# The Scattering of Electromagnetic Waves from Two-Dimensional Randomly Rough Surfaces

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# Outline

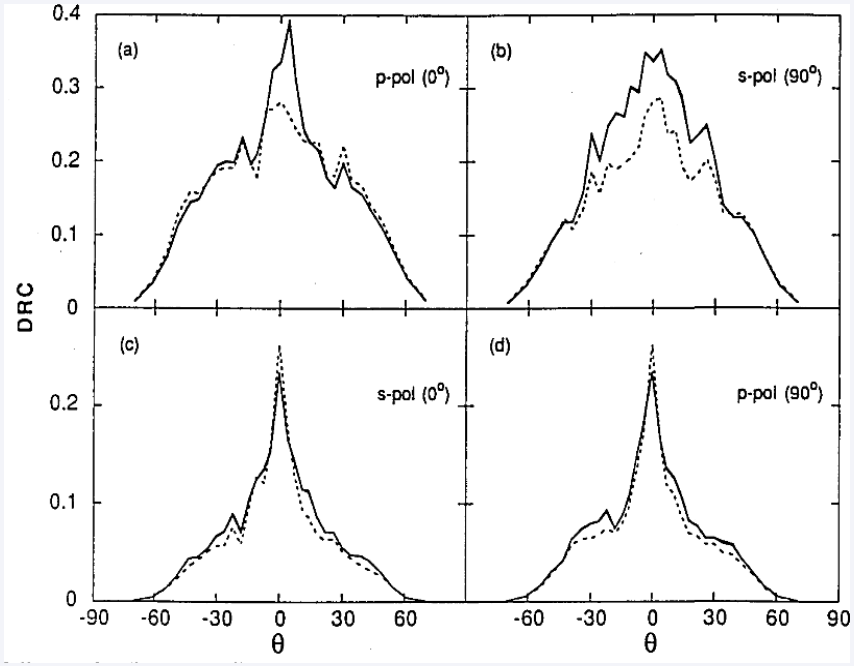
# Motivation

Why are we doing this work?

- Scattering from one-dimensional surface is well studied numerically and theoretically
- Most naturally occurring surfaces are two-dimensional
- Several scattering effects are unique to two-dimensional surfaces
- Relatively little work has been devoted to numerical studies of wave-scattering from for two-dimensionally surfaces

# Original Work

- Method of moments and Neumann–Liouville iteration $O(MN^2)$: P. Tran and A. A. Maradudin, Opt. Commun. **110**, 269 (1994).




The contribution to the mean differential reflection coefficient from the incoherent component of the scattered light for scattering from a metal surface ($\delta = \lambda$, $a = 2\lambda$ with $\lambda = 1\mu$m).

# Subsequent Work

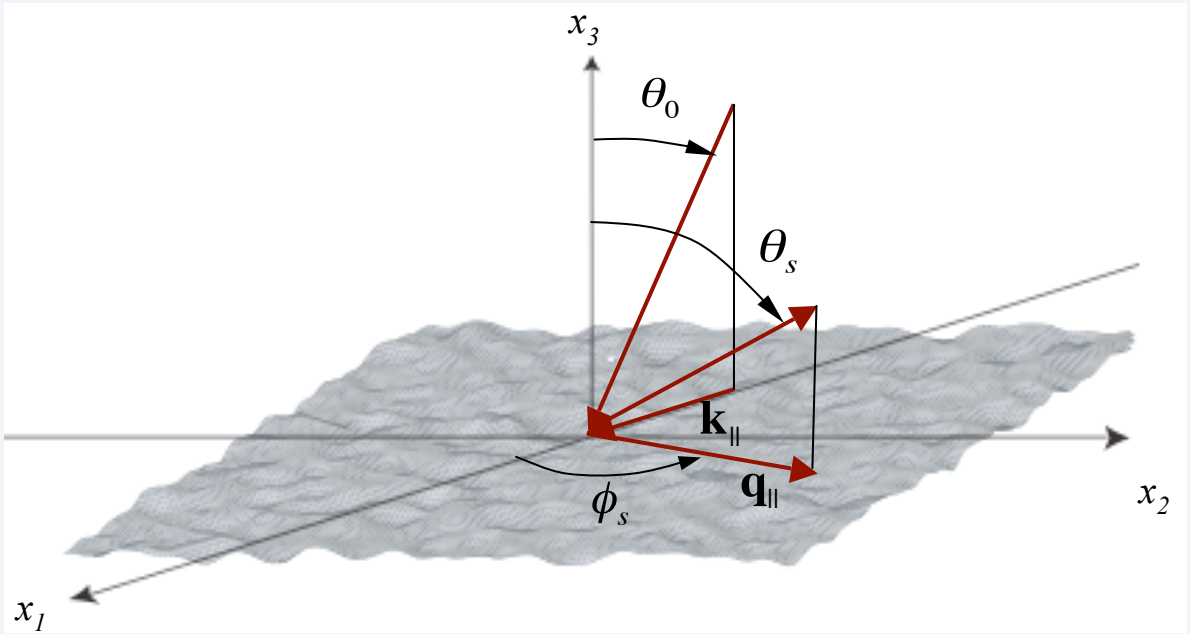
- Sparse-matrix flat-surface iterative approach $O(N^2)$
  K. Pak, L. Tsang, and J. Johnson, J. Opt. Soc. Am. A**14**, 1515 (1997)
- Steepest-descent fast-multipole method $O(N)$
  V. Jandhayala, B. Shanker, E. Michielssen, and W. C. Chew, J. Opt. Soc. Am. A**15**, 1887 (1998)
- Sparse-matrix flat-surface iterative approach plus an impedance boundary condition and a biconjugate gradient stabilized iterative approach $O(N^2)$
  G. Soriano and M. Saillard, J. Opt. Soc. Am. A**18**, 124 (2001).

# Scattering System

The scattering system we will be considering consists of the two media:

- vacuum ($x_3 > \zeta(\mathbf{x}_{||})$)
- metal ($x_3 < \zeta(\mathbf{x}_{||})$)

separated by a randomly rough interface located at $x_3 = \zeta(\mathbf{x}_{||})$.




The parallel wave-vector: $\mathbf{q}_{||} = \frac{\omega}{c} \sin\theta_s \left(\cos\phi_s, \sin\phi_s, 0\right)$

# Formulation of the Problem

## Problem Formulation

Given the

- statistical properties of the randomly rough surface $x_3 = \zeta(\mathbf{x}_{\|})$
- incident field (of given angle of incidence, polarization, wavelength and beam type) impinging onto the surface

then what is the

- angular distribution and polarization of the scattered light.

In the following we will demonstrate how this question can be addressed and show numerical simulation results for the scattered light.

# Properties of the Surface Profile Function

The surface profile function $\zeta(\mathbf{x}_\parallel)$ is a *single–valued function* of $\mathbf{x}_\parallel = (x_1, x_2, 0)$ that is differentiable with respect to $x_1$ and $x_2$, and constitutes a stationary, zero-mean, isotropic, Gaussian random process defined by

$$\langle \zeta(\mathbf{x}_\parallel)\zeta(\mathbf{x}'_\parallel)\rangle = \delta^2 W(|\mathbf{x}_\parallel - \mathbf{x}'_\parallel|).$$

The angle brackets denote an average over the ensemble of realizations of the surface profile function.

$$\delta = \langle \zeta^2(\mathbf{x}_\parallel)\rangle^{\frac{1}{2}}$$

is the rms height of the surface.

We will here assume a Gaussian height-height correlation function

$$W(|\mathbf{x}_\parallel|) = \exp(-x_\parallel^2/a^2),$$

where $a$ is the transverse correlation length of the surface roughness.

# Some Notation

The *surface currents* $\mathbf{J}_E(\mathbf{x}_\parallel|\omega)$ and $\mathbf{J}_H(\mathbf{x}_\parallel|\omega)$ are defined by

$$\begin{aligned}
\mathbf{J}_E(\mathbf{x}_\parallel|\omega) &= [\mathbf{n}\times\mathbf{E}^>(\mathbf{x}|\omega)]\Big|_{x_3=\zeta(\mathbf{x}_\parallel)} = [\mathbf{n}\times\mathbf{E}^<(\mathbf{x}|\omega)]\Big|_{x_3=\zeta(\mathbf{x}_\parallel)} \\
\mathbf{J}_H(\mathbf{x}_\parallel|\omega) &= [\mathbf{n}\times\mathbf{H}^>(\mathbf{x}|\omega)]\Big|_{x_3=\zeta(\mathbf{x}_\parallel)} = [\mathbf{n}\times\mathbf{H}^<(\mathbf{x}|\omega)]\Big|_{x_3=\zeta(\mathbf{x}_\parallel)},
\end{aligned}$$

and the *surface charge densities* $\rho_E(\mathbf{x}_\parallel|\omega)$ and $\rho_H(\mathbf{x}_\parallel|\omega)$ are defined by

$$\begin{aligned}
\rho_E(\mathbf{x}_\parallel|\omega) &= \frac{1}{\varepsilon(\omega)}\mathbf{n}\cdot\mathbf{E}^>(\mathbf{x}|\omega)\Big|_{x_3=\zeta(\mathbf{x}_\parallel)} = \mathbf{n}\cdot\mathbf{E}^<(\mathbf{x}|\omega)\Big|_{x_3=\zeta(\mathbf{x}_\parallel)}, \\
\rho_H(\mathbf{x}_\parallel\omega) &= \mathbf{n}\cdot\mathbf{H}^>(\mathbf{x}|\omega)\Big|_{x_3=\zeta(\mathbf{x}_\parallel)} = \mathbf{n}\cdot\mathbf{H}^<(\mathbf{x}|\omega)\Big|_{x_3=\zeta(\mathbf{x}_\parallel)},
\end{aligned}$$

where $\mathbf{n} = (-\zeta_1(\mathbf{x}_\parallel), -\zeta_2(\mathbf{x}_\parallel), 1)$, with $\zeta_\alpha(\mathbf{x}_\parallel) \equiv \frac{\partial}{\partial x_\alpha}\zeta(\mathbf{x}_\parallel)$ $\alpha = 1,2$.

## Source Function

The functions $\mathbf{J}_E$, $\mathbf{J}_H$, $\rho_E$ and $\rho_H$ are the *Source Functions* in our formulation of the scattering problem

# Stratton-Chu formulas for the fields in the vacuum

The starting point for the analysis is the so-called Stratton-Chu formulas (Stratton, 1941) that can be derived from *Green's second identity*.

In the vacuum they read:

$$\begin{aligned}
\theta(x_3-\zeta(\mathbf{x}_{\|}))\mathbf{E}^{>}(\mathbf{x}|\omega) &= \mathbf{E}(\mathbf{x}|\omega)_{inc} \\
&\quad + \frac{1}{4\pi}\int d^2x'_{\|}\Big\{\nabla' g_0(\mathbf{x}|\mathbf{x}')\rho_E(\mathbf{x}'_{\|}|\omega) \\
&\quad + i\frac{\omega}{c}g_0(\mathbf{x}|\mathbf{x}')\mathbf{J}_H(\mathbf{x}'_{\|}|\omega) - \nabla' g_0(\mathbf{x}|\mathbf{x}')\times\mathbf{J}_E(\mathbf{x}'_{\|}|\omega)\Big\}\Big|_{x'_3=\zeta(\mathbf{x}'_{\|})}
\end{aligned}$$

$$\begin{aligned}
\theta(x_3-\zeta(\mathbf{x}_{\|}))\mathbf{H}^{>}(\mathbf{x}|\omega) &= \mathbf{H}(\mathbf{x}|\omega)_{inc} \\
&\quad + \frac{1}{4\pi}\int d^2x'_{\|}\Big\{\nabla' g_0(\mathbf{x}|\mathbf{x}')\rho_H(\mathbf{x}'_{\|}|\omega) \\
&\quad - i\frac{\omega}{c}g_0(\mathbf{x}|\mathbf{x}')\mathbf{J}_E(\mathbf{x}'_{\|}|\omega) - \nabla' g_0(\mathbf{x}|\mathbf{x}')\times\mathbf{J}_H(\mathbf{x}'_{\|}|\omega)\Big\}\Big|_{x'_3=\zeta(\mathbf{x}'_{\|})}
\end{aligned}$$

where $g_0(\mathbf{x}|\mathbf{x}')$ is the free-space Green's function to be given later.

In these equations $\theta(z)$ is the Heaviside unit step function, the Green's functions

$$g_0(\mathbf{x}|\mathbf{x}') = \frac{\exp[i(\omega/c)|\mathbf{x}-\mathbf{x}'|]}{|\mathbf{x}-\mathbf{x}'|} = \int \frac{d^2q_\parallel}{(2\pi)^2}\frac{2\pi i}{\alpha_0(q_\parallel)}\exp[i\mathbf{q}_\parallel\cdot(\mathbf{x}_\parallel-\mathbf{x}'_\parallel)+i\alpha_0(q_\parallel)|x_3-x'_3|],$$

$$g_\varepsilon(\mathbf{x}|\mathbf{x}') = \frac{\exp[-|\mathbf{x}-\mathbf{x}'|/d(\omega)]}{|\mathbf{x}-\mathbf{x}'|} = \int \frac{d^2q_\parallel}{(2\pi)^2}\frac{2\pi}{\beta(q_\parallel)}\exp[i\mathbf{q}_\parallel\cdot(\mathbf{x}_\parallel-\mathbf{x}'_\parallel)-\beta(\mathbf{q}_\parallel)|x_3-x'_3|],$$

where the skin-depth in the metal is given by

$$d(\omega) = \frac{c}{\omega}(-\varepsilon(\omega))^{-\frac{1}{2}} \qquad Re\, d(\omega) > 0,\ Im\, d(\omega) > 0,$$

and where

$$\alpha_0(q_\parallel) = \left[\frac{\omega^2}{c^2} - q_\parallel^2\right]^{\frac{1}{2}} \qquad Re\,\alpha_0(q_\parallel) > 0,\ Im\,\alpha_0(q_\parallel) > 0$$

$$\beta(q_\parallel) = \left[q_\parallel^2 + \frac{1}{d^2(\omega)}\right]^{\frac{1}{2}} \qquad Re\,\beta(q_\parallel) > 0,\ Im\,\beta(q_\parallel) < 0.$$

# Stratton-Chu Formulas for the fields in the metal

Similar relations also hold in the metal:

$$\theta(\zeta(\mathbf{x}_\parallel)-x_3)\mathbf{E}^<(\mathbf{x}|\omega) = \frac{1}{4\pi}\int d^2x'_\parallel \Big\{-\nabla' g_\varepsilon(\mathbf{x}|\mathbf{x}')\rho_E(\mathbf{x}'_\parallel|\omega) - i\frac{\omega}{c} g_\varepsilon(\mathbf{x}|\mathbf{x}')\mathbf{J}_H(\mathbf{x}'_\parallel|\omega) + \nabla' g_\varepsilon(\mathbf{x}|\mathbf{x}')\times\mathbf{J}_E(\mathbf{x}'_\parallel|\omega)\Big\}\Big|_{x'_3=\zeta(\mathbf{x}'_\parallel)}$$

$$\theta(\zeta(\mathbf{x}_\parallel)-x_3)\mathbf{H}^<(\mathbf{x}|\omega) = \frac{1}{4\pi}\int d^2x'_\parallel \Big\{-\nabla' g_\varepsilon(\mathbf{x}|\mathbf{x}')\rho_H(\mathbf{x}'_\parallel|\omega) + i\varepsilon(\omega)\frac{\omega}{c} g_\varepsilon(\mathbf{x}|\mathbf{x}')\mathbf{J}_E(\mathbf{x}'_\parallel|\omega) + \nabla' g_\varepsilon(\mathbf{x}|\mathbf{x}')\times\mathbf{J}_H(\mathbf{x}'_\parallel|\omega)\Big\}\Big|_{x'_3=\zeta(\mathbf{x}'_\parallel)}$$

where $g_\varepsilon(\mathbf{x}|\mathbf{x}')$ is a Green's function to be given previously.

**Interpretation of the Stratton-Chu Formulas**

The Stratton-Chu formulas express the fact that the fields at any point in space can be obtained by (surface) integration *once* the *surface currents*, $\mathbf{J}_E(\mathbf{x}_\parallel|\omega)$ and $\mathbf{J}_H(\mathbf{x}_\parallel|\omega)$, and *surface charge densities*, $\rho_E(\mathbf{x}_\parallel|\omega)$ and $\rho_E(\mathbf{x}_\parallel|\omega)$, are *all* known.

Hence the real work consists of determining these source functions!
How this can be done we will come back to in a moment.

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Hence the real work consists of determining these source functions!
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# Franz Formulas

## Franz Formulas

Franz Formulas are obtained from the Stratton-Chu formulas by eliminating the *surface charge densities*, $\rho_E(\mathbf{x}_\parallel|\omega)$ and $\rho_H(\mathbf{x}_\parallel|\omega)$.

The derivation goes like this: From the Stratton-Chu formulas for $\mathbf{E}^>(\mathbf{x}|\omega)$

$$\theta(x_3-\zeta(\mathbf{x}_\parallel))\mathbf{E}^>(\mathbf{x}|\omega)=\mathbf{E}(\mathbf{x}|\omega)_{inc} + \frac{1}{4\pi}\int d^2x'_\parallel\{\nabla' g_0(\mathbf{x}|\mathbf{x}')\rho_E(\mathbf{x}'_\parallel|\omega)+i\frac{\omega}{c}g_0(\mathbf{x}|\mathbf{x}')\mathbf{J}_H(\mathbf{x}'_\parallel|\omega)-\nabla' g_0(\mathbf{x}|\mathbf{x}')\times\mathbf{J}_E(\mathbf{x}'_\parallel|\omega)\}\Big|_{x'_3=\zeta(\mathbf{x}'_\parallel)}$$

one can eliminate the term containing $\rho_E(\mathbf{x}'_\parallel|\omega)$ in the integrand on the right-hand side of this equation by assuming that $\mathbf{x}$ is in the region $x_3>\zeta(\mathbf{x}_\parallel)$, using the relation $\nabla' g_0(\mathbf{x}|\mathbf{x}')=-\nabla g_0(\mathbf{x}|\mathbf{x}')$, taking the curl of both sides of the resulting equation, and then using the relation $\nabla\times\mathbf{E}(\mathbf{x}|\omega)=i(\omega/c)\mathbf{H}(\mathbf{x}|\omega)$. In this way we obtain

$$\theta(x_3-\zeta(\mathbf{x}_\parallel))\mathbf{H}^>(\mathbf{x}|\omega)=\mathbf{H}(\mathbf{x}|\omega)_{inc} + \frac{1}{4\pi}\nabla\times\int d^2x'_\parallel\, g_0(\mathbf{x}|\mathbf{x}')\Big|_{x'_3=\zeta(\mathbf{x}'_\parallel)}\mathbf{J}_H(\mathbf{x}'_\parallel|\omega)-\frac{ic}{4\pi\omega}\nabla\times\nabla\times\int d^2x'_\parallel\, g_0(\mathbf{x}|\mathbf{x}')\Big|_{x'_3=\zeta(\mathbf{x}'_\parallel)}\mathbf{J}_E(\mathbf{x}'_\parallel|\omega)$$

that is the the Franz formula for $\mathbf{H}^>(\mathbf{x}|\omega)$.

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that is the the Franz formula for $\mathbf{H}^>(\mathbf{x}|\omega)$.

In a similar way one obtains the Franz formula for $\mathbf{E}^{>}(\mathbf{x}|\omega)$ (in the vacuum)

$$\begin{aligned}\theta(x_3-\zeta(\mathbf{x}_{\|}))\mathbf{E}^{>}(\mathbf{x}|\omega) &= \mathbf{E}(\mathbf{x}|\omega)_{inc} \\ &+\frac{1}{4\pi}\nabla\times\int d^2x'_{\|}\,g_0(\mathbf{x}|\mathbf{x}')\Big|_{x'_3=\zeta(\mathbf{x}'_{\|})}\mathbf{J}_E(\mathbf{x}'_{\|}|\omega) \\ &+\frac{ic}{4\pi\omega}\nabla\times\nabla\times\int d^2x'_{\|}\,g_0(\mathbf{x}|\mathbf{x}')\Big|_{x'_3=\zeta(\mathbf{x}'_{\|})}\mathbf{J}_H(\mathbf{x}'_{\|}|\omega).\end{aligned}$$

Similar formulas hold for the fields in the metal [$\mathbf{E}^{<}(\mathbf{x}|\omega)$ and $\mathbf{H}^{<}(\mathbf{x}|\omega)$]

The Franz formulas are particularly useful for obtaining the scattered fields since they depend only on $\mathbf{J}_E(\mathbf{x}_{\|}|\omega)$ and $\mathbf{J}_H(\mathbf{x}_{\|}|\omega)$.

# The Scattered Fields

The scattered magnetic and electric fields in the far zone ($x_3 \gg \zeta(\mathbf{x}_\parallel)$) are obtained from Franz formulae by using asymptotics of the Greens functions ($\mathscr{E}_\nu = \mathscr{E} \cdot \hat{\gamma}_\nu$):

$$\mathbf{E}(\mathbf{x}|\omega)_{sc} = \int \frac{d^2 q_\parallel}{(2\pi)^2} \left[\mathscr{E}_p(\mathbf{q}_+,\omega)\hat{\gamma}_p(\mathbf{q}_+,\omega) + \mathscr{E}_s(\mathbf{q}_+,\omega)\hat{\gamma}_s(\mathbf{q}_+,\omega)\right] \exp[i\mathbf{q}_+ \cdot \mathbf{x}],$$

$$\mathbf{H}(\mathbf{x}|\omega)_{sc} = \int \frac{d^2 q_\parallel}{(2\pi)^2} \left[\mathscr{E}_p(\mathbf{q}_+,\omega)\hat{\gamma}_s(\mathbf{q}_+,\omega) - \mathscr{E}_s(\mathbf{q}_+,\omega)\hat{\gamma}_p(\mathbf{q}_+,\omega)\right] \exp[i\mathbf{q}_+ \cdot \mathbf{x}].$$

In these expressions the *polarization vectors* are defined by $(\mathbf{q}_\pm,\omega) = \mathbf{q}_\parallel \pm \alpha(q_\parallel,\omega)\hat{\mathbf{x}}_3)$

$$\hat{\gamma}_s(\mathbf{q}_\pm,\omega) = \frac{\mathbf{q}_\pm \times \hat{\mathbf{x}}_3}{|\mathbf{q}_\pm \times \hat{\mathbf{x}}_3|} = \hat{\mathbf{q}}_\parallel \times \hat{\mathbf{x}}_3,$$

$$\hat{\gamma}_p(\mathbf{q}_\pm,\omega) = \hat{\gamma}_s(\mathbf{q}_\pm,\omega) \times \hat{\mathbf{q}}_\pm = \frac{q_\parallel \hat{\mathbf{x}}_3 \mp \alpha_0(q_\parallel,\omega)\hat{\mathbf{q}}_\parallel}{\omega/c},$$

and the corresponding scattering amplitudes are

$$\mathscr{E}_p(\mathbf{q}_\pm,\omega) = \frac{(\omega/c)}{2\alpha_0(q_\parallel)} \int d^2 x_\parallel \, e^{-i\mathbf{q}_\parallel \cdot \mathbf{x}_\parallel - i\alpha_0(q_\parallel)\zeta(\mathbf{x}_\parallel)} \left[\hat{\gamma}_s(\mathbf{q}_\pm,\omega) \cdot \mathbf{J}_H(\mathbf{x}_\parallel|\omega) + \hat{\gamma}_p(\mathbf{q}_\pm,\omega) \cdot \mathbf{J}_E(\mathbf{x}_\parallel|\omega)\right],$$

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**Note**

The surface currents , $\mathbf{J}_E(\mathbf{x}_\parallel|\omega)$ and $\mathbf{J}_H(\mathbf{x}_\parallel|\omega)$, are needed in order to define the *scattered fields* (and related observables)

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# The Mean Differential Reflection Coefficient

The *physical observable* we will consider in the *mean differential reflection coefficient*

**Definition: The Differential Reflection Coefficient**

The Differential Reflection Coefficient (DRC) is defined as the fraction of the total time-averaged flux incident on the surface that is scattered into the element of solid angle $d\Omega_s$ about the scattering direction $(\theta_s, \phi_s)$.

The (total time averaged) scattered flux is given by

$$\begin{aligned} P_{sc} &= \int d^2x_{\parallel} Re\,(\mathbf{S}^c_{sc})_3 = \frac{c}{8\pi}\int d^2x_{\parallel} Re\,[\mathbf{E}(\mathbf{x}|\omega)_{sc} \times \mathbf{H}^*(\mathbf{x}|\omega)_{sc}]_3 \\ &= \frac{c^2}{8\pi\omega}\int\limits_{q_{\parallel}<\frac{\omega}{c}} \frac{d^2q_{\parallel}}{(2\pi)^2}\alpha_0(q_{\parallel})\left[\left|\mathscr{E}_p(\mathbf{q}_+,\omega)\right|^2 + \left|\mathscr{E}_s(\mathbf{q}_+,\omega)\right|^2\right] \equiv \int d\Omega_s\, p_{sc}(\theta_s,\phi_s), \end{aligned}$$

and the incident flux, of polarization $\alpha$, by $P^{(\alpha)}_{inc} = \frac{c^2}{8\pi\omega}p^{(\alpha)}_{inc}$ where $p^{(\alpha)}_{inc} = S(\omega/c)\cos\theta_0$ for a plane incident wave.

Hence, according to it definition, the mean differential reflection coefficient is given by

$$\left\langle \frac{\partial R_{\beta\alpha}}{\partial\Omega_s} \right\rangle = \left\langle \frac{p^{(\beta)}_{sc}(\theta_s,\phi_s)}{P^{(\alpha)}_{inc}} \right\rangle = \frac{1}{4\pi^2}\left(\frac{\omega}{c}\right)^3 \cos^2\theta_s \frac{\left\langle \left|\mathscr{E}_\beta(\mathbf{q}_{\parallel},\omega)\right|^2 \right\rangle}{p^{(\alpha)}_{inc}}$$

The mean differential reflection coefficient can be separated into two terms; a *coherent* and an *incoherent* component.

If we write $\mathscr{E}_\beta(\mathbf{q}_\pm,\omega)$ as the sum of its mean value and the fluctuation about the mean,

$$\mathscr{E}_\beta(\mathbf{q}_\pm,\omega) = \langle\mathscr{E}_\beta(\mathbf{q}_\pm,\omega)\rangle + [\mathscr{E}_\beta(\mathbf{q}_\pm,\omega) - \langle\mathscr{E}_\beta(\mathbf{q}_\pm,\omega)\rangle],$$

each term contributes separately to the mean differential reflection coefficient,

$$\begin{aligned}
\left\langle \frac{\partial R_{\beta\alpha}}{\partial\Omega_s} \right\rangle &= \frac{1}{4\pi^2}\left(\frac{\omega}{c}\right)^3 \cos^2\theta_s \frac{1}{p_{inc}^{(\alpha)}} |\langle\mathscr{E}_\beta(\mathbf{q}_+,\omega)\rangle|^2 \\
&\quad + \frac{1}{4\pi^2}\left(\frac{\omega}{c}\right)^3 \cos^2\theta_s \frac{1}{p_{inc}^{(\alpha)}} [\langle|\mathscr{E}_\beta(\mathbf{q}_+,\omega)|^2\rangle - |\langle\mathscr{E}_\beta(\mathbf{q}_+,\omega)\rangle|^2] \\
&= \left\langle \frac{\partial R_{\beta\alpha}}{\partial\Omega_s} \right\rangle_{coh} + \left\langle \frac{\partial R_{\beta\alpha}}{\partial\Omega_s} \right\rangle_{incoh}
\end{aligned}$$

Note that the lateral wave-vector is relates the polar ($\theta_s$) and azimuthal angles ($\phi_s$) of scattering *via*

$$\mathbf{q}_\parallel = \frac{\omega}{c}\sin\theta_s\,(\cos\phi_s, \sin\phi_s, 0)\,.$$

The first term gives the contribution to the mean differential reflection coefficient from the light scattered coherently (specularly)

**Coherent Component**

$$\left\langle \frac{\partial R_{\beta\alpha}}{\partial \Omega_s} \right\rangle_{coh} = \frac{1}{4\pi^2} \left(\frac{\omega}{c}\right)^3 \cos^2\theta_s \frac{1}{p_{inc}^{(\alpha)}} |\langle \mathscr{E}_\beta(\mathbf{q}_+, \omega)\rangle|^2.$$

The second term gives the contribution to the mean differential reflection coefficient from the light scattered incoherently (diffusely)

**Incoherent Component**

$$\left\langle \frac{\partial R_{\beta\alpha}}{\partial \Omega_s} \right\rangle_{incoh} = \frac{1}{4\pi^2} \left(\frac{\omega}{c}\right)^3 \cos^2\theta_s \frac{1}{p_{inc}^{(\alpha)}} [\langle |\mathscr{E}_\beta(\mathbf{q}_+, \omega)|^2\rangle - |\langle \mathscr{E}_\beta(\mathbf{q}_+, \omega)\rangle|^2].$$

To solve the scattering problem, it is imperative to be able to determine the source currents, $\mathbf{J}_E(\mathbf{x}_\parallel|\omega)$ and $\mathbf{J}_H(\mathbf{x}_\parallel|\omega)$.

However, how could this be done?

# Equations satisfied by $\mathbf{J}_E(\mathbf{x}_\parallel|\omega)$ and $\mathbf{J}_H(\mathbf{x}_\parallel|\omega)$

We will now see how the equations satisfied by $\mathbf{J}_E(\mathbf{x}_\parallel|\omega)$ and $\mathbf{J}_H(\mathbf{x}_\parallel|\omega)$ can be derived.

Evaluate the Stratton-Chu equation for $\mathbf{H}^>(\mathbf{x}|\omega)$ at $x_3 = \zeta(\mathbf{x}_\parallel) + \eta$ and at $x_3 = \zeta(\mathbf{x}_\parallel) - \eta$, where $\eta$ is a positive infinitesimal, and add the resulting equations gives:

$$\mathbf{H}^>(\mathbf{x}|\omega)\Big|_{x_3=\zeta(\mathbf{x}_\parallel)} = 2\mathbf{H}(\mathbf{x}|\omega)_{inc}\Big|_{x_3=\zeta(\mathbf{x}_\parallel)} + \frac{1}{2\pi} P \int d^2x'_\parallel [\![\nabla' g_0(\mathbf{x}|\mathbf{x}')]\!] \rho_H(\mathbf{x}'_\parallel|\omega)$$
$$- \frac{1}{2\pi} P \int d^2x'_\parallel [\![\nabla' g_0(\mathbf{x}|\mathbf{x}')]\!] \times \mathbf{J}_H(\mathbf{x}'_\parallel|\omega) - \frac{i}{2\pi}\frac{\omega}{c} \int d^2x'_\parallel [\![g_0(\mathbf{x}|\mathbf{x}')]\!] \mathbf{J}_E(\mathbf{x}'_\parallel|\omega).$$

where $P$ denotes the Cauchy principal value, and

$$[\![f(\mathbf{x}|\mathbf{x}')]\!] = f(\mathbf{x}|\mathbf{x}')\Big|_{\substack{x_3=\zeta(\mathbf{x}_\parallel) \\ x'_3=\zeta(\mathbf{x}'_\parallel)}}.$$

A similar equation obtained in exactly the same way, holds for $\mathbf{E}^>(\mathbf{x}|\omega)$, but will not be given explicitly here.

To obtain the equation satisfied by $\mathbf{J}_H(\mathbf{x}_\parallel|\omega)$, we take the vector cross product between the surface normal vector $\mathbf{n}$ and both sides of the eq. for $\mathbf{H}^>(\mathbf{x}|\omega)|_{x_3=\zeta(\mathbf{x}_\parallel)}$ (previous slide) with the result [recall that $\mathbf{J}_H(\mathbf{x}_\parallel|\omega) = [\mathbf{n}\times\mathbf{H}^>(\mathbf{x}|\omega)]|_{x_3=\zeta(\mathbf{x}_\parallel)}$]

$$\begin{aligned}\mathbf{J}_H(\mathbf{x}_\parallel|\omega) &= 2\mathbf{J}_H^{(i)}(\mathbf{x}_\parallel|\omega) + \frac{1}{2\pi}P\int d^2x'_\parallel \mathbf{n}\times[\![\nabla' g_0(\mathbf{x}|\mathbf{x}')]\!]\rho_H(\mathbf{x}'_\parallel|\omega)\\ &\quad -\frac{1}{2\pi}P\int d^2x'_\parallel \mathbf{n}\times([\![\nabla' g_0(\mathbf{x}|\mathbf{x}')]\!]\times\mathbf{J}_H(\mathbf{x}'_\parallel|\omega))\\ &\quad -\frac{i}{2\pi}\frac{\omega}{c}\int d^2x'_\parallel [\![g_0(\mathbf{x}|\mathbf{x}')]\!](\mathbf{n}\times\mathbf{J}_E(\mathbf{x}'_\parallel|\omega)),\end{aligned}$$

where

$$J_H^{(i)}(\mathbf{x}_\parallel|\omega) = \mathbf{n}\times\mathbf{H}(\mathbf{x}|\omega)_{inc}\Big|_{x_3=\zeta(\mathbf{x}_\parallel)},$$

A similar procedure applied to the equation for $\mathbf{E}^>(\mathbf{x}|\omega)$ gives

$$\begin{aligned}\mathbf{J}_E(\mathbf{x}_\parallel|\omega) &= -\frac{1}{2\pi}P\int d^2x'_\parallel \mathbf{n}\times[\![\nabla' g_\varepsilon(\mathbf{x}|\mathbf{x}')]\!]\rho_E(\mathbf{x}'_\parallel|\omega)\\ &\quad +\frac{1}{2\pi}P\int d^2x'_\parallel \mathbf{n}\times([\![\nabla' g_\varepsilon(\mathbf{x}|\mathbf{x}')]\!]\times\mathbf{J}_E(\mathbf{x}'_\parallel|\omega))\\ &\quad -\frac{i}{2\pi}\frac{\omega}{c}\int d^2x'_\parallel [\![g_\varepsilon(\mathbf{x}|\mathbf{x}')]\!](\mathbf{n}\times\mathbf{J}_H(\mathbf{x}'_\parallel|\omega)).\end{aligned}$$

Hence, we have a set of six coupled integral equations (for eight components).

However, not all components of $\mathbf{J}_E(\mathbf{x}_\parallel|\omega)$ and $\mathbf{J}_H(\mathbf{x}_\parallel|\omega)$ are independent!

From their definitions:

$$\begin{aligned} \mathbf{n}\cdot\mathbf{J}_E(\mathbf{x}_\parallel|\omega) &= 0, \\ \mathbf{n}\cdot\mathbf{J}_H(\mathbf{x}_\parallel|\omega) &= 0, \end{aligned}$$

it follows only two components of $\mathbf{J}_E(\mathbf{x}_\parallel|\omega)$ and $\mathbf{J}_H(\mathbf{x}_\parallel|\omega)$ are independent.

We will take $J_E(\mathbf{x}_\parallel,\omega)_{1,2}$ and $J_H(\mathbf{x}_\parallel|\omega)_{1,2}$ as the independent components, while $[\zeta_i(\mathbf{x}_\parallel) = \partial\zeta((\mathbf{x}_\parallel)/\partial x_i\ ]$

$$\begin{aligned} J_E(\mathbf{x}_\parallel|\omega)_3 &= \zeta_1(\mathbf{x}_\parallel)J_E(\mathbf{x}_\parallel|\omega)_1 + \zeta_2(\mathbf{x}_\parallel)J_E(\mathbf{x}_\parallel|\omega)_2 \\ J_H(\mathbf{x}_\parallel|\omega)_3 &= \zeta_1(\mathbf{x}_\parallel)J_H(\mathbf{x}_\parallel|\omega)_1 + \zeta_2(\mathbf{x}_\parallel)J_H(\mathbf{x}_\parallel|\omega)_2. \end{aligned}$$

However, the system is still not complete since we have four equations for six unknowns!

# Elimination of $\rho_H(\mathbf{x}_{||}|\omega)$ and $\rho_E(\mathbf{x}_{||}|\omega)$

Several options exist for making the system of equations complete:

1. Derive additional equations for the surface charge densities, $\rho_H(\mathbf{x}_{||}|\omega)$ and $\rho_E(\mathbf{x}_{||}|\omega)$.
   - This is not so attractive since it increases the number of equations to be solved. [see P.Tran *et al.*, J. Opt. Soc. Am A **11**, 1668 (1994)]
2. However, the surface charge densities $\rho_E(\mathbf{x}_{||}|\omega)$ and $\rho_H(\mathbf{x}_{||}|\omega)$ be eliminated by the use of the (continuity) relations:

$$\begin{aligned} \rho_H(\mathbf{x}_{||}|\omega) &= -\frac{c}{i\omega}\nabla\cdot\mathbf{J}_E(\mathbf{x}_{||}|\omega), \\ \rho_E(\mathbf{x}_{||}|\omega) &= \frac{c}{i\omega\varepsilon(\omega)}\nabla\cdot\mathbf{J}_H(\mathbf{x}_{||}|\omega). \end{aligned}$$

## The Final Source Functions

As a result, the remaining source functions are those of $\mathbf{J}_E(\mathbf{x}_{||}|\omega)_{1,2}$ and $\mathbf{J}_H(\mathbf{x}_{||}|\omega)_{1,2}$, *i.e.* *four unknown components in total* that satisfies complete set of coupled integral equations.

# The Final Coupled Integral Equations

The resulting *coupled integral equations* for $\mathbf{J}_E(\mathbf{x}_\parallel|\omega)_{1,2}$ and $\mathbf{J}_H(\mathbf{x}_\parallel|\omega)_{1,2}$, after eliminating $\rho_E(\mathbf{x}_\parallel|\omega)$ and $\rho_H(\mathbf{x}_\parallel|\omega)$, read:

$$
\begin{aligned}
J_H(\mathbf{x}_\parallel|\omega)_1 \;=\;& 2J_H^{(i)}(\mathbf{x}_\parallel|\omega)_1 \\
&- \frac{1}{2\pi}P\int d^2x'_\parallel \left\{ [\![ \frac{\partial}{\partial x'_1} g_0(\mathbf{x}|\mathbf{x}') ]\!] \zeta_1(\mathbf{x}'_\parallel) + \zeta_2(\mathbf{x}_\parallel) [\![ \frac{\partial}{\partial x'_2} g_0(\mathbf{x}|\mathbf{x}') ]\!] - [\![ \frac{\partial}{\partial x'_3} g_0(\mathbf{x}|\mathbf{x}') ]\!] \right\} J_H(\mathbf{x}'_\parallel|\omega)_1 \\
&\quad - \frac{1}{2\pi}P\int d^2x'_\parallel [\![ \frac{\partial}{\partial x'_1} g_0(\mathbf{x}|\mathbf{x}') ]\!] (\zeta_2(\mathbf{x}'_\parallel) - \zeta_2(\mathbf{x}_\parallel)) J_H(\mathbf{x}'_\parallel|\omega)_2 \\
&+ \frac{i\omega}{2\pi c}\int d^2x'_\parallel \zeta_2(\mathbf{x}_\parallel) [\![ g_0(\mathbf{x}|\mathbf{x}') ]\!] \zeta_1(\mathbf{x}'_\parallel) J_E(\mathbf{x}'_\parallel|\omega)_1 \\
&- \frac{ic}{2\pi\omega}P\int d^2x'_\parallel \left\{ [\![ \frac{\partial}{\partial x'_2} g_0(\mathbf{x}|\mathbf{x}') ]\!] + \zeta_2(\mathbf{x}_\parallel) [\![ \frac{\partial}{\partial x'_3} g_0(\mathbf{x}|\mathbf{x}') ]\!] \right\} \frac{\partial}{\partial x'_1} J_E(\mathbf{x}'_\parallel|\omega)_1 \\
&+ \frac{i\omega}{2\pi c}\int d^2x'_\parallel [\![ g_0(\mathbf{x}|\mathbf{x}') ]\!] (1 + \zeta_2(\mathbf{x}_\parallel)\zeta_2(\mathbf{x}'_\parallel)) J_E(\mathbf{x}'_\parallel|\omega)_2 \\
&- \frac{ic}{2\pi\omega}P\int d^2x'_\parallel \left\{ [\![ \frac{\partial}{\partial x'_2} g_0(\mathbf{x}|\mathbf{x}') ]\!] + \zeta_2(\mathbf{x}_\parallel) [\![ \frac{\partial}{\partial x'_3} g_0(\mathbf{x}|\mathbf{x}') ]\!] \right\} \frac{\partial}{\partial x'_2} J_E(\mathbf{x}'_\parallel|\omega)_2
\end{aligned}
$$

$$
\begin{aligned}
J_H(\mathbf{x}_\parallel|\omega)_2 \;=\;& 2J_H^{(i)}(\mathbf{x}_\parallel|\omega)_2 - \frac{1}{2\pi}P\int d^2x'_\parallel [\![ \frac{\partial}{\partial x'_2} g_0(\mathbf{x}|\mathbf{x}') ]\!] (\zeta_1(\mathbf{x}'_\parallel) - \zeta_1(\mathbf{x}_\parallel)) J_H(\mathbf{x}'_\parallel|\omega)_1 \\
&- \frac{1}{2\pi}P\int d^2x'_\parallel \left\{ \zeta_1(\mathbf{x}_\parallel) [\![ \frac{\partial}{\partial x'_1} g_0(\mathbf{x}|\mathbf{x}') ]\!] + [\![ \frac{\partial}{\partial x'_2} g_0(\mathbf{x}|\mathbf{x}') ]\!] \zeta_2(\mathbf{x}'_\parallel) - [\![ \frac{\partial}{\partial x'_3} g_0(\mathbf{x}|\mathbf{x}') ]\!] \right\} J_H(\mathbf{x}'_\parallel|\omega)_2 \\
&- \frac{i\omega}{2\pi c}\int d^2x'_\parallel [\![ g_0(\mathbf{x}|\mathbf{x}') ]\!] (1 + \zeta_1(\mathbf{x}_\parallel)\zeta_1(\mathbf{x}'_\parallel)) J_E(\mathbf{x}_\parallel|\omega)_1 \\
&+ \frac{ic}{2\pi\omega}P\int d^2x'_\parallel \left\{ [\![ \frac{\partial}{\partial x'_1} g_0(\mathbf{x}|\mathbf{x}') ]\!] + \zeta_1(\mathbf{x}_\parallel) [\![ \frac{\partial}{\partial x'_3} g_0(\mathbf{x}|\mathbf{x}') ]\!] \right\} \frac{\partial}{\partial x'_1} J_E(\mathbf{x}_\parallel|\omega)_1 \\
&- \frac{i\omega}{2\pi c}\int d^2x'_\parallel \zeta_1(\mathbf{x}_\parallel) [\![ g_0(\mathbf{x}|\mathbf{x}') ]\!] \zeta_2(\mathbf{x}'_\parallel) J_E(\mathbf{x}'_\parallel|\omega)_2 \\
&+ \frac{ic}{2\pi\omega}P\int d^2x'_\parallel \left\{ [\![ \frac{\partial}{\partial x'_1} g_0(\mathbf{x}|\mathbf{x}') ]\!] + \zeta_1(\mathbf{x}_\parallel) [\![ \frac{\partial}{\partial x'_3} g_0(\mathbf{x}|\mathbf{x}') ]\!] \right\} \frac{\partial}{\partial x'_2} J_E(\mathbf{x}'_\parallel|\omega)_2
\end{aligned}
$$

$$
\begin{aligned}
J_E(\mathbf{x}_\parallel|\omega)_1 &= \frac{1}{2\pi}P\int d^2x'_\parallel \left\{ [\![ \frac{\partial}{\partial x'_1} g_\varepsilon(\mathbf{x}|\mathbf{x}') ]\!] \zeta_1(\mathbf{x}'_\parallel)\zeta_2(\mathbf{x}_\parallel) [\![ \frac{\partial}{\partial x'_2} g_\varepsilon(\mathbf{x}|\mathbf{x}') ]\!] - [\![ \frac{\partial}{\partial x'_3} g_\varepsilon(\mathbf{x}|\mathbf{x}') ]\!] \right\} J_E(\mathbf{x}'_\parallel|\omega)_1 \\
&+ \frac{1}{2\pi}P\int d^2x'_\parallel [\![ \frac{\partial}{\partial x'_1} g_\varepsilon(\mathbf{x}|\mathbf{x}') ]\!] (\zeta_2(\mathbf{x}'_\parallel) - \zeta_2(\mathbf{x}_\parallel)) J_E(\mathbf{x}'_\parallel|\omega)_2 \\
&+ \frac{i\omega}{2\pi c}\int d^2x'_\parallel \zeta_2(\mathbf{x}_\parallel) [\![ g_\varepsilon(\mathbf{x}|\mathbf{x}') ]\!] \zeta_1(\mathbf{x}'_\parallel) J_H(\mathbf{x}'_\parallel|\omega)_1 \\
&- \frac{ic}{2\pi\omega\varepsilon(\omega)}P\int d^2x'_\parallel \left\{ [\![ \frac{\partial}{\partial x'_2} g_\varepsilon(\mathbf{x}|\mathbf{x}') ]\!] + \zeta_2(\mathbf{x}_\parallel) [\![ \frac{\partial}{\partial x'_3} g_\varepsilon(\mathbf{x}|\mathbf{x}') ]\!] \right\} \frac{\partial}{\partial x'_1} J_H(\mathbf{x}'_\parallel|\omega)_1 \\
&+ \frac{i\omega}{2\pi c}\int d^2\mathbf{x}'_\parallel [\![ g_\varepsilon(\mathbf{x}|\mathbf{x}') ]\!] (1 + \zeta_2(\mathbf{x}_\parallel)\zeta_2(\mathbf{x}'_\parallel)) J_H(\mathbf{x}'_\parallel|\omega)_2 \\
&- \frac{ic}{2\pi\varepsilon(\omega)}P\int d^2x'_\parallel \left\{ [\![ \frac{\partial}{\partial x'_2} g_\varepsilon(\mathbf{x}|\mathbf{x}') ]\!] + \zeta_2(\mathbf{x}_\parallel) [\![ \frac{\partial}{\partial x'_3} g_\varepsilon(\mathbf{x}|\mathbf{x}') ]\!] \right\} \frac{\partial}{\partial x'_2} J_H(\mathbf{x}'_\parallel|\omega)_2
\end{aligned}
$$

$$
\begin{aligned}
J_E(\mathbf{x}_\parallel|\omega)_2 &= \frac{1}{2\pi}P\int d^2x'_\parallel [\![ \frac{\partial}{\partial x'_2} g_\varepsilon(\mathbf{x}|\mathbf{x}') ]\!] (\zeta_1(\mathbf{x}'_\parallel) - \zeta_1(\mathbf{x}_\parallel)) J_E(\mathbf{x}'_\parallel|\omega)_1 \\
&+ \frac{1}{2\pi}P\int d^2x'_\parallel \left\{ \zeta_1(\mathbf{x}_\parallel) [\![ \frac{\partial}{\partial x'_1} g_\varepsilon(\mathbf{x}|\mathbf{x}') ]\!] + [\![ \frac{\partial}{\partial x'_2} g_\varepsilon(\mathbf{x}|\mathbf{x}') ]\!] \zeta_2(\mathbf{x}'_\parallel) - [\![ \frac{\partial}{\partial x'_3} g_\varepsilon(\mathbf{x}|\mathbf{x}') ]\!] \right\} J_E(\mathbf{x}'_\parallel|\omega)_2 \\
&- \frac{i\omega}{2\pi c}\int d^2x'_\parallel [\![ g_\varepsilon(\mathbf{x}|\mathbf{x}') ]\!] (1 + \zeta_1(\mathbf{x}_\parallel)\zeta_1(\mathbf{x}'_\parallel)) J_H(\mathbf{x}'_\parallel|\omega)_1 \\
&+ \frac{ic}{2\pi\omega\varepsilon(\omega)}P\int d^2x'_\parallel \left\{ [\![ \frac{\partial}{\partial x'_1} g_\varepsilon(\mathbf{x}|\mathbf{x}') ]\!] + \zeta_1(\mathbf{x}_\parallel) [\![ \frac{\partial}{\partial x'_3} g_\varepsilon(\mathbf{x}|\mathbf{x}') ]\!] \right\} \frac{\partial}{\partial x'_1} J_H(\mathbf{x}'_\parallel|\omega)_1 \\
&- \frac{i\omega}{2\pi c}\int d^2x'_\parallel \zeta_1(\mathbf{x}_\parallel) [\![ g_\varepsilon(\mathbf{x}|\mathbf{x}') ]\!] \zeta_2(\mathbf{x}'_\parallel) J_H(\mathbf{x}'_\parallel|\omega)_2 \\
&+ \frac{ic}{2\pi\omega\varepsilon(\omega)}P\int d^2x'_\parallel \left\{ [\![ \frac{\partial}{\partial x'_1} g_\varepsilon(\mathbf{x}|\mathbf{x}') ]\!] + \zeta_1(\mathbf{x}_\parallel) [\![ \frac{\partial}{\partial x'_3} g_\varepsilon(\mathbf{x}|\mathbf{x}') ]\!] \right\} \frac{\partial}{\partial x'_2} J_H(\mathbf{x}'_\parallel|\omega)_2 .
\end{aligned}
$$

# Comments: Coupled Integral Equations

## Rigorous Equations

The just presented Coupled Integral Equations for $J_E(\mathbf{x}_{||}|\omega)_i$ and $J_H(\mathbf{x}_{||}|\omega)_i$ $(i = 1, 2)$ are *rigorous*, *i.e.* no approximations have been done in order to obtain them!

However, open questions remains:

1. How to handle terms like $\partial J_E(\mathbf{x}_{||}|\omega)_i/\partial x_i$ and $\partial J_H(\mathbf{x}_{||}|\omega)_i/\partial x_i$?
2. Is it practically feasible to implement and solve (within finite time) these (dense set of) equations [memory consumption being the main issue scaling like $(4N^2)^2 = 16N^4$ where $N$ denotes the linear no. of surface points]?
3. What about the memory footprint needed to hold the system matrix?
   - with $N = 100$ the memory requirement is 12Gb
   - with $N = 128$ the memory requirement is 32Gb
   - with $N = 256$ the memory requirement is 512Gb
   - with $N = 512$ the memory requirement is 8,192Gb

   For the case of a perfect conductor, the memory consumption is reduced by a factor four.

## Special Case: Perfect Conductor

An important special case is that of a Perfect Conductor for which $\mathbf{J}_E(\mathbf{x}_{||}|\omega) = 0$.
Hence, only two non-trivial source functions remains, *i.e.* $J_H(\mathbf{x}_{||}|\omega)_i$ $(i = 1, 2)$.

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# Numerical Solution of the Coupled Integral Equations

To solve these integral equations the following steps are taken:

1. The surface profile function $\zeta(\mathbf{x}_\parallel)$ is generated on a $N \times N$ square grid of points covering an area $L \times L$. The generation is carried out by the use of a two-dimensional filtering method.
2. We convert the coupled integral equations into matrix equations
   - The integrations over $\mathbf{x}'_\parallel$ are replaced by sums by the use of a two-dimensional version of the extended midpoint integration scheme, based on the grid points at which $\zeta(\mathbf{x}_\parallel)$ is generated. The integration mesh size is therefore $\Delta x = L/N$.
   - The variable $\mathbf{x}_\parallel$ is given the values corresponding to the positions of these grid points.
3. The resulting matrix equations are then solved by the *biconjugate gradient stabilized method*.
4. The solution is used to calculate the scattering amplitude $\mathscr{E}_\beta(\mathbf{q}_+, \omega)$ and $|\mathscr{E}_\beta(\mathbf{q}_+, \omega)|^2$.

Statistical averages are done by doing the following for each surface realization:

1. The procedure now is to generate a large number $N_\zeta$ of realizations of the surface profile function $\zeta(\mathbf{x}_\parallel)$, and for each realization to solve the scattering problem for an incident field of $p$ or $s$ polarization.
2. An arithmetic average of the $N_p$ results for these quantities yields the functions $\left|\langle \mathscr{E}_\beta(\mathbf{q}_+, \omega)\rangle\right|^2$ and $\langle|\mathscr{E}_\beta(\mathbf{q}_+, \omega)|^2\rangle$ entering the expressions for the mean differential reflection coefficient.

# Numerical Results : Perfect Conductor

For a Perfect Electric Conductor (PEC) one has $\mathbf{J}_E = 0$ so that the coupled integral equations to be solved is

$$\mathbf{J}_H(\mathbf{x}_\parallel|\omega) = 2\mathbf{J}_H^{(i)}(\mathbf{x}_\parallel|\omega) + \frac{1}{2\pi} P \int d^2x'_\parallel \, \mathbf{n} \times \Big( [\![ \nabla' g_0(\mathbf{x}|\mathbf{x}') ]\!] \times \mathbf{J}_H(\mathbf{x}'_\parallel|\omega) \Big),$$

or in component form

$$\begin{aligned} J_H(\mathbf{x}_\parallel|\omega)_1 = {} & 2J_H^{(i)}(\mathbf{x}_\parallel|\omega)_1 - \frac{1}{2\pi} P \int d^2x'_\parallel \Big\{ \Big[ g_3^{(0)}(\mathbf{x}_\parallel|\mathbf{x}'_\parallel) - g_1^{(0)}(\mathbf{x}_\parallel|\mathbf{x}'_\parallel)\zeta_1(\mathbf{x}'_\parallel) - \zeta_2(\mathbf{x}_\parallel) g_2^{(0)}(\mathbf{x}_\parallel|\mathbf{x}'_\parallel) \Big] J_H(\mathbf{x}'_\parallel|\omega)_1 \\ & + g_1^{(0)}(\mathbf{x}_\parallel|\mathbf{x}'_\parallel) \Big[ \zeta_2(\mathbf{x}_\parallel) - \zeta_2(\mathbf{x}'_\parallel) \Big] J_H(\mathbf{x}'_\parallel|\omega)_2 \Big\} \\ J_H(\mathbf{x}_\parallel|\omega)_2 = {} & 2J_H^{(i)}(\mathbf{x}_\parallel|\omega)_2 - \frac{1}{2\pi} P \int d^2x'_\parallel \Big\{ g_2^{(0)}(\mathbf{x}_\parallel|\mathbf{x}'_\parallel) \Big[ \zeta_1(\mathbf{x}_\parallel) - \zeta_1(\mathbf{x}'_\parallel) \Big] J_H(\mathbf{x}'_\parallel|\omega)_1 \\ & + \Big[ g_3^{(0)}(\mathbf{x}_\parallel|\mathbf{x}'_\parallel) - g_2^{(0)}(\mathbf{x}_\parallel|\mathbf{x}'_\parallel)\zeta_2(\mathbf{x}'_\parallel) - \zeta_1(\mathbf{x}_\parallel) g_1^{(0)}(\mathbf{x}_\parallel|\mathbf{x}'_\parallel) \Big] J_H(\mathbf{x}'_\parallel|\omega)_2 \Big\}, \end{aligned}$$

where

$$\begin{aligned} g_0(\mathbf{x}|\mathbf{x}') &= \frac{\exp\left[ i \frac{\omega}{c} |\mathbf{x} - \mathbf{x}'| \right]}{|\mathbf{x} - \mathbf{x}'|}, \\ g_l^{(0)}(\mathbf{x}_\parallel|\mathbf{x}'_\parallel) &= [\![ \frac{\partial}{\partial x_l} g_0(\mathbf{x}|\mathbf{x}') ]\!] = (x_l - x'_l) \left[ \frac{i(\omega/c)}{|\mathbf{x} - \mathbf{x}'|^2} - \frac{1}{|\mathbf{x} - \mathbf{x}'|^3} \right] \exp[i(\omega/c)|\mathbf{x} - \mathbf{x}'|] \Big|_{\substack{x_3 = \zeta(\mathbf{x}_\parallel) \\ x'_3 = \zeta(\mathbf{x}'_\parallel)}}, \\ \zeta_i(\mathbf{x}_\parallel) &= \frac{\partial \zeta(\mathbf{x}_\parallel)}{\partial x_i}. \end{aligned}$$

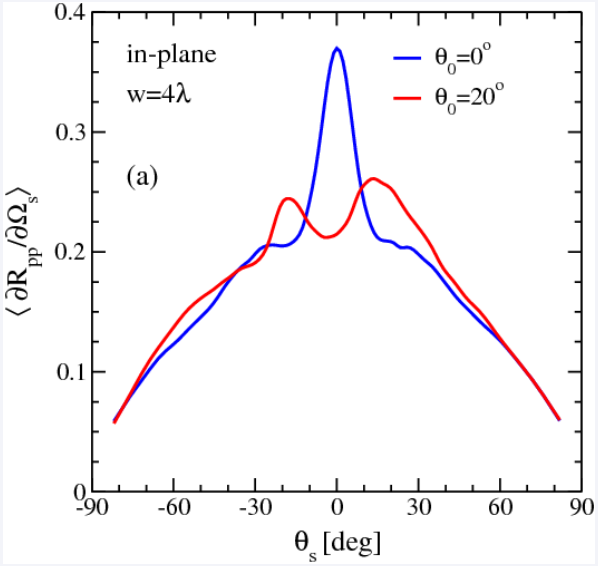
# Numerical Results : Perfect Conductor

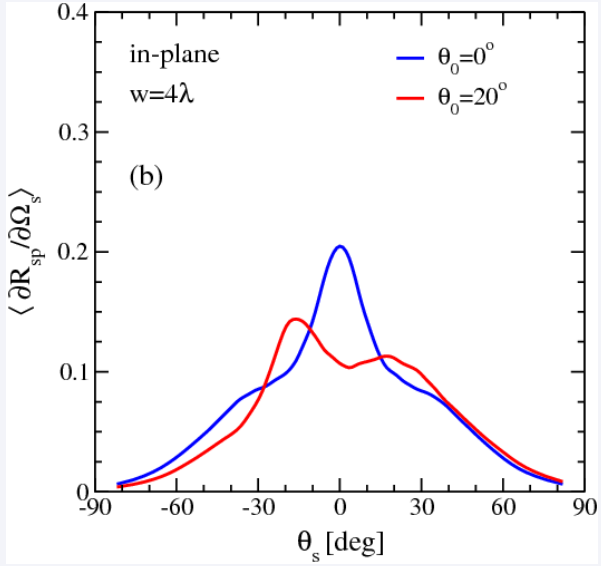
We start by presenting results for the wave-scattering from a Perfectly Conducting surface.

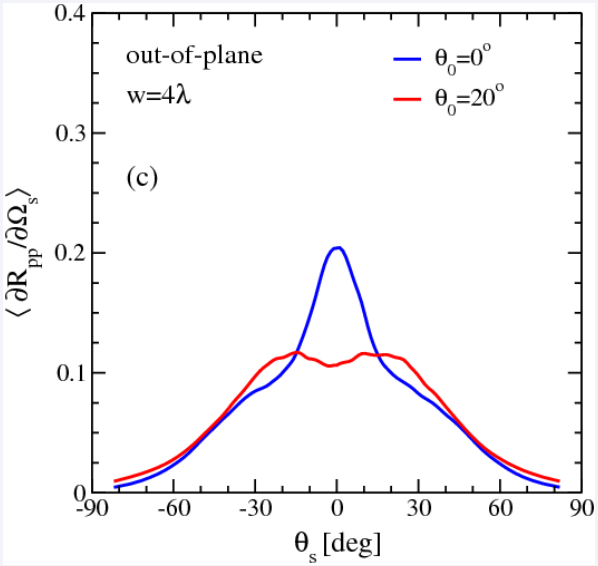
- Surface Properties : $\zeta(\mathbf{x}_{\|})$ is a Gaussian random process of Gaussian correlation function
- Surface statistical properties : $\delta = \lambda$, $a = 2\lambda$
- Surface length : $L = 16\lambda$
- Surface points : $N = 112$; $\Delta x = \lambda/7$
- No. realizations : $N_{\zeta} = 30,000$
- Incident field : p- or s-polarized plane wave

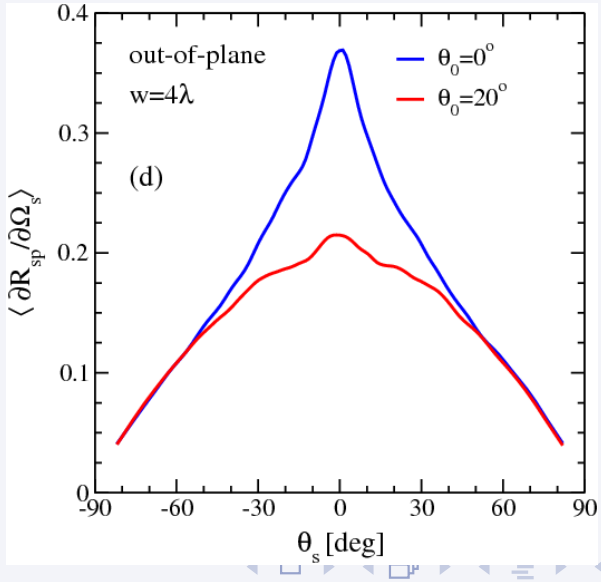
# Perfect Conductor; p-polarization

p-polarized plane incident wave; $\delta = \lambda$ and $a = 2\lambda$; $N_\zeta = 30,000$



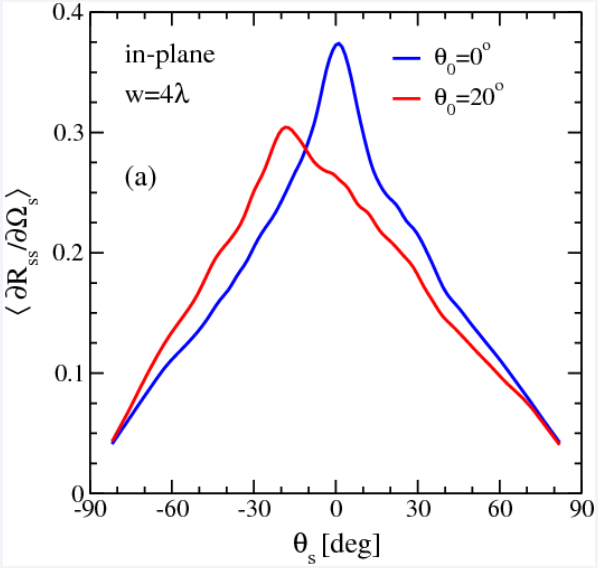



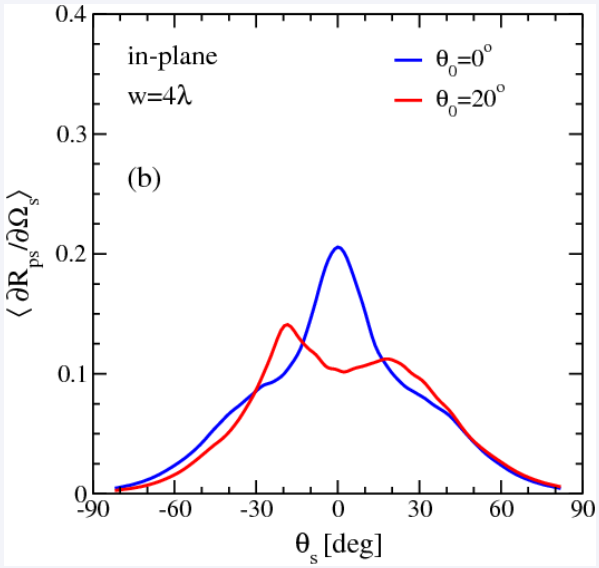



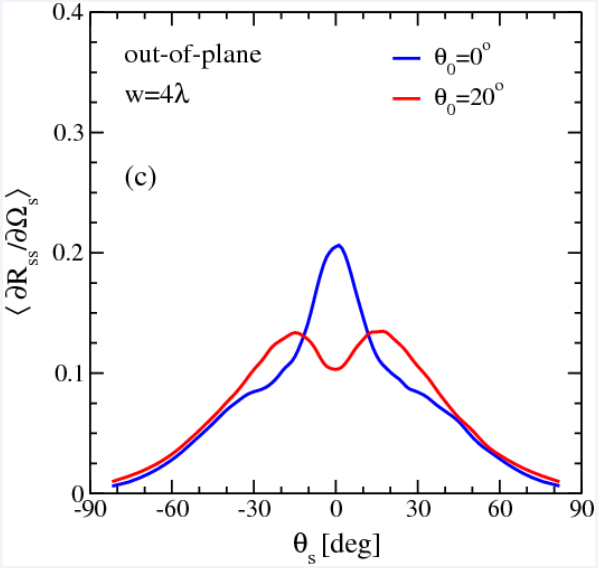



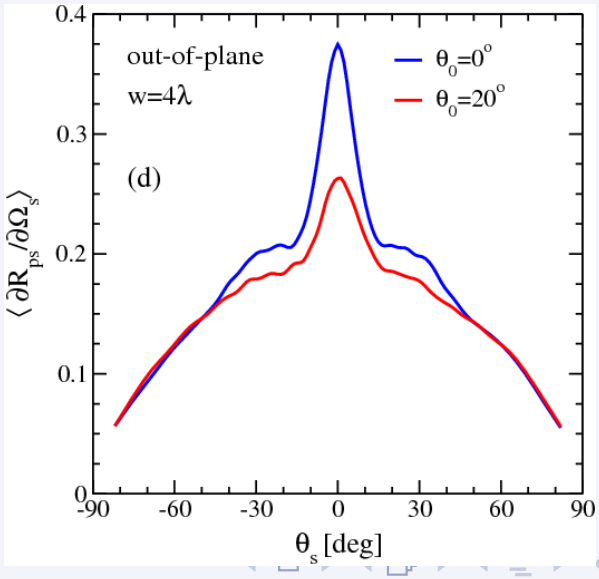
# Perfect Conductor; s-polarization

s-polarized plane incident wave; $\delta = \lambda$ and $a = 2\lambda$; $N_\zeta = 30,000$



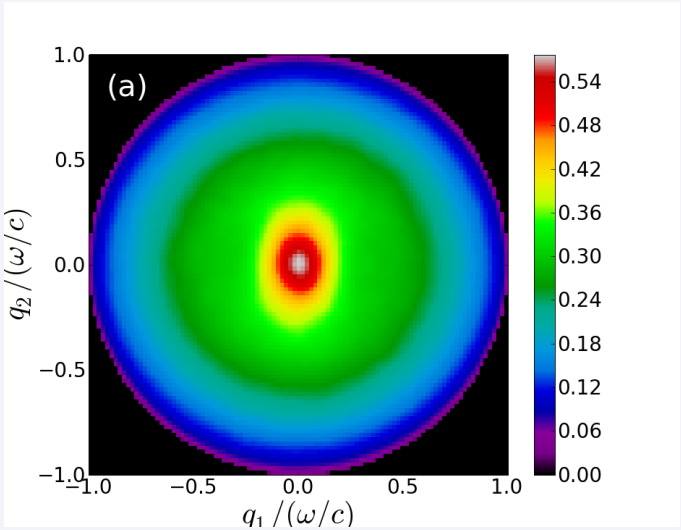



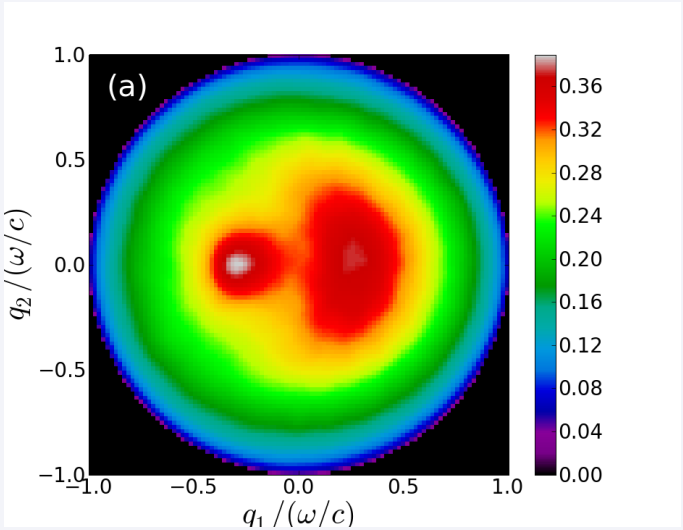



# Perfect Conductor; Angular distribution

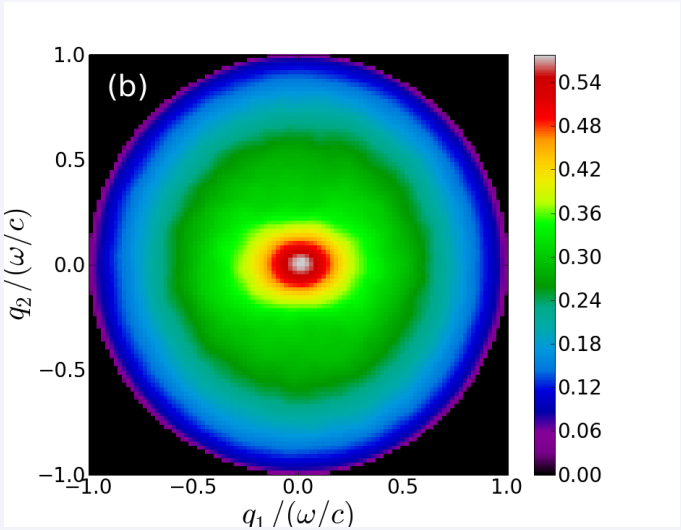
$\delta = \lambda$ and $a = 2\lambda$, the sum of s- and p-polarized scattered light is recorded



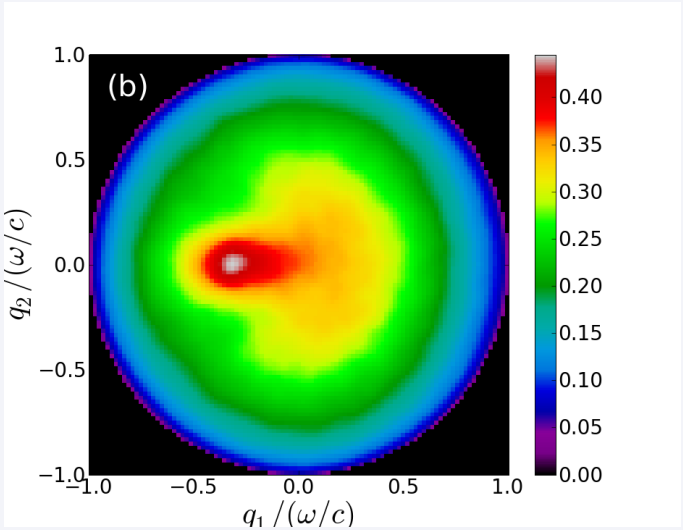

(a) p-polarization; $\theta_0 = 0^o$




(b) p-polarization; $\theta_0 = 20^o$



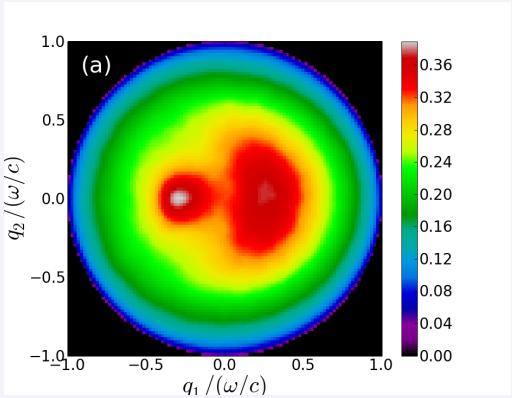

(c) s-polarization; $\theta_0 = 0^o$



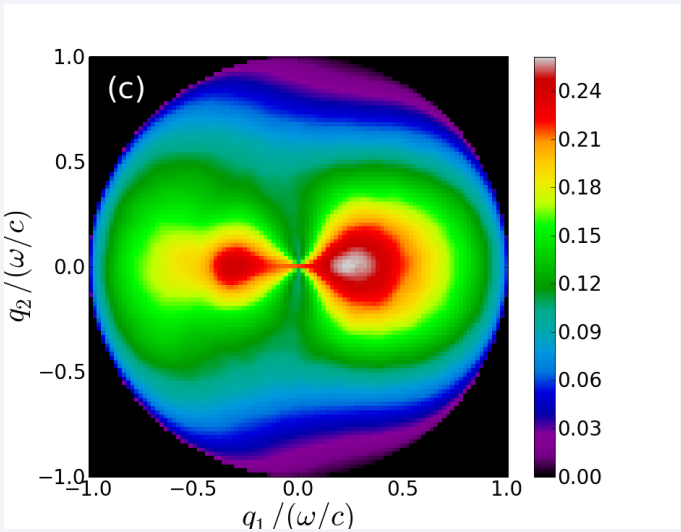

(d) s-polarization $\theta_0 = 20^o$

# Perfect Conductor: How is the scattered field polarized?

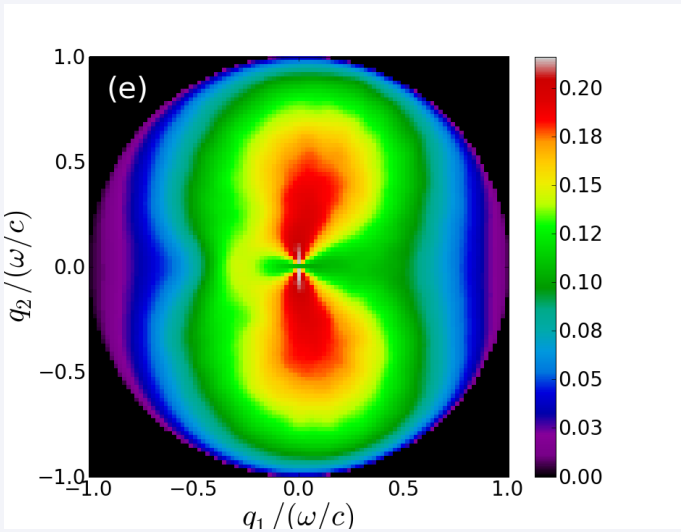
One may wonder how the scattered light is polarized! Take for instance the results we showed previously for $\theta_0 = 20^o$ and p-polarized incident light.




(e) $p \rightarrow sp$

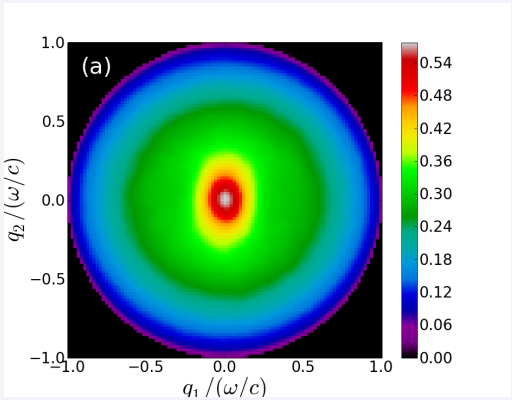


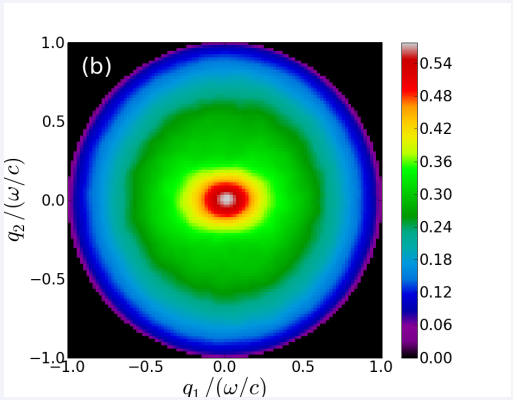

(f) $p \rightarrow p$

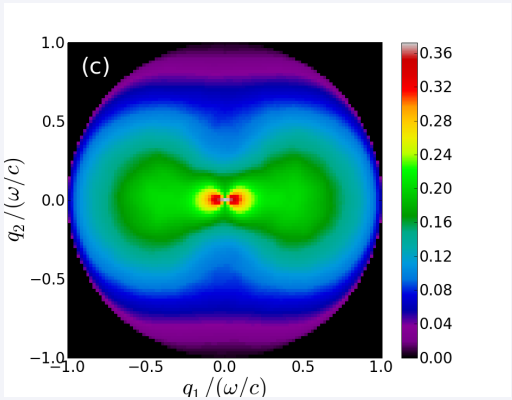


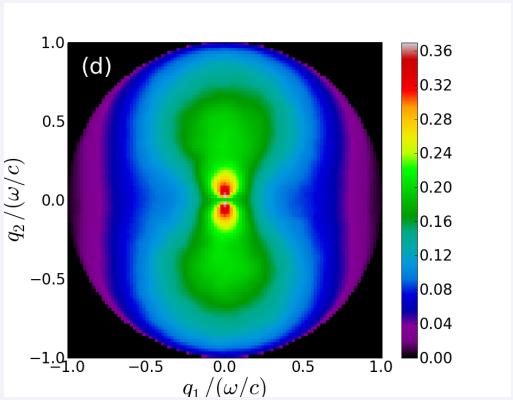

(g) $p \rightarrow s$

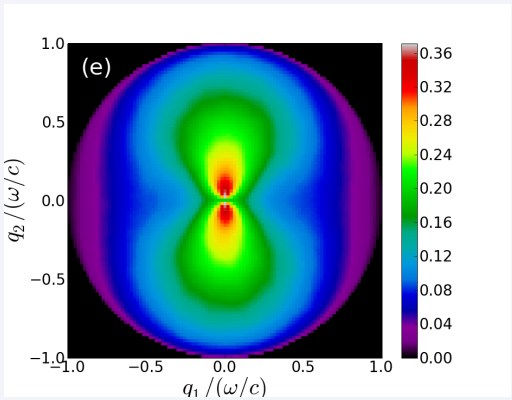
# Perfect Conductor; s-polarization

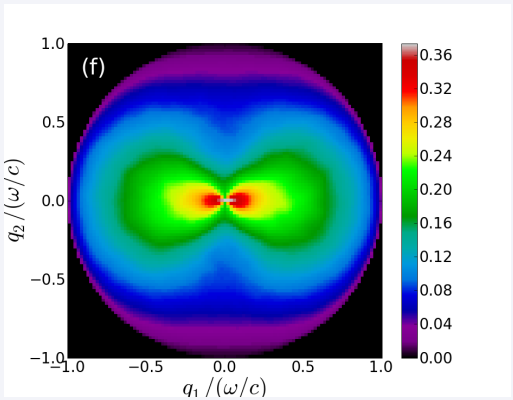


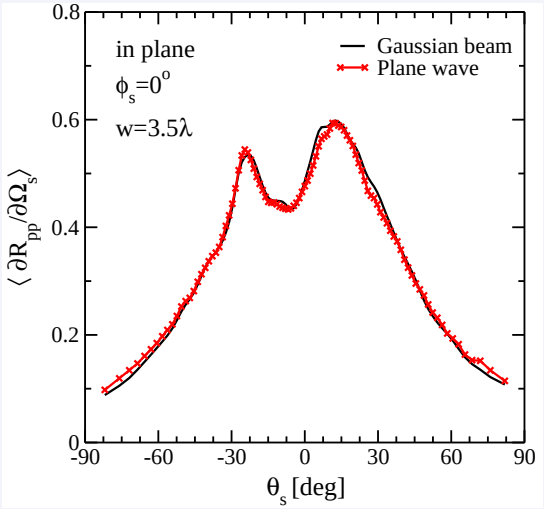



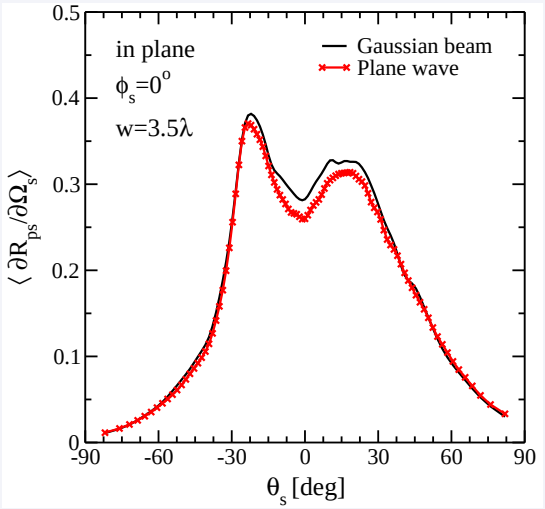



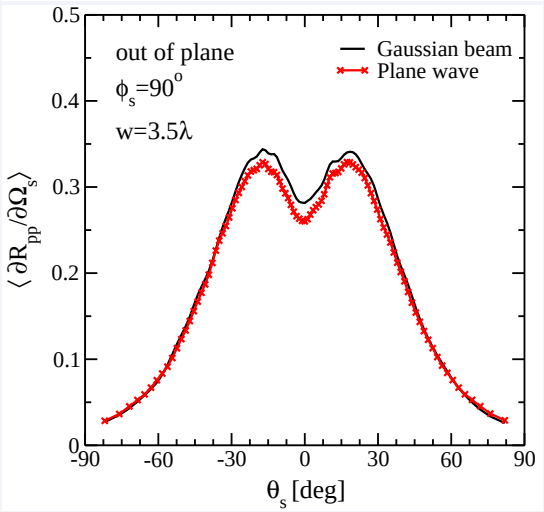



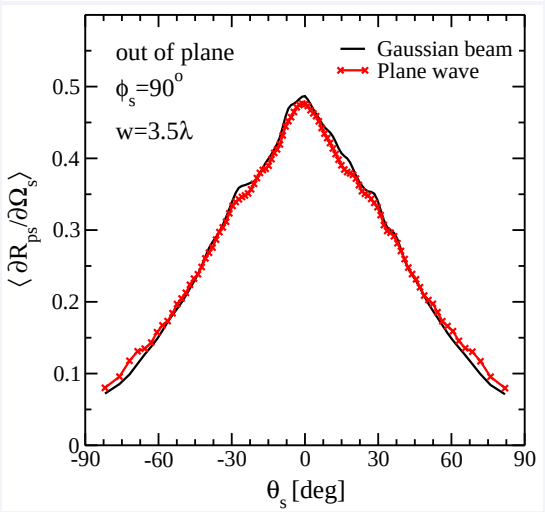
# Numerical Results : What About Edge Effects?

A concern when using a plane incident wave is edge effects.




At least for the parameters used in the numerical simulations, there seems to be only limited effects of the edges when using a plane incident wave.

# Computational Details

The total CPU time per sample used for these calculations was : 76s per angles of incidence (on a 2.83GHz Intel Core2 (Q95550))

1. The surface roughness $x_3 = \zeta(x_{\|})$
   - Linear dimensions : $L = 16\lambda$
   - Discretization (on $N \times N$ grid) : $N = 112$ (*i.e.* $\Delta x = \lambda/7$)
2. The CPU time per sample was distributed as:
   - 36s : Setting up the matrix system :
   - 31s : Solving the matrix system by BiCGStab method
   - 9s : for calculating the scattering amplitudes
   - "0s " : for the rest of the calculation (can be neglected))
3. Memory footprint was about $4.69Gb$
4. Unitarity (energy conservation) is within a few percent of 1

# Computational Details : Performance

| $N$ | $t_{tot}$[s] | $t_A$[s] | $t_{A\mathbf{x}=\mathbf{b}}$[s] | | $t_{\mathscr{E}}$[s] | $\mathscr{N}$ | $\mathscr{M}_A$[Gb] |
|---|---|---|---|---|---|---|---|
| | | | BiCGStab | LU | | | |
| 64 | 10.5 | 4.0 | 3.5 | 127 | 3.0 | 8192 | 0.50 |
| 80 | 22.0 | 9.5 | 8.0 | 474 | 4.5 | 12800 | 1.22 |
| 100 | 58.5 | 23.0 | 28.5 | 1780 | 7.0 | 20000 | 2.98 |
| 112 | 76.0 | 36.0 | 31.0 | 3540 | 9.0 | 25088 | 4.69 |

Table: The CPU time spent on various stages of the calculations for one realization of the surface profile function and one angle of incidence. All CPU times are measured in seconds, and the numbers have been rounded to the closest half second, and they refer to a machine running an Intel Core2 CPU (Q9550) operating at $2.83\,$GHz and running the Linux operating system. The surface was discretized on a $N \times N$ grid of points. The reported CPU times are: the total CPU time spent for simulating one surface realization for one angle of incidence including reading and writing of data ($t_{tot}$); the setup of the system matrix of the linear system $A\mathbf{x} = \mathbf{b}$ determining the surface currents ($t_A$); the time to solve this system by a the iterative BiCGStab method or the direct LU decomposition method ($t_{A\mathbf{x}=\mathbf{b}}$); and finally the time to calculate the reflection amplitudes, $\mathscr{E}(\mathbf{q}_+, \omega)$ for both scattered polarizations on a grid of $101 \times 101$ points ($t_{\mathscr{E}}$). The number of unknowns to be solved for is $\mathscr{N} = 2N^2$, where the memory (in Gigabytes (Gb)) required to hold the complex system matrix $A$, using single precision, is denoted by $\mathscr{M}_A \propto \mathscr{N}^2 = 4N^4$.

# Computational Details : Accuracy

Fraction of incident energy that is scattered by the surface:

$$\mathscr{U}_{\alpha}^{\beta}(\theta_0,\phi_0) \quad = \quad \int d\Omega_s \left\langle \frac{\partial R_{\beta\alpha}}{\partial \Omega_s} \right\rangle,$$

where $\alpha$ and $\beta$ denotes the polarization of the incident and scattered light Since a PEC does not absorb energy, one must have

$$\mathscr{U}(\theta_0,\phi_0) = \sum_{\alpha=p,s} \mathscr{U}_{\alpha}(\theta_0,\phi_0) = \sum_{\beta=p,s}\sum_{\alpha=p,s} \mathscr{U}_{\alpha}^{\beta}(\theta_0,\phi_0) = 1.$$

| $\theta_0$ [deg] | $\alpha$ | $\mathscr{U}$ | $\mathscr{U}_{\text{incoh}}$ | $\mathscr{U}_{\text{coh}}[10^{-4}]$ | $\mathscr{U}_{\alpha}^{p}/\mathscr{U}$ | $\mathscr{U}_{\alpha}^{s}/\mathscr{U}$ |
|---|---|---|---|---|---|---|
| 0 | $p$ | 0.9976 | 0.9975 | 0.9 | 0.5054 | 0.4946 |
| 20 | $p$ | 0.9962 | 0.9961 | 0.9 | 0.5315 | 0.4686 |
| 40 | $p$ | 0.9951 | 0.9947 | 3.8 | 0.5407 | 0.4592 |
| 0 | $s$ | 0.9970 | 0.9967 | 3.1 | 0.5021 | 0.4979 |
| 20 | $s$ | 0.9966 | 0.9963 | 2.8 | 0.4939 | 0.5061 |
| 40 | $s$ | 0.9953 | 0.9948 | 4.9 | 0.4834 | 0.5166 |

Table: The energy conservation for various polar angles of incidence ($\theta_0$) and incidence polarizations ($\alpha$) for the surface parameters given in the text. The surface and scattering amplitude were discretized on $112 \times 112$ and $101 \times 101$ grids, respectively. (**??**).

# Impedance Boundary Condition

One may approximatively treat the scattering from a metal by use of the (no-rigorous) impedance boundary condition. This is achieved by introducing the (local) Impedance Tensor $K_{ij}^{(0)}(\mathbf{x}_\parallel|\omega)$ defined by

$$J_E(\mathbf{x}_\parallel|\omega)_i = K_{ij}^{(0)}(\mathbf{x}_\parallel|\omega)\, J_H(\mathbf{x}_\parallel|\omega)_j \qquad j=1,2$$

where the Impedance Tensor is expanded in the skin depth, $d(\omega)$, of the metal

$$\begin{aligned}
K_{11}^{(0)}(\mathbf{x}_\parallel|\omega) &= i\frac{\omega d}{c\phi}\left\{\zeta_1\zeta_2 - \frac{d}{2\phi^3}\left[\zeta_{11}\left(1+\zeta_2^2\right)\zeta_1\zeta_2 - 2\zeta_{12}\left(1+\zeta_1^2\right)\left(1+\zeta_2^2\right) + \zeta_{22}\left(1+\zeta_1^2\right)\zeta_1\zeta_2\right] + O(d^2)\right\}\\
K_{12}^{(0)}(\mathbf{x}_\parallel|\omega) &= i\frac{\omega d}{c\phi}\left\{\left(1+\zeta_2^2\right) - \frac{d}{2\phi^3}\left[\zeta_{11}\left(1+\zeta_2^2\right)^2 - 2\zeta_{12}\left(1+\zeta_2^2\right)\zeta_1\zeta_2 - \zeta_{22}\left[\left(1+\zeta_1^2\right)\left(1+\zeta_2^2\right) - 2\zeta_1^2\zeta_2^2\right]\right] + O(d^2)\right\}\\
K_{21}^{(0)}(\mathbf{x}_\parallel|\omega) &= i\frac{\omega d}{c\phi}\left\{-\left(1+\zeta_1^2\right) - \frac{d}{2\phi^3}\left[\zeta_{11}\left[\left(1+\zeta_1^2\right)\left(1+\zeta_2^2\right) - 2\zeta_1^2\zeta_2^2\right] + 2\zeta_{12}\left(1+\zeta_1^2\right)\zeta_1\zeta_2 - \zeta_{22}\left(1+\zeta_1^2\right)^2\right] + O(d^2)\right\}\\
K_{22}^{(0)}(\mathbf{x}_\parallel|\omega) &= i\frac{\omega d}{c\phi}\left\{-\zeta_1\zeta_2 - \frac{d}{2\phi^3}\left[-\zeta_{11}\left(1+\zeta_2^2\right)\zeta_1\zeta_2 + 2\zeta_{12}\left(1+\zeta_1^2\right)\left(1+\zeta_2^2\right) - \zeta_{22}\left(1+\zeta_1^2\right)\zeta_1\zeta_2\right] + O(d^2)\right\},
\end{aligned}$$

with

$$d(\omega) = \frac{\omega}{c}\frac{1}{\sqrt{-\varepsilon(\omega)}}, \qquad Re\, d(\omega) > 0, \quad Im\, d(\omega) > 0,$$

and

$$\zeta_{ij} \equiv \frac{\partial^2\zeta(\mathbf{x}_\parallel)}{\partial x_i \partial x_j}, \qquad \phi(\mathbf{x}_\parallel) = \left[1+\zeta_1^2(\mathbf{x}_\parallel)+\zeta_2^2(\mathbf{x}_\parallel)\right]^{\frac{1}{2}}.$$

**The Advantage of the Impedance Boundary Conditions**

By using the Impedance Boundary Conditions the memory footprint of the simulations is not increased, and one may approximately treat scattering from good conductors

# Impedance Boundary Condition

One may approximatively treat the scattering from a metal by use of the (no-rigorous) impedance boundary condition. This is achieved by introducing the (local) Impedance Tensor $K_{ij}^{(0)}(\mathbf{x}_\parallel|\omega)$ defined by

$$J_E(\mathbf{x}_\parallel|\omega)_i = K_{ij}^{(0)}(\mathbf{x}_\parallel|\omega)\, J_H(\mathbf{x}_\parallel|\omega)_j \qquad j=1,2$$

where the Impedance Tensor is expanded in the skin depth, $d(\omega)$, of the metal

$$\begin{aligned}
K_{11}^{(0)}(\mathbf{x}_\parallel|\omega) &= i\frac{\omega d}{c\phi}\left\{\zeta_1\zeta_2 - \frac{d}{2\phi^3}\left[\zeta_{11}\left(1+\zeta_2^2\right)\zeta_1\zeta_2 - 2\zeta_{12}\left(1+\zeta_1^2\right)\left(1+\zeta_2^2\right) + \zeta_{22}\left(1+\zeta_1^2\right)\zeta_1\zeta_2\right] + O(d^2)\right\}\\
K_{12}^{(0)}(\mathbf{x}_\parallel|\omega) &= i\frac{\omega d}{c\phi}\left\{\left(1+\zeta_2^2\right) - \frac{d}{2\phi^3}\left[\zeta_{11}\left(1+\zeta_2^2\right)^2 - 2\zeta_{12}\left(1+\zeta_2^2\right)\zeta_1\zeta_2 - \zeta_{22}\left[\left(1+\zeta_1^2\right)\left(1+\zeta_2^2\right) - 2\zeta_1^2\zeta_2^2\right]\right] + O(d^2)\right\}\\
K_{21}^{(0)}(\mathbf{x}_\parallel|\omega) &= i\frac{\omega d}{c\phi}\left\{-\left(1+\zeta_1^2\right) - \frac{d}{2\phi^3}\left[\zeta_{11}\left[\left(1+\zeta_1^2\right)\left(1+\zeta_2^2\right) - 2\zeta_1^2\zeta_2^2\right] + 2\zeta_{12}\left(1+\zeta_1^2\right)\zeta_1\zeta_2 - \zeta_{22}\left(1+\zeta_1^2\right)^2\right] + O(d^2)\right\}\\
K_{22}^{(0)}(\mathbf{x}_\parallel|\omega) &= i\frac{\omega d}{c\phi}\left\{-\zeta_1\zeta_2 - \frac{d}{2\phi^3}\left[-\zeta_{11}\left(1+\zeta_2^2\right)\zeta_1\zeta_2 + 2\zeta_{12}\left(1+\zeta_1^2\right)\left(1+\zeta_2^2\right) - \zeta_{22}\left(1+\zeta_1^2\right)\zeta_1\zeta_2\right] + O(d^2)\right\},
\end{aligned}$$

with

$$d(\omega) = \frac{\omega}{c}\frac{1}{\sqrt{-\varepsilon(\omega)}}, \qquad Re\, d(\omega) > 0, \quad Im\, d(\omega) > 0,$$

and

$$\zeta_{ij} \equiv \frac{\partial^2 \zeta(\mathbf{x}_\parallel)}{\partial x_i \partial x_j}, \qquad \phi(\mathbf{x}_\parallel) = \left[1 + \zeta_1^2(\mathbf{x}_\parallel) + \zeta_2^2(\mathbf{x}_\parallel)\right]^{\frac{1}{2}}.$$

**The Advantage of the Impedance Boundary Conditions**

By using the Impedance Boundary Conditions the memory footprint of the simulations is not increased, and one may approximately treat scattering from good conductors

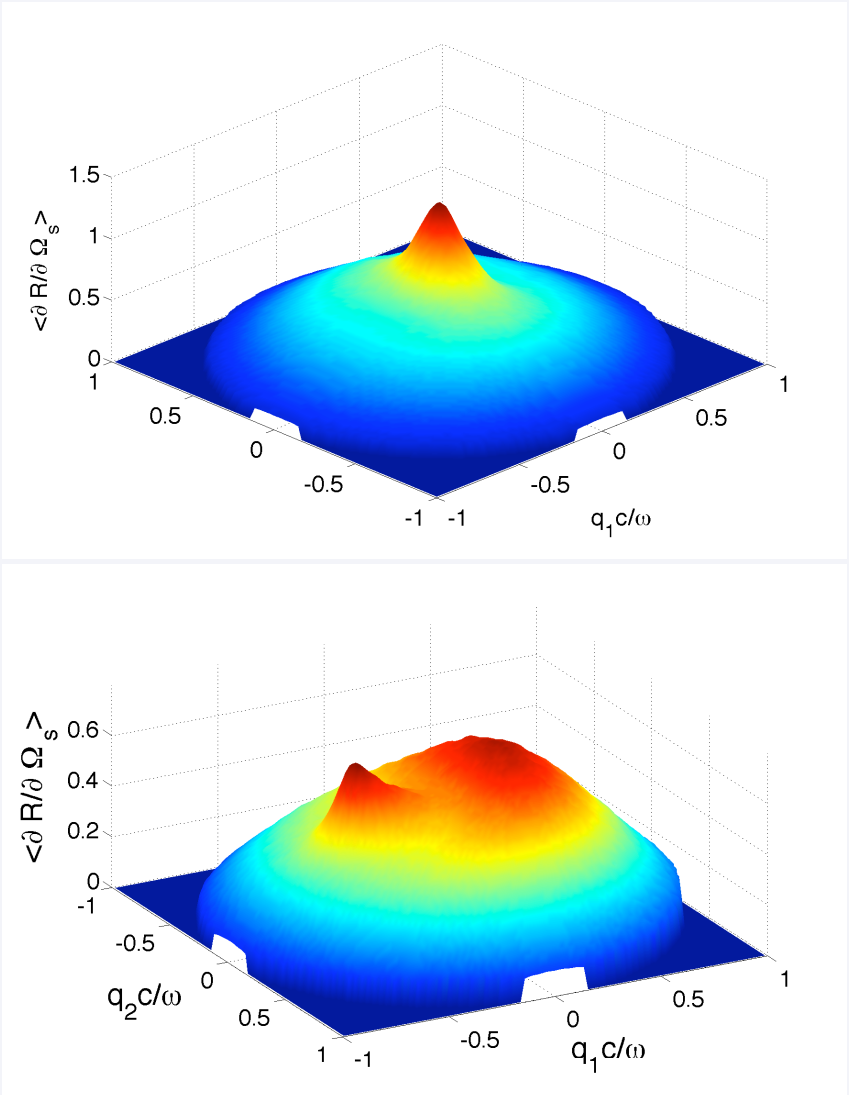
When using the Impedance Boundary Conditions, the Coupled Integral Equations reduce to:

$$
\begin{aligned}
J_H(\mathbf{x}_\parallel|\omega)_1 &= 2J_H^{(i)}(\mathbf{x}_\parallel|\omega)_1 \\
&- \frac{1}{2\pi}P\int d^2x'_\parallel \left\{ [\![ \frac{\partial}{\partial x'_1} g_0(\mathbf{x}|\mathbf{x}')]\!]\zeta_1(\mathbf{x}'_\parallel) + \zeta_2(\mathbf{x}_\parallel)[\![ \frac{\partial}{\partial x'_2} g_0(\mathbf{x}|\mathbf{x}')]\!] - [\![ \frac{\partial}{\partial x'_3} g_0(\mathbf{x}|\mathbf{x}')]\!] \right\} J_H(\mathbf{x}'_\parallel|\omega)_1 \\
&+ \frac{i\omega}{2\pi c}\int d^2x'_\parallel \zeta_2(\mathbf{x}_\parallel)[\![ g_0(\mathbf{x}|\mathbf{x}')]\!]\zeta_1(\mathbf{x}'_\parallel)K_{11}^{(0)}(\mathbf{x}'_\parallel|\omega)J_H(\mathbf{x}'_\parallel|\omega)_1 \\
&+ \frac{i\omega}{2\pi c}\int d^2x'_\parallel [\![ g_0(\mathbf{x}|\mathbf{x}')]\!](1+\zeta_2(\mathbf{x}_\parallel)\zeta_2(\mathbf{x}'_\parallel))K_{21}^{(0)}(\mathbf{x}'_\parallel|\omega)J_H(\mathbf{x}'_\parallel|\omega)_1 \\
&- \frac{ic}{2\pi\omega}P\int d^2x'_\parallel \left\{ [\![ \frac{\partial}{\partial x'_2} g_0(\mathbf{x}|\mathbf{x}')]\!] + \zeta_2(\mathbf{x}_\parallel)[\![ \frac{\partial}{\partial x'_3} g_0(\mathbf{x}|\mathbf{x}')]\!] \right\} \left\{ \frac{\partial}{\partial x'_1}[K_{11}^{(0)}(\mathbf{x}'_\parallel|\omega)J_H(\mathbf{x}'_\parallel|\omega)_1] + \frac{\partial}{\partial x'_2}[K_{21}^{(0)}(\mathbf{x}'_\parallel|\omega)J_H(\mathbf{x}'_\parallel|\omega)_1] \right\} \\
&- \frac{1}{2\pi}P\int d^2x'_\parallel [\![ \frac{\partial}{\partial x'_1} g_0(\mathbf{x}|\mathbf{x}')]\!](\zeta_2(\mathbf{x}'_\parallel) - \zeta_2(\mathbf{x}_\parallel))J_H(\mathbf{x}'_\parallel|\omega)_2 \\
&+ \frac{i\omega}{2\pi c}\int d^2x'_\parallel \zeta_2(\mathbf{x}_\parallel)[\![ g_0(\mathbf{x}|\mathbf{x}')]\!]\zeta_1(\mathbf{x}'_\parallel)K_{12}^{(0)}(\mathbf{x}'_\parallel|\omega)J_H(\mathbf{x}_\parallel|\omega)_2 \\
&+ \frac{i\omega}{2\pi c}\int d^2x'_\parallel [\![ g_0(\mathbf{x}|\mathbf{x}')]\!](1+\zeta_2(\mathbf{x}_\parallel)\zeta_2(\mathbf{x}'_\parallel))K_{22}^{(0)}(\mathbf{x}'_\parallel|\omega)J_H(\mathbf{x}'_\parallel|\omega)_2 \\
&- \frac{ic}{2\pi\omega}P\int d^2x'_\parallel \left\{ [\![ \frac{\partial}{\partial x'_2} g_0(\mathbf{x}|\mathbf{x}')]\!] + \zeta_2(\mathbf{x}_\parallel)[\![ \frac{\partial}{\partial x'_3} g_0(\mathbf{x}|\mathbf{x}')]\!] \right\} \left\{ \frac{\partial}{\partial x'_1}[K_{12}^{(0)}(\mathbf{x}'_\parallel|\omega)J_H(\mathbf{x}'_\parallel|\omega)_2] + \frac{\partial}{\partial x'_2}[K_{22}^{(0)}(\mathbf{x}'_\parallel|\omega)J_H(\mathbf{x}'_\parallel|\omega)_2] \right\}
\end{aligned}
$$

$$
\begin{aligned}
J_H(\mathbf{x}_\parallel|\omega)_2 &= 2J_H^{(i)}(\mathbf{x}_\parallel|\omega)_2 - \frac{1}{2\pi}P\int d^2x'_\parallel [\![ \frac{\partial}{\partial x'_2} g_0(\mathbf{x}|\mathbf{x}')]\!](\zeta_1(\mathbf{x}'_\parallel) - \zeta_1(\mathbf{x}_\parallel))J_H(\mathbf{x}'_\parallel|\omega)_1 \\
&- \frac{i\omega}{2\pi c}\int d^2x'_\parallel [\![ g_0(\mathbf{x}|\mathbf{x}')]\!](1+\zeta_1(\mathbf{x}_\parallel)\zeta_1(\mathbf{x}'_\parallel))K_{11}^{(0)}(\mathbf{x}'_\parallel|\omega)J_H(\mathbf{x}'_\parallel|\omega)_1 \\
&- \frac{i\omega}{2\pi c}\int d^2x'_\parallel \zeta_1(\mathbf{x}_\parallel)[\![ g_0(\mathbf{x}|\mathbf{x}')]\!]\zeta_2(\mathbf{x}'_\parallel)K_{21}^{(0)}(\mathbf{x}'_\parallel|\omega)J_H(\mathbf{x}'_\parallel|\omega)_1 \\
&+ \frac{ic}{2\pi\omega}P\int d^2x'_\parallel \left\{ [\![ \frac{\partial}{\partial x'_1} g_0(\mathbf{x}|\mathbf{x}')]\!] + \zeta_1(\mathbf{x}_\parallel)[\![ \frac{\partial}{\partial x'_3} g_0(\mathbf{x}|\mathbf{x}')]\!] \right\} \left\{ \frac{\partial}{\partial x'_1}[K_{11}^{(0)}(\mathbf{x}'_\parallel|\omega)J_H(\mathbf{x}'_\parallel|\omega)_1] + \frac{\partial}{\partial x'_2}[K_{21}^{(0)}(\mathbf{x}'_\parallel|\omega)J_H(\mathbf{x}'_\parallel|\omega)_1] \right\} \\
&- \frac{1}{2\pi}P\int d^2x'_\parallel \left\{ \zeta_1(\mathbf{x}_\parallel)[\![ \frac{\partial}{\partial x'_1} g_0(\mathbf{x}|\mathbf{x}')]\!] + [\![ \frac{\partial}{\partial x'_2} g_0(\mathbf{x}|\mathbf{x}')]\!]\zeta_2(\mathbf{x}'_\parallel) - [\![ \frac{\partial}{\partial x'_3} g_0(\mathbf{x}|\mathbf{x}')]\!] \right\} J_H(\mathbf{x}'_\parallel|\omega)_2 \\
&- \frac{i\omega}{2\pi c}\int d^2x'_\parallel [\![ g_0(\mathbf{x}|\mathbf{x}')]\!](1+\zeta_1(\mathbf{x}_\parallel)\zeta_1(\mathbf{x}'_\parallel))K_{12}^{(0)}(\mathbf{x}'_\parallel|\omega)J_H(\mathbf{x}'_\parallel|\omega)_2 \\
&- \frac{i\omega}{2\pi c}\int d^2x'_\parallel \zeta_1(\mathbf{x}_\parallel)[\![ g_0(\mathbf{x}|\mathbf{x}')]\!]\zeta_2(\mathbf{x}'_\parallel)K_{22}^{(0)}(\mathbf{x}'_\parallel|\omega)J_H(\mathbf{x}'_\parallel|\omega)_2 \\
&+ \frac{ic}{2\pi\omega}P\int d^2x'_\parallel \left\{ [\![ \frac{\partial}{\partial x'_1} g_0(\mathbf{x}|\mathbf{x}')]\!] + \zeta_1(\mathbf{x}_\parallel)[\![ \frac{\partial}{\partial x'_3} g_0(\mathbf{x}|\mathbf{x}')]\!] \right\} \left\{ \frac{\partial}{\partial x'_1}[K_{12}^{(0)}(\mathbf{x}'_\parallel|\omega)J_H(\mathbf{x}'_\parallel|\omega)_2] + \frac{\partial}{\partial x'_2}[K_{22}^{(0)}(\mathbf{x}'_\parallel|\omega)J_H(\mathbf{x}'_\parallel|\omega)_2] \right\}.
\end{aligned}
$$

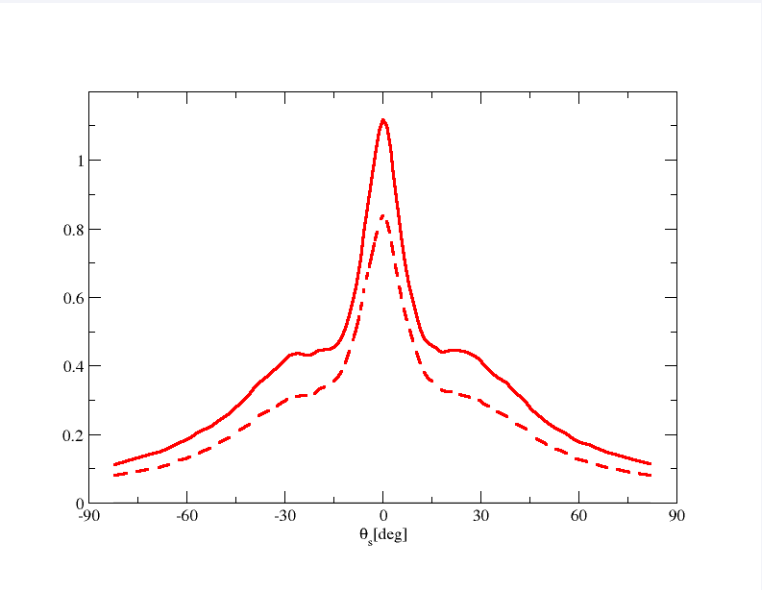
# Numerical results: Metal (Impedance Boundary Condition)

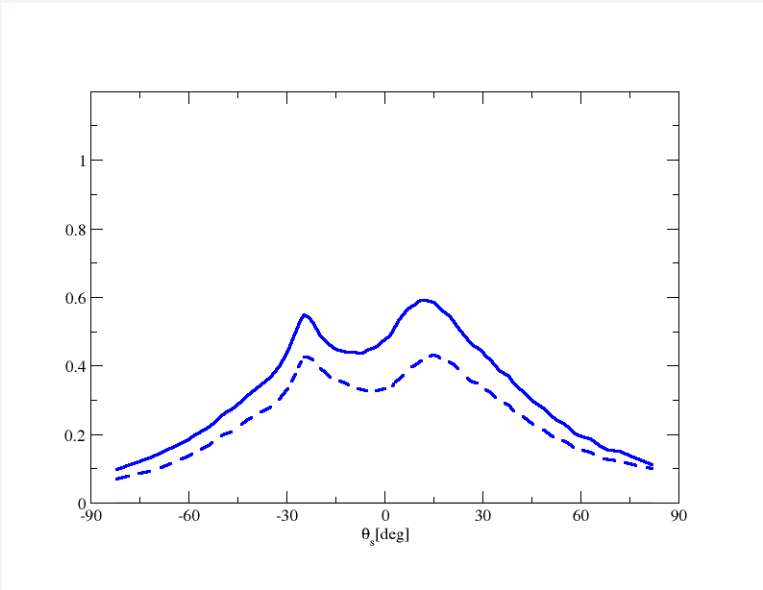
Results for a randomly rough silver surface of $\delta = \lambda$, $a = 2\lambda$ and $\lambda = 1\mu$m. At this wavelength $\varepsilon(\omega) = -45.68 + i2.9$. The incident wave was a p-polarized plane wave.



# Comparison of the Perfect Conductor and Metal case for $\lambda = 1\mu$m

Using the same parameters as for as before ($\delta = \lambda$, $a = 2\lambda$, $\lambda = 1\mu$m and $\varepsilon(\omega) = -45.68 + i2.9$) we below compare the simulation results for a Perfect Conductor (solid lines) to those of a silver (bashed lines) for $\theta_0 = 0^o$(left) and $\theta_0 = 25^o$(right).



# Computational Details

The total CPU time per sample used for these calculations was : 153s for two angles of incidence (on a 2.83GHz Intel Core2 Quad (Q9550))

1. The surface roughness $x_3 = \zeta(x_{\|})$
   - Linear dimensions : $L = 14.286\lambda$
   - Discretization (on $N \times N$ grid) : $N = 100$ (*i.e.* $\Delta x = \lambda/7$)
2. The CPU time per sample was distributed as:
   - 87s : Setting up the matrix system
   - 26.5s : Solving the matrix system by BiCGStab method
   - 6.5s : for calculating the scattering amplitudes
   - 1s : for the rest of the calculation
3. Memory footprint was about $3Gb$

# Directions for Further Research

Directions for further research are:

1. Fully parallelize the code in order to
   - speed up the calculations
   - rigorously handle the scattering problem for metallic/dielectric surfaces
   - study transmission
2. more detailed studies of polarization properties of the scattered light
3. comparison to angular resolved *experimental* measurments
4. Stokes and Mueller-matrix formalism
5. speckle correlations

# Conclusions

- We have successfully solved numerically the two-dimensional rough surface scattering problem for both perfect conductors and metals.
- We have shown that the use of the Stratton-Chu formulas, an impedance boundary condition for a two-dimensional rough metal surface, the method of moments and the biconjugate gradient stabilized iterative method provides a nearly exact solution to the problem of the scattering of an electromagnetic wave from a two-dimensional, randomly rough metal surface, with a modest expenditure of computational time, even though it is an $O(N^2)$ method.
- For the first time (to the best of our knowledge) the *fill* angular distribution of the scattered light has been obtained by a rigorous computer simulation approach
- At least for the roughness parameters we have assumed, the use of a plane wave for the incident field instead of a beam of finite width does not degrade the quality of the calculated mean differential reflection coefficient, and significantly simplifies the calculations.
- The use of numerical differentiation (needed for metals) of the unknown functions to avoid hypersingular kernels in the integral equations for the source functions poses no computational problems.

Thank you for your Attention!

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