbf{x}|\omega)_{\mathrm{inc}} + \frac{1}{4\pi}\int \mathrm{d}^2x'_\parallel \;\; [\nabla g_0(\mathbf{x}|\mathbf{x}')]|_{x'_3=\zeta(\mathbf{x}'_\parallel)} \times \mathbf{J}_H(\mathbf{x}'_\parallel|\omega),$$

where

- $\theta(z)$ is the Heaviside unit step function
- $\mathbf{H}(\mathbf{x}|\omega)_{\mathrm{inc}}$ is the magnetic component of the incident field
- $\mathbf{J}_H(\mathbf{x}_\parallel|\omega) = [\mathbf{n} \times \mathbf{H}^>(\mathbf{x}|\omega)]|_{x_3=\zeta(\mathbf{x}_\parallel)}$
- $\mathbf{n} = (-\zeta_1(\mathbf{x}_\parallel), -\zeta_2(\mathbf{x}_\parallel), 1)$
- $\zeta_i(\mathbf{x}_\parallel) = \frac{\partial \zeta(\mathbf{x}_\parallel)}{\partial x_j}, \qquad j = 1, 2$

and $g_0(\mathbf{x}|\mathbf{x}')$ is the scalar free-space Green's function

$$g_0(\mathbf{x}|\mathbf{x}') = \frac{\exp\left[\mathrm{i}\frac{\omega}{c}\left|\mathbf{x}-\mathbf{x}'\right|\right]}{|\mathbf{x}-\mathbf{x}'|}$$
$$= \int \frac{\mathrm{d}^2 q_\parallel}{(2\pi)^2}\, \frac{2\pi\mathrm{i}}{\alpha_0(q_\parallel)} \exp\left[\mathrm{i}\mathbf{q}_\parallel \cdot (\mathbf{x}_\parallel - \mathbf{x}'_\parallel)\right] \exp\left[\mathrm{i}\alpha_0(q_\parallel)|x_3 - x'_3|\right],$$

with

$$\alpha_0(q_\parallel) = [(\omega/c)^2 - q_\parallel^2]^{\frac{1}{2}}, \qquad \mathrm{Re}\,\alpha_0(q_\parallel) > 0, \mathrm{Im}\,\alpha_0(q_\parallel) > 0.$$

We evaluate the equation at $x_3 = \zeta(\mathbf{x}_\parallel) + \eta$ and at $x_3 = \zeta(\mathbf{x}_\parallel) - \eta$, where η is a positive infinitesimal, add the resulting two equations, and take the cross product of the sum with $\mathbf{n}$, and obtain

$$\mathbf{J}_H(\mathbf{x}_\parallel|\omega) = 2\mathbf{J}_H^{(i)}(\mathbf{x}_\parallel|\omega) + \frac{1}{2\pi} P \int \mathrm{d}^2 x'_\parallel \; \mathbf{n} \times \left\{ [\![\boldsymbol{\nabla} g_0(\mathbf{x}|\mathbf{x}')]\!] \times \mathbf{J}_H(\mathbf{x}'_\parallel|\omega) \right\},$$

where

$$\mathbf{J}_H^{(i)}(\mathbf{x}_\parallel|\omega)_{\mathrm{inc}} = \mathbf{n} \times \mathbf{H}(\mathbf{x}|\omega)_{\mathrm{inc}}|_{x_3=\zeta(\mathbf{x}_\parallel)},$$

P denotes the Cauchy principal value, and

$$[\![f(\mathbf{x}|\mathbf{x}')]\!] = f(\mathbf{x}|\mathbf{x}') \Big|_{\substack{x_3=\zeta(\mathbf{x}_\parallel) \\ x'_3=\zeta(\mathbf{x}'_\parallel)}}.$$

This is the equation satisfied by $\mathbf{J}_H(\mathbf{x}_\parallel|\omega)$.

In view of the definition $\mathbf{J}_H(\mathbf{x}_\parallel|\omega) = [\mathbf{n} \times \mathbf{H}^>(\mathbf{x}|\omega)]|_{x_3=\zeta(\mathbf{x}_\parallel)}$ it follows that

$$\mathbf{n} \cdot \mathbf{J}_H(\mathbf{x}_\parallel|\omega) = 0,$$

so that only two components of $\mathbf{J}_H(\mathbf{x}_\parallel|\omega)$ are independent. We choose $\mathbf{J}_H(\mathbf{x}_\parallel|\omega)_{1,2}$ as the independent components, while

$$J_H(\mathbf{x}_\parallel|\omega)_3 = \zeta_1(\mathbf{x}_\parallel)J_H(\mathbf{x}_\parallel|\omega)_1 + \zeta_2(\mathbf{x}_\parallel)J_H(\mathbf{x}_\parallel|\omega)_2.$$

Equations satisfied by $\mathbf{J}_H(\mathbf{x}_\parallel|\omega)_{1,2}$

$$
\begin{aligned}
J_H(\mathbf{x}_\parallel|\omega)_1 =& 2J_H^{(i)}(\mathbf{x}_\parallel|\omega)_1 \\
& - \frac{1}{2\pi} P \int \mathrm{d}^2\mathbf{x}'_\parallel \left\{ \left[g_3^{(0)}(\mathbf{x}_\parallel|\mathbf{x}'_\parallel) - g_1^{(0)}(\mathbf{x}_\parallel|\mathbf{x}'_\parallel)\zeta_1(\mathbf{x}'_\parallel) - \zeta_2(\mathbf{x}_\parallel) g_2^{(0)}(\mathbf{x}_\parallel|\mathbf{x}'_\parallel) \right] J_H(\mathbf{x}'_\parallel|\omega)_1 \right. \\
& \left. + g_1^{(0)}(\mathbf{x}_\parallel|\mathbf{x}'_\parallel) \left[\zeta_2(\mathbf{x}_\parallel) - \zeta_2(\mathbf{x}'_\parallel) \right] J_H(\mathbf{x}'_\parallel|\omega)_2 \right\} \\
J_H(\mathbf{x}_\parallel|\omega)_2 =& 2J_H^{(i)}(\mathbf{x}_\parallel|\omega)_2 \\
& - \frac{1}{2\pi} P \int \mathrm{d}^2 x'_\parallel \left\{ g_2^{(0)}(\mathbf{x}_\parallel|\mathbf{x}'_\parallel) \left[\zeta_1(\mathbf{x}_\parallel) - \zeta_1(\mathbf{x}'_\parallel) \right] J_H(\mathbf{x}'_\parallel|\omega)_1 \right. \\
& \left. + \left[g_3^{(0)}(\mathbf{x}_\parallel|\mathbf{x}'_\parallel) - g_2^{(0)}(\mathbf{x}_\parallel|\mathbf{x}'_\parallel)\zeta_2(\mathbf{x}'_\parallel) - \zeta_1(\mathbf{x}_\parallel) g_1^{(0)}(\mathbf{x}_\parallel|\mathbf{x}'_\parallel) \right] J_H(\mathbf{x}'_\parallel|\omega)_2 \right\}
\end{aligned}
$$

where

$$
\begin{aligned}
g_\ell^{(0)}(\mathbf{x}_\parallel|\mathbf{x}'_\parallel) =& \left[\!\left[\frac{\partial}{\partial x_\ell} g_0(\mathbf{x}|\mathbf{x}') \right]\!\right] \\
=& (x_\ell - x'_\ell) \left[\frac{i(\omega/c)}{|\mathbf{x}-\mathbf{x}'|^2} - \frac{1}{|\mathbf{x}-\mathbf{x}'|^3} \right] \exp[i(\omega/c)|\mathbf{x}-\mathbf{x}'|] \Big|_{\substack{x_3=\zeta(\mathbf{x}_\parallel) \\ x'_3=\zeta(\mathbf{x}'_\parallel)}}
\end{aligned}
$$

Incident Field

Gaussian Beam

The electric vector of an incident field that is p polarized is

$$\mathbf{E}_p(\mathbf{x}|\omega)_{\mathrm{inc}} = \frac{w^2}{2\pi} \int_{q_\parallel < \frac{\omega}{c}} \mathrm{d}^2 q_\parallel \; \hat{\mathcal{E}}_p^{(i)}(\mathbf{q}_-|\omega) \exp\left[\mathrm{i}\mathbf{q}_- \cdot \mathbf{x}\right] \exp\left[-\frac{w^2}{2}(\mathbf{q}_\parallel - \mathbf{k}_\parallel)^2\right].$$

The electric vector of an incident field that is s polarized

$$\mathbf{E}_s(\mathbf{x}|\omega)_{\mathrm{inc}} = \frac{w^2}{2\pi} \int_{q_\parallel < \frac{\omega}{c}} \mathrm{d}^2 q_\parallel \; \hat{\mathcal{E}}_s^{(i)}(\mathbf{q}_-|\omega) \exp\left[\mathrm{i}\mathbf{q}_- \cdot \mathbf{x}\right] \exp\left[-\frac{w^2}{2}(\mathbf{q}_\parallel - \mathbf{k}_\parallel)^2\right].$$

where $\mathbf{q}_\pm = \mathbf{q}_\parallel \pm \alpha_0(\mathbf{q}_\parallel)\hat{\mathbf{x}}_3$ and

$$\hat{\mathcal{E}}_p^{(i)}(\mathbf{q}_-|\omega) = \frac{\alpha_0(q_\parallel)\hat{\mathbf{x}}_1 + q_1\hat{\mathbf{x}}_3}{[q_1^2 + \alpha_0^2(q_\parallel)]^{\frac{1}{2}}}$$

$$\hat{\mathcal{E}}_s^{(i)}(\mathbf{q}_-|\omega) = \frac{q_1 q_2\hat{\mathbf{x}}_1 - [q_1^2 + \alpha_0^2(q_\parallel)]\hat{\mathbf{x}}_2 - q_2\alpha_0(q_\parallel)\hat{\mathbf{x}}_3}{\frac{\omega}{c}[q_1^2 + \alpha_0^2(q_\parallel)]^{\frac{1}{2}}}.$$

The magnetic vector of the incident field that is p polarized is

$$\mathbf{H}_p(\mathbf{x}|\omega)_{\mathrm{inc}} = \frac{w^2}{2\pi} \int_{q_\parallel < \frac{\omega}{c}} \mathrm{d}^2 q_\parallel \; \hat{\mathcal{E}}_s^{(i)}(\mathbf{q}_- |\omega) \exp\left[\mathrm{i}\mathbf{q}_- \cdot \mathbf{x}\right] \exp\left[-\frac{w^2}{2}(\mathbf{q}_\parallel - \mathbf{k}_\parallel)^2\right].$$

The magnetic vector of the incident field that is s polarized is

$$\mathbf{H}_s(\mathbf{x}|\omega)_{\mathrm{inc}} = -\frac{w^2}{2\pi} \int_{q_\parallel < \frac{\omega}{c}} \mathrm{d}^2 q_\parallel \; \hat{\mathcal{E}}_p^{(i)}(\mathbf{q}_- |\omega) \exp\left[\mathrm{i}\mathbf{q}_- \cdot \mathbf{x}\right] \exp\left[-\frac{w^2}{2}(\mathbf{q}_\parallel - \mathbf{k}_\parallel)^2\right].$$

The magnitude of the total time-averaged incident flux of polarization ν is

$$P_{\mathrm{inc}}^{(\nu)} = -\,\mathrm{Re}\int \mathrm{d}^2 x_{\|}\,\frac{c}{8\pi}\left[\mathbf{E}_{\nu}(\mathbf{x}_{\|}|\omega)_{\mathrm{inc}}\times\mathbf{H}_{\nu}(\mathbf{x}_{\|}|\omega)_{\mathrm{inc}}^{*}\right]_3 = \frac{c^2}{8\pi\omega}p_{\mathrm{inc}}$$

where

$$\begin{aligned}
p_{\mathrm{inc}} =& w^4 \int\limits_{q_{\|}<\frac{\omega}{c}} \mathrm{d}^2 q_{\|}\;\alpha_0(q_{\|})\exp\left[-w^2(\mathbf{q}_{\|}-\mathbf{k}_{\|})^2\right] \\
=& 2\pi w^4\left(\frac{\omega}{c}\right)^3\exp(-w^2k_{\|}^2)\int\limits_0^{\frac{\pi}{2}}\mathrm{d}\theta\,\sin\theta\cos^2\theta\; I_0\left(2w^2\frac{\omega k_{\|}}{c}\sin\theta\right) \\
&\times\exp\left[-\left(\frac{\omega w}{c}\right)^2\sin^2\theta\right],
\end{aligned}$$

and $I_0(x)$ is the modified Bessel function of the first kind and zero order.

The Scattered Field

The Stratton-Chu equations for $\mathbf{H}^>(\mathbf{x}|\omega)$:

$$\theta(x_3 - \zeta(\mathbf{x}_\parallel))\,\mathbf{H}^>(\mathbf{x}|\omega) = \mathbf{H}(\mathbf{x}|\omega)_{\mathrm{inc}} + \frac{1}{4\pi}\int \mathrm{d}^2x'_\parallel\ \left[\nabla g_0(\mathbf{x}|\mathbf{x}')\right]\Big|_{x'_3=\zeta(\mathbf{x}'_\parallel)} \times \mathbf{J}_H(\mathbf{x}'_\parallel|\omega),$$

The magnetic vector of the scattered field is therefore

$$\mathbf{H}^>(\mathbf{x}|\omega)_{\mathrm{sc}} = \frac{1}{4\pi}\int \mathrm{d}^2x'_\parallel\ \left[\nabla g_0(\mathbf{x}|\mathbf{x}')\right]\Big|_{x'_3=\zeta(\mathbf{x}'_\parallel)} \times \mathbf{J}_H(\mathbf{x}'_\parallel|\omega)$$

$$g_0(\mathbf{x}|\mathbf{x}') = \int \frac{\mathrm{d}^2q_\parallel}{(2\pi)^2}\,\frac{2\pi\mathrm{i}}{\alpha_0(q_\parallel)}\exp\left[\mathrm{i}\mathbf{q}_\parallel\cdot(\mathbf{x}_\parallel - \mathbf{x}'_\parallel)\right]\,\exp\left[\mathrm{i}\alpha_0(q_\parallel)|x_3 - x'_3|\right].$$

If we write the electric vector of the scattered field in the form

$$\mathbf{E}(\mathbf{x}|\omega)_{\mathrm{sc}} = \int \frac{\mathrm{d}^2q_\parallel}{(2\pi)^2}\ \left[\mathcal{E}_p(\mathbf{q}_+|\omega)\hat{\gamma}_p(\mathbf{q}_+|\omega) + \mathcal{E}_s(\mathbf{q}_+|\omega)\hat{\gamma}_s(\mathbf{q}_+|\omega)\right]\exp\left[\mathrm{i}\mathbf{q}_+\cdot\mathbf{x}\right],$$

where

$$\hat{\gamma}_p(\mathbf{q}_+|\omega) = \frac{-\alpha_0(q_\parallel)\hat{\mathbf{q}}_\parallel + q_\parallel\hat{\mathbf{x}}_3}{\omega/c,} \qquad \hat{\gamma}_s(\mathbf{q}_+|\omega) = \hat{\mathbf{q}}_\parallel\times\hat{\mathbf{x}}_3,$$

we can write the magnetic vector of the scattered field as

$$\mathbf{H}(\mathbf{x}|\omega)_{\mathrm{sc}} = \int \frac{\mathrm{d}^2 q_\parallel}{(2\pi)^2} \left[\mathcal{E}_p(\mathbf{q}_+|\omega)\hat{\gamma}_s(\mathbf{q}_+|\omega) - \mathcal{E}_s(\mathbf{q}_+|\omega)\hat{\gamma}_p(\mathbf{q}_+|\omega)\right] \exp\left(\mathrm{i}\mathbf{q}_+ \cdot \mathbf{x}\right).$$

We find that $(\nu = p, s)$

$$\mathcal{E}_\nu(\mathbf{q}_+|\omega) = -\frac{(\omega/c)}{2\alpha_0(q_\parallel)} \int \mathrm{d}^2 x_\parallel \; \hat{\gamma}_\nu(\mathbf{q}_+|\omega) \cdot \mathbf{J}_H(\mathbf{x}_\parallel|\omega) \exp\left[-\mathrm{i}\mathbf{q}_\parallel \cdot \mathbf{x}_\parallel - \mathrm{i}\alpha_0(q_\parallel)\zeta(\mathbf{x}_\parallel)\right]$$

The total time-averaged scattered flux

$$
\begin{aligned}
P_{sc} &= \int \mathrm{d}^2 x_{\|} \, \mathrm{Re}\, (\mathbf{S}^c_{sc})_3 \\
&= \frac{c}{8\pi} \int \mathrm{d}^2 x_{\|} \, \mathrm{Re}\, [\mathbf{E}(\mathbf{x}|\omega)_{sc} \times \mathbf{H}^*(\mathbf{x}|\omega)_{sc}]_3 \\
&= \frac{c^2}{8\pi\omega} \int_{q_{\|} < \frac{\omega}{c}} \frac{\mathrm{d}^2 q_{\|}}{(2\pi)^2} \, \alpha_0(q_{\|}) \left[\left| \mathcal{E}_p(\mathbf{q}_+|\omega) \right|^2 + \left| \mathcal{E}_s(\mathbf{q}_+|\omega) \right|^2 \right] .
\end{aligned}
$$

We write $\mathbf{q}_{\|}$ in terms of the polar and azimuthal scattering angles θ_s and ϕ_s as

$$
\mathbf{q}_{\|} = \frac{\omega}{c} \sin\theta_s (\cos\phi_s, \sin\phi_s, 0).
$$

Then

$$
P_{sc} = \frac{c}{8\pi} \left(\frac{\omega}{2\pi c} \right)^2 \int \mathrm{d}\Omega_s \, \cos\theta_s \left[\left| \mathcal{E}_p(\mathbf{q}_+|\omega) \right|^2 + \left| \mathcal{E}_s(\mathbf{q}_+|\omega) \right|^2 \right] .
$$

The Differential Reflection Coefficient

The differential reflection coefficient $(\partial R/\partial\Omega_s)$ is defined such that $(\partial R/\partial\Omega_s)\mathrm{d}\Omega_s$ is the fraction of the total time-averaged flux incident on the surface that is scattered into the element of solid angle $\mathrm{d}\Omega_s$ about the scattering direction (θ_s, ϕ_s).

The differential reflection coefficient for the scattering of light of polarization β, the projection of whose wave vector on the mean scattering surface is $\mathbf{k}_{\|}$, into light of polarization α, the projection of whose wave vector on the mean scattering surface is $\mathbf{q}_{\|}$, is given by

$$\frac{\partial R_{\alpha\beta}}{\partial\Omega_s} = \left(\frac{\omega}{2\pi c}\right)^2 \frac{\omega}{c} \cos^2\theta_s \frac{\left|\mathcal{E}_\alpha(\mathbf{q} + |\omega)\right|^2}{p_{\mathrm{inc}}}$$

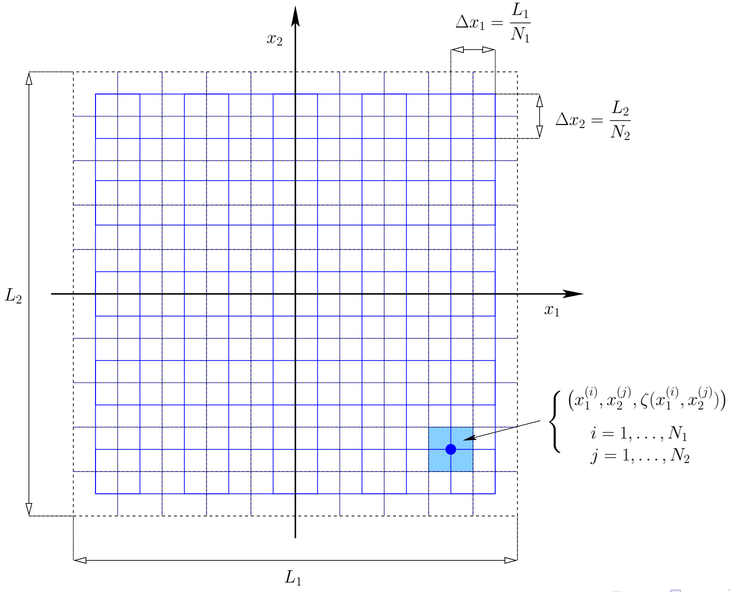
Since we are scattering from a randomly rough surface, it is this average of this expression over the ensemble of realizations of the surface profile function that we wish to calculate:

$$\left\langle \frac{\partial R_{\alpha\beta}}{\partial \Omega_s} \right\rangle = \left(\frac{\omega}{2\pi c}\right)^2 \frac{\omega}{c} \cos^2\theta_s \frac{\left\langle \left|\mathcal{E}_\alpha(\mathbf{q} + |\omega)\right|^2 \right\rangle}{p_{\mathrm{inc}}}.$$

The contribution to the mean differential reflection coefficient from the light that has been scattered incoherently (diffusely) is

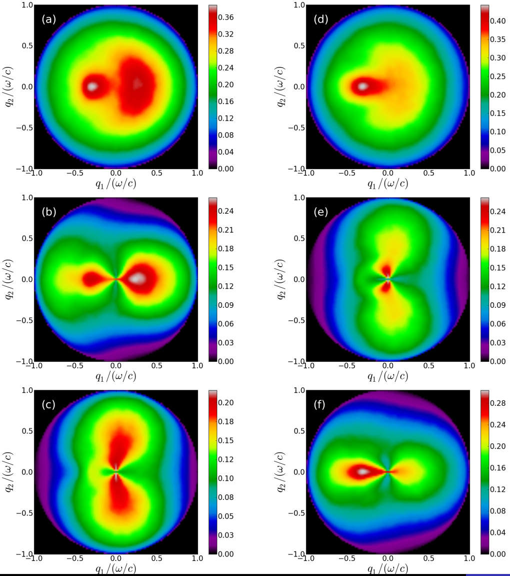
$$\left\langle \frac{\partial R_{\alpha\beta}}{\partial \Omega_s} \right\rangle_{\mathrm{incoh}} = \left(\frac{\omega}{2\pi c}\right)^2 \frac{\omega}{c} \cos^2\theta_s \frac{\left\langle \left|\mathcal{E}_\alpha(\mathbf{q} + |\omega)\right|^2 \right\rangle - \left|\left\langle \mathcal{E}_\alpha(\mathbf{q} + |\omega) \right\rangle\right|^2}{p_{\mathrm{inc}}}.$$

The Grid used in the Numerical Solution for $\mathbf{J}_H(\mathbf{x}|\omega)_{1,2}$



An Isotropic Roughness Power Spectrum

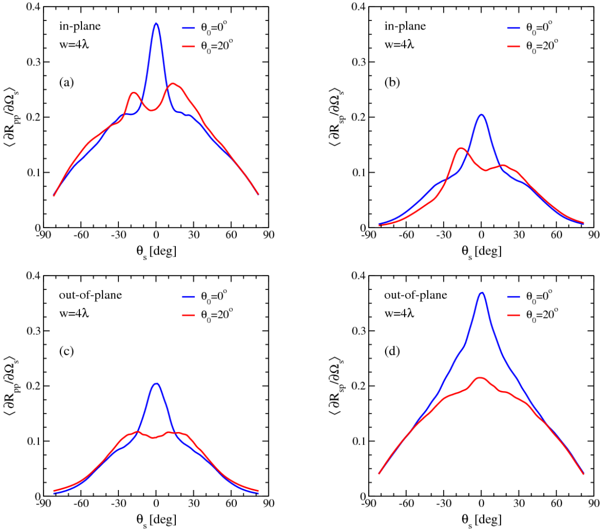
The full angular intensity distributions



- $W(\mathbf{x}_\parallel) = \exp\left(-\frac{x_\parallel^2}{a^2}\right)$
- $g(\mathbf{k}_\parallel) = \pi a^2 \exp\left(-\frac{k_\parallel^2 a^2}{4}\right)$
- Parameters:
  - $(\theta_0, \phi_0) = (20^\circ, 0^\circ)$
  - $\delta = \lambda$
  - $a = 2\lambda$
  - $w = 4\lambda$
  - $L^2 = 16\lambda \times 16\lambda$
  - $\Delta x = \lambda/7$
  - $N_p = 12\,000$

A Metallic Surface

In-plane and out-of-plane intensity distributions for *p* polarized incident light



All parameters as on previous slide.

Unitarity

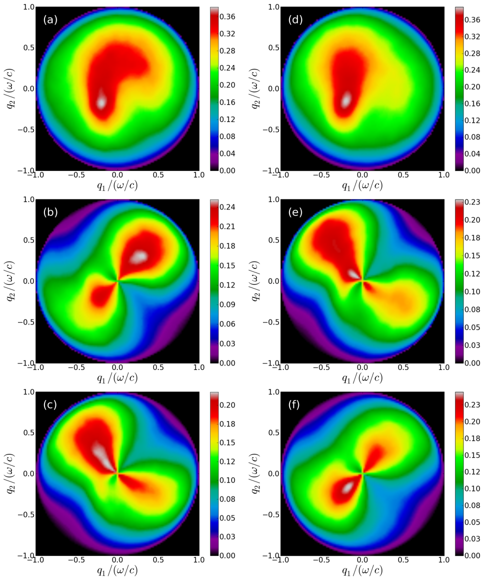
The fraction of the total energy flux scattered from the surface to that incident on it — the unitarity — is defined as:

$$\mathcal{U}_\beta(\theta_0, \phi_0) = \sum_{\alpha=p,s} \int \mathrm{d}\Omega_s \left\langle \frac{\partial R_{\alpha\beta}}{\partial \Omega_s} \right\rangle = 1, \qquad \beta = p, s$$

For the numerical simulation results just presented we obtained:

$$\mathcal{U}_p(20^\circ, 0^\circ) = 0.9962$$
$$\mathcal{U}_s(20^\circ, 0^\circ) = 0.9966.$$

An Anisotropic Roughness Power Spectrum



- $W(\mathbf{x}_\parallel) = \exp\left(-\frac{x_1^2}{a_1^2} - \frac{x_2^2}{a_2^2}\right)$
- $g(\mathbf{k}_\parallel) = \pi a_1 a_2 \exp\left(-\frac{k_1^2 a_1^2}{4} - \frac{k_2^2 a_2^2}{4}\right.$
- Parameters:
  - $(\theta_0, \phi_0) = (20^\circ, 45^\circ)$
  - $a_1 = \lambda$; $a_2 = 1.5\lambda$;
  - $\delta = \lambda$
  - $w = 4\lambda$
  - $L^2 = 16\lambda \times 16\lambda$
  - $\Delta x = \lambda/7$
  - $N_p = 6\,000$

Outline

A Penetrable Surface
Franz Formulas

$x_3 > \zeta(\mathbf{x}_\parallel)$:

$$\mathbf{H}^>(\mathbf{x}|\omega) = \mathbf{H}(\mathbf{x}|\omega)_{\text{inc}} + \frac{1}{4\pi}\boldsymbol{\nabla}\times\int \mathrm{d}^2x'_\parallel\; g_0(\mathbf{x}|\mathbf{x}')|_{x'_3=\zeta(\mathbf{x}'_\parallel)}\mathbf{J}_H(\mathbf{x}'_\parallel|\omega) - \frac{\mathrm{i}c}{4\pi\omega}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\int \mathrm{d}^2x'_\parallel\; g_0(\mathbf{x}|\mathbf{x}')|_{x'_3=\zeta(\mathbf{x}'_\parallel)}\mathbf{J}_E(\mathbf{x}'_\parallel|\omega)$$

$$\mathbf{E}^>(\mathbf{x}|\omega) = \mathbf{E}(\mathbf{x}|\omega)_{\text{inc}} + \frac{1}{4\pi}\boldsymbol{\nabla}\times\int \mathrm{d}^2x'_\parallel\; g_0(\mathbf{x}|\mathbf{x}')|_{x'_3=\zeta(\mathbf{x}'_\parallel)}\mathbf{J}_E(\mathbf{x}'_\parallel|\omega) + \frac{\mathrm{i}c}{4\pi\omega}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\int \mathrm{d}^2x'_\parallel\; g_0(\mathbf{x}|\mathbf{x}')|_{x'_3=\zeta(\mathbf{x}'_\parallel)}\mathbf{J}_H(\mathbf{x}'_\parallel|\omega).$$

$x_3 < \zeta(\mathbf{x}_\parallel)$:

$$\mathbf{H}^<(\mathbf{x}|\omega) = -\frac{1}{4\pi}\boldsymbol{\nabla}\times\int \mathrm{d}^2x'_\parallel\; g_\varepsilon(\mathbf{x}|\mathbf{x}')|_{x'_3=\zeta(\mathbf{x}'_\parallel)}\mathbf{J}_H(\mathbf{x}'_\parallel|\omega) + \frac{\mathrm{i}c}{4\pi\omega}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\int \mathrm{d}^2x'_\parallel\; g_\varepsilon(\mathbf{x}|\mathbf{x}')|_{x'_3=\zeta(\mathbf{x}'_\parallel)}\mathbf{J}_E(\mathbf{x}'_\parallel|\omega)$$

$$\mathbf{E}^<(\mathbf{x}|\omega) = -\frac{1}{4\pi}\boldsymbol{\nabla}\times\int \mathrm{d}^2x'_\parallel\; g_\varepsilon(\mathbf{x}|\mathbf{x}')|_{x'_3=\zeta(\mathbf{x}'_\parallel)}\mathbf{J}_E(\mathbf{x}'_\parallel|\omega) - \frac{\mathrm{i}c}{4\pi\omega\varepsilon(\omega)}\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\int \mathrm{d}^2x_\parallel\; g_\varepsilon(\mathbf{x}|\mathbf{x}')|_{x'_3=\zeta(\mathbf{x}'_\parallel)}\mathbf{J}_H(\mathbf{x}'_\parallel|\omega).$$

$$\mathbf{J}_H(\mathbf{x}_\parallel|\omega) = \left[\mathbf{n} \times \mathbf{H}^>(\mathbf{x}|\omega)\right]\Big|_{x_3=\zeta(\mathbf{x}_\parallel)} = \left[\mathbf{n} \times \mathbf{H}^<(\mathbf{x}|\omega)\right]\Big|_{x_3=\zeta(\mathbf{x}_\parallel)}$$

$$\mathbf{J}_E(\mathbf{x}_\parallel|\omega) = \left[\mathbf{n} \times \mathbf{E}^>(\mathbf{x}|\omega)\right]\Big|_{x_3=\zeta(\mathbf{x}_\parallel)} = \left[\mathbf{n} \times \mathbf{E}^<(\mathbf{x}|\omega)\right]\Big|_{x_3=\zeta(\mathbf{x}_\parallel)}$$

$$\mathbf{n} =(-\zeta_1(\mathbf{x}_\parallel, -\zeta_2(\mathbf{x}_\parallel, 1)$$

$$g_0(\mathbf{x}|\mathbf{x}') =\frac{\exp\left[\mathrm{i}\frac{\omega}{c}\left|\mathbf{x}-\mathbf{x}'\right|\right]}{\left|\mathbf{x}-\mathbf{x}'\right|}$$

$$=\int \frac{\mathrm{d}^2 q_\parallel}{(2\pi)^2}\, \frac{2\pi\mathrm{i}}{\alpha_0(q_\parallel)} \exp\left[\mathrm{i}\mathbf{q}_\parallel \cdot (\mathbf{x}_\parallel - \mathbf{x}'_\parallel)\right] \exp\left[\mathrm{i}\alpha_0(q_\parallel)|x_3 - x'_3|\right]$$

$$g_\varepsilon(\mathbf{x}|\mathbf{x}') =\frac{\exp\left[-\left|\mathbf{x}-\mathbf{x}'\right|/d(\omega)\right]}{\left|\mathbf{x}-\mathbf{x}'\right|}$$

$$=\int \frac{\mathrm{d}^2 q_\parallel}{(2\pi)^2}\, \frac{2\pi}{\beta(q_\parallel)} \exp\left[\mathrm{i}\mathbf{q}_\parallel \cdot (\mathbf{x}_\parallel - \mathbf{x}'_\parallel)\right] \exp\left[-\beta(q_\parallel)|x_3 - x'_3|\right]$$

$$\beta(q_\parallel) = \left[q_\parallel^2 + \frac{1}{d^2(\omega)}\right], \qquad \mathrm{Re}\,\beta(q_\parallel) > 0, \mathrm{Im}\,\beta(q_\parallel) < 0$$

$$d(\omega) =\frac{c/\omega}{[-\varepsilon(\omega)]^{\frac{1}{2}}}, \qquad \mathrm{Re}\,d(\omega) > 0, \mathrm{Im}\,d(\omega) > 0$$

The Müller Equations

$$\begin{aligned}
\mathbf{J}_H(\mathbf{x}_\parallel|\omega) =& \mathbf{J}_H(\mathbf{x}_\parallel|\omega)_{\text{inc}} + \frac{1}{4\pi} P \int \mathrm{d}^2 x'_\parallel \, [\![\mathbf{n} \times \{ \boldsymbol{\nabla} \times [g_0(\mathbf{x}|\mathbf{x}') - g_\varepsilon(\mathbf{x}|\mathbf{x}')] \mathbf{J}_H(\mathbf{x}'_\parallel|\omega) \}]\!] \\
& - \frac{\mathrm{i}c}{4\pi\omega} \int \mathrm{d}^2 x'_\parallel \, [\![\mathbf{n} \times \{ \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times [g_0(\mathbf{x}|\mathbf{x}') - g_\varepsilon(\mathbf{x}|\mathbf{x}')] \mathbf{J}_E(\mathbf{x}'_\parallel|\omega) \}]\!] \\
\mathbf{J}_E(\mathbf{x}_\parallel|\omega) =& 2 \frac{\mathbf{J}_E(\mathbf{x}_\parallel|\omega)_{\text{inc}}}{1+\varepsilon(\omega)} + \frac{2}{4\pi[1+\varepsilon(\omega)]} P \int \mathrm{d}^2 x'_\parallel \, [\![\mathbf{n} \times \{ \boldsymbol{\nabla} \times [g_0(\mathbf{x}|\mathbf{x}') - \varepsilon(\omega) g_\varepsilon(\mathbf{x}|\mathbf{x}')] \mathbf{J}_E(\mathbf{x}'_\parallel|\omega) \}]\!] \\
& + \frac{2\mathrm{i}c}{4\pi\omega[1+\varepsilon(\omega)]} \int \mathrm{d}^2 x'_\parallel \, [\![\mathbf{n} \times \{ \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times [g_0(\mathbf{x}|\mathbf{x}') - g_\varepsilon(\mathbf{x}|\mathbf{x}')] \mathbf{J}_H(\mathbf{x}'_\parallel|\omega) \}]\!],
\end{aligned}$$

$$\begin{aligned}
\mathbf{J}_H(\mathbf{x}_\parallel|\omega)_{\text{inc}} =& \mathbf{n} \times \mathbf{H}(\mathbf{x}|\omega)_{\text{inc}} \big|_{x_3=\zeta(\mathbf{x}_\parallel)} \\
\mathbf{J}_E(\mathbf{x}_\parallel|\omega)_{\text{inc}} =& \mathbf{n} \times \mathbf{E}(\mathbf{x}|\omega)_{\text{inc}} \big|_{x_3=\zeta(\mathbf{x}_\parallel)} \\
\mathbf{n} \cdot \mathbf{J}_{E,H}(\mathbf{x}_\parallel|\omega) =& 0 \\
\mathbf{J}_{E,H}(\mathbf{x}_\parallel|\omega)_3 =& \zeta_1(\mathbf{x}_\parallel) \mathbf{J}_{E,H}(\mathbf{x}_\parallel|\omega)_1 + \zeta_2(\mathbf{x}_\parallel) \mathbf{J}_{E,H}(\mathbf{x}_\parallel|\omega)_2
\end{aligned}$$

$\mathbf{J}_{E,H}(\mathbf{x}_\parallel|\omega)_{1,2}$ satisfy a system of 4 coupled, inhomogeneous, two-dimesional integral equations.

Local Impedance Boundary Condition

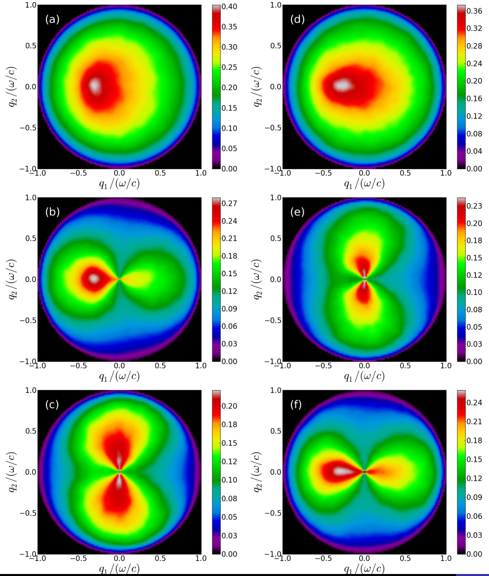
$$\mathbf{J}_E(\mathbf{x}_\parallel|\omega)_i = K_{ij}(\mathbf{x}_\parallel|\omega)\mathbf{J}_H(\mathbf{x}_\parallel|\omega)_j$$

$$\begin{aligned} K_{ij}(\mathbf{x}_\parallel|\omega) = {} & \mathrm{i}\frac{\omega}{c}\frac{d(\omega)}{\phi(\mathbf{x}_\parallel)}\begin{pmatrix} \zeta_1\zeta_2 & (1+\zeta_2^2) \\ -(1+\zeta_1^2) & -\zeta_1\zeta_2 \end{pmatrix} \\ & - \mathrm{i}\frac{\omega}{c}\frac{d^2(\omega)}{2\phi^3(\mathbf{x}_\parallel)}\begin{pmatrix} \begin{matrix} \zeta_{11}(1+\zeta_2^2)\zeta_1\zeta_2 \\ -2\zeta_{12}(1+\zeta_1^2)(1+\zeta_2^2) \\ +\zeta_{22}(1+\zeta_1^2)\zeta_1\zeta_2 \end{matrix} & \begin{matrix} \zeta_{11}(1+\zeta_2^2)^2 - 2\zeta_{12}(1+\zeta_2^2)\zeta_1\zeta_2 \\ -\zeta_{22}\left[(1+\zeta_1^2)(1+\zeta_2^2) - 2\zeta_1^2\zeta_2^2\right] \end{matrix} \\ \begin{matrix} \zeta_{11}\left[(1+\zeta_1^2)(1+\zeta_2^2) - 2\zeta_1^2\zeta_2^2\right] \\ +2\zeta_{12}(1+\zeta_1^2)\zeta_1\zeta_2 - \zeta_{22}(1+\zeta_1^2)^2 \end{matrix} & \begin{matrix} -\zeta_{11}(1+\zeta_2^2)\zeta_1\zeta_2 \\ +2\zeta_{12}(1+\zeta_1^2)(1+\zeta_2^2) \\ -\zeta_{22}(1+\zeta_1^2)\zeta_1\zeta_2 \end{matrix} \end{pmatrix} \\ & + \mathcal{O}(d^3(\omega)) \end{aligned}$$

$$\phi(\mathbf{x}_\parallel) = \left[1 + \zeta_1^2(\mathbf{x}_\parallel + \zeta_2^2(\mathbf{x}_\parallel\right]^{\frac{1}{2}}, \qquad \zeta_i(\mathbf{x}_\parallel) = \frac{\partial\zeta(\mathbf{x}_\parallel)}{\partial x_i}, \qquad \zeta_{ij}(\mathbf{x}_\parallel) = \frac{\partial^2\zeta(\mathbf{x}_\parallel)}{\partial x_i \partial x_j}, \quad \text{etc.}$$

A Metallic Surface

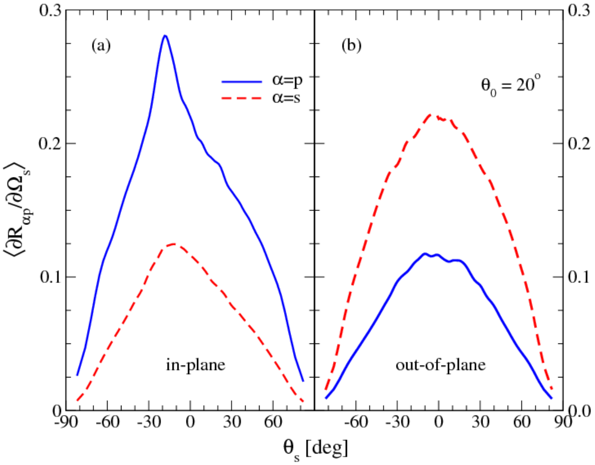
The full angular intensity distributions



- $W(\mathbf{x}_\parallel) = \exp\left(-\frac{x_\parallel^2}{a^2}\right)$
- $g(\mathbf{k}_\parallel) = \pi a^2 \exp\left(-\frac{k_\parallel^2 a^2}{4}\right)$
- Parameters:
  - $(\theta_0, \phi_0) = (20^\circ, 0^\circ)$
  - $\lambda = 0.6328\mu\text{m}$
  - $\varepsilon(\omega) = -16.0 + 1.088i$
  - $\delta = \lambda/4$
  - $a = \lambda/2$
  - $w = 4\lambda$
  - $L^2 = 16\lambda \times 16\lambda$
  - $\Delta x = \lambda/7$
  - $N_p = 5\,000$

A Metallic Surface

In-plane and out-of-plane intensity distributions for *p* polarized incident light



All parameters as on previous slide.

Outline

Solution of the Reduced Rayleigh Equation

$x_3 > \zeta(\mathbf{x}_\parallel)$

$$\mathbf{E}(\mathbf{x};t) = \left[\mathbf{E}^{(i)}(\mathbf{x}|\omega) + \mathbf{E}^{(s)}(\mathbf{x}|\omega)\right] \exp(-\mathrm{i}\omega t)$$

where

$$\mathbf{E}^{(i)}(\mathbf{x}|\omega) = \left\{ \frac{c}{\omega} \left[\alpha_0(k_\parallel)\hat{\mathbf{k}}_\parallel + k_\parallel \hat{\mathbf{x}}_3\right] \mathcal{E}_p^{(i)}(\mathbf{k}_\parallel) + [\hat{\mathbf{k}}_\parallel \times \hat{\mathbf{x}}_3]\, \mathcal{E}_s^{(i)}(\mathbf{k}_\parallel) \right\} \times \exp\left[\mathrm{i}\mathbf{k}_\parallel \cdot \mathbf{x}_\parallel - \mathrm{i}\alpha_0(k_\parallel)x_3\right],$$

$$\mathbf{E}^{(s)}(\mathbf{x}|\omega) = \int \frac{\mathrm{d}^2 q_\parallel}{(2\pi)^2} \left\{ \frac{c}{\omega} \left[-\alpha_0(q_\parallel)\hat{\mathbf{q}}_\parallel + q_\parallel \hat{\mathbf{x}}_3\right] \mathcal{E}_p^{(s)}(\mathbf{q}_\parallel) + [\hat{\mathbf{q}}_\parallel \times \hat{\mathbf{x}}_3]\, \mathcal{E}_s^{(s)}(\mathbf{q}_\parallel) \right\} \times \exp\left[\mathrm{i}\mathbf{q}_\parallel \cdot \mathbf{x}_\parallel + \mathrm{i}\alpha_0(q_\parallel)x_3\right].$$

A linear relation exists between the amplitudes $\mathcal{E}_{\alpha}^{(s)}(\mathbf{q}_{\|})$ and $\mathcal{E}_{\beta}^{(i)}(\mathbf{k}_{\|})$, which we write as $(\alpha = p, s,\ \beta = p, s)$

$$\mathcal{E}_{\alpha}^{(s)}(\mathbf{q}_{\|}) = \sum_{\beta} R_{\alpha\beta}(\mathbf{q}_{\|}|\mathbf{k}_{\|})\mathcal{E}_{\beta}^{(i)}(\mathbf{k}_{\|}).$$

The contribution to the differential reflection coefficient from the incoherent (diffuse) component of the scattered light is then

$$\left\langle \frac{\partial R_{\alpha\beta}}{\partial \Omega_s} \right\rangle_{\text{incoh}} = \frac{1}{S}\left(\frac{\omega}{2\pi c}\right)^2 \frac{\cos^2\theta_s}{\cos\theta_0}\left[\left\langle \left|R_{\alpha\beta}(\mathbf{q}_{\|}|\mathbf{k}_{\|})\right|^2 \right\rangle - \left|\left\langle R_{\alpha\beta}(\mathbf{q}_{\|}|\mathbf{k}_{\|})\right\rangle\right|^2\right]$$

where

$$\mathbf{k}_{\|} = \frac{\omega}{c}\sin\theta_0(\cos\phi_0, \sin\phi_0, 0), \qquad \mathbf{q}_{\|} = \frac{\omega}{c}\sin\theta_s(\cos\phi_s, \sin\phi_s, 0).$$

The Reduced Rayleigh equations

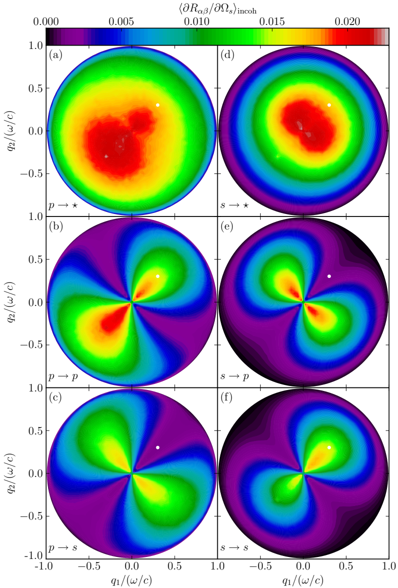
$$\int \frac{\mathrm{d}^2 q_{\|}}{(2\pi)^2} \, \frac{I(\alpha(p_{\|}) - \alpha_0(q_{\|})|\mathbf{p}_{\|} - \mathbf{q}_{\|})}{\alpha(p_{\|}) - \alpha_0(q_{\|})} \, \mathbf{M}(\mathbf{p}_{\|}|\mathbf{q}_{\|})\mathbf{R}(\mathbf{q}_{\|}|\mathbf{k}_{\|}) = -\frac{I(\alpha(p_{\|}) + \alpha_0(k_{\|})|\mathbf{p}_{\|} - \mathbf{k}_{\|})}{\alpha(p_{\|}) + \alpha_0(k_{\|})}\mathbf{N}(\mathbf{p}_{\|}|\mathbf{k}_{\|}),$$

$$\begin{aligned} I(\gamma|\mathbf{Q}_{\|}) &= \int \mathrm{d}^2 x_{\|} \exp\left[-\mathrm{i}\gamma\zeta(\mathbf{x}_{\|})\right] \exp\left[-\mathrm{i}\mathbf{Q}_{\|} \cdot \mathbf{x}_{\|}\right] \\ &= (2\pi)^2 \delta(\mathbf{Q}_{\|}) + \sum_{n=1}^{\infty} \frac{(-i\gamma)^n}{n!} \int \mathrm{d}^2 x_{\|} \zeta^n(\mathbf{x}_{\|}) \exp(-\mathrm{i}\mathbf{Q}_{\|} \cdot \mathbf{x}_{\|}) \end{aligned}$$

$$\mathbf{M}(\mathbf{p}_{\|}|\mathbf{q}_{\|}) = \begin{pmatrix} [p_{\|} q_{\|} + \alpha(p_{\|})\hat{\mathbf{p}}_{\|} \cdot \hat{\mathbf{q}}_{\|} \alpha_0(q_{\|})] & -\frac{\omega}{c}\alpha(p_{\|}) \, [\hat{\mathbf{p}}_{\|} \times \hat{\mathbf{q}}_{\|}]_3 \\ \frac{\omega}{c}[\hat{\mathbf{p}}_{\|} \times \hat{\mathbf{q}}_{\|}]_3 \, \alpha_0(q_{\|}) & \frac{\omega^2}{c^2}\hat{\mathbf{p}}_{\|} \cdot \hat{\mathbf{q}}_{\|} \end{pmatrix}$$

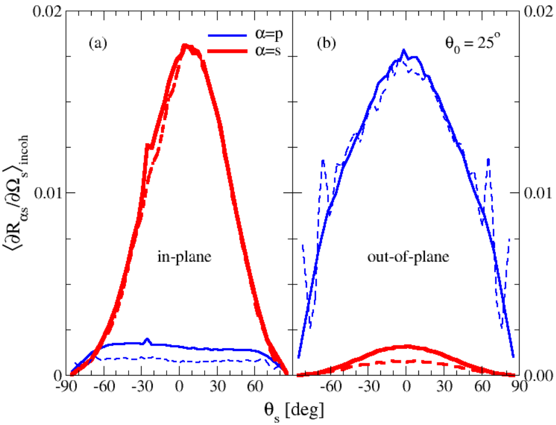
$$\mathbf{N}(\mathbf{p}_{\|}|\mathbf{k}_{\|}) = \begin{pmatrix} [p_{\|} k_{\|} - \alpha(p_{\|})\hat{\mathbf{p}}_{\|} \cdot \hat{\mathbf{k}}_{\|} \alpha_0(k_{\|})] & -\frac{\omega}{c}\alpha(p_{\|}) \, [\hat{\mathbf{p}}_{\|} \times \hat{\mathbf{k}}_{\|}]_3 \\ -\frac{\omega}{c}[\hat{\mathbf{p}}_{\|} \times \hat{\mathbf{k}}_{\|}]_3 \, \alpha_0(k_{\|}) & \frac{\omega^2}{c^2}\hat{\mathbf{p}}_{\|} \cdot \hat{\mathbf{k}}_{\|} \end{pmatrix}.$$

A Metallic Surface

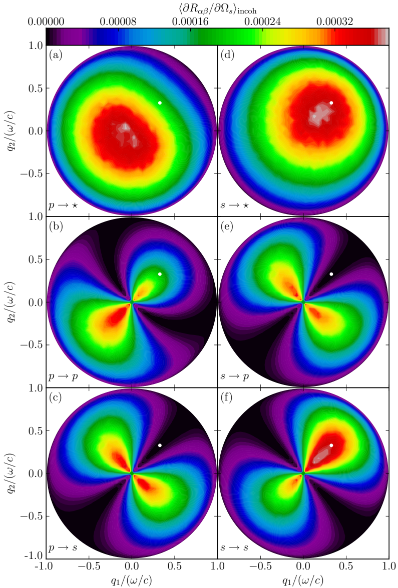


- $W(\mathbf{x}_\parallel) = \exp\left(-\frac{x_\parallel^2}{a^2}\right)$
- $g(\mathbf{k}_\parallel) = \pi a^2 \exp\left(-\frac{k_\parallel^2 a^2}{4}\right)$
- Parameters:
  - $(\theta_0, \phi_0) = (20^\circ, 45^\circ)$
  - $\lambda = 0.6328\mu\text{m}$
  - $\varepsilon(\omega) = -16.0 + 1.088\text{i}$
  - $\delta = \lambda/40$
  - $a = \lambda/2$
  - $L^2 = 25\lambda \times 25\lambda$
  - $\Delta x = \lambda/40$
  - $Q = 6.4\omega/c$
  - $N_p = 10\,000$

Comparison of the results obtained by the Rayleigh approach (solid lines) and rigorous approach (dashed lines):



A Dielectric Surface



- $W(\mathbf{x}_\parallel) = \exp\left(-\frac{x_\parallel^2}{a^2}\right)$
- $g(\mathbf{k}_\parallel) = \pi a^2 \exp\left(-\frac{k_\parallel^2 a^2}{4}\right)$
- Parameters:
  - $(\theta_0, \phi_0) = (20^\circ, 45^\circ)$
  - $\lambda = 0.6328\mu$m
  - $\varepsilon(\omega) = 2.64$
  - $\delta = \lambda/40$
  - $a = \lambda/2$
  - $L^2 = 25\lambda \times 25\lambda$
  - $\Delta x = \lambda/40$
  - $Q = 8\omega/c$
  - $N_p = 6\,000$

Outline

Discussion and Conclusions

- The use of the method of moments and the biconjugate gradient stabilized method provides a formally exact solution to the scattering of *p*- and *s*-polarized light from a two-dimensional randomly rough perfectly conducting surface with a modest expenditure of computational time.
- The addition of an impedance boundary condition on a two-dimensional rough surface to these two methods provides a formally exact solution to the scattering of polarized light from a two-dimensional randomly rough metallic surface, also with a modest expenditure of computational time.
- Rigorous calculations of the full angular distribution of the intensity of light scattered from strongly rough perfectly conducting and lossless metallic surfaces have been carried out.
- The accuracy of the methods used in our calculations, and the adequacy of the discretization of the mean scattering surface are indicated by the results that conservation of energy in the scattering process is satisfied within an error smaller than 0.5%.

Discussion and Conclusions (continued...)

- A rigorous numerical solution of the reduced Rayleigh equation for the scattering of *p*- and *s*-polarized light from a penetrable surface has been carried out. Good agreement between the results obtained in this fashion and those obtained by the use of the rigorous computational methods indicate that the simpler approach yields accurate results for the scattering from surfaces that are not very rough. The limits of validity of this equation have yet to be determined.
- The door is now open to rigorous computational studies of other properties of electromagnetic waves scattered from two-dimensional randomly rough surfaces. These include calculations of ellipsometric parameters of metallic and dielectric surfaces, transmission through dielectric surfaces, all the elements of the Mueller matrix, and scattering from coated surfaces.

References

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  The Scattering of Electromagnetic Waves from Two-Dimensional Randomly Rough Penetrable Surfaces
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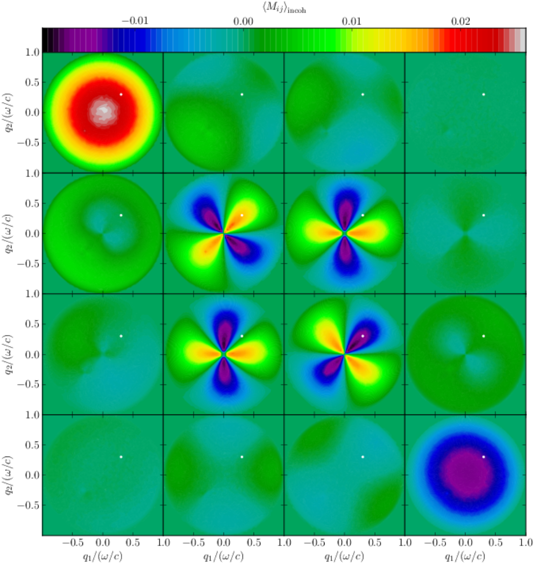
  The Scattering of Electromagnetic Waves from Two-Dimensional Randomly Rough Perfectly Conducting Surfaces: The Full Angular Intensity Distribution
  Phys. Rev. A **81**, 013806 (2010).

The Mueller Matrix



- $W(\mathbf{x}_{\|}) = \exp\left(-\frac{x_{\|}^2}{a^2}\right)$
- $g(\mathbf{k}_{\|}) = \pi a^2 \exp\left(-\frac{k_{\|}^2 a^2}{4}\right)$
- Parameters:
  - $(\theta_0, \phi_0) = (25^\circ, 45^\circ)$
  - $\lambda = 0.4579\mu\text{m}$
  - $\varepsilon(\omega) = -7.5 + i0.24$
  - $\delta = \lambda/40$
  - $a = \lambda/4$
  - $L^2 = 25\lambda \times 25\lambda$
  - $\Delta x = \lambda/40$
  - $Q = 6.4\omega/c$
  - $N_p = 10\,000$