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**Exam in TFY4240 Electromagnetic Theory**

Dec 9, 2011
09:00–13:00

Allowed help: Alternativ **C**
Authorized calculator and mathematical formula book

This problem set consists of 5=1one page=0.

This exam consists of two problems each containing several sub-problems. Each of the sub-problems will be given approximately equal weight during grading, except point 1c and 1f that will be given double weight. For your information, I estimate that you will spend more time on the first problem relative the second.

I will be available for questions related to the problems themselves (though not the answers!). The first round (of two), I plan to do around 10am, and the other one, about two hours later.

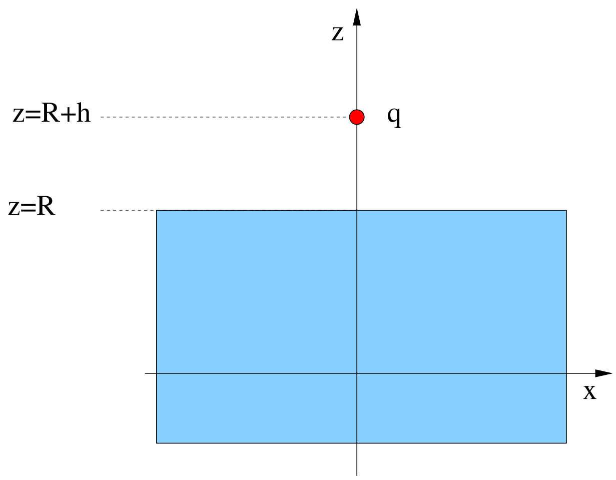
The problems are given in English only. Should you have any language problems related to the exam set, do not hesitate to ask. For your answers, you are free to use either English or Norwegian.

Good luck to all of you!

**Problem 1.**

This problem is devoted to the study of electrostatics for various geometries, and we will apply the method of images.

**a)** Describe in words what is meant by the method of images and for what region the scalar potential obtained by this method can be used.




A change $q > 0$ is situated a distance $\Delta z = h > 0$ above a *grounded* conducting half space. A coordinate system is defined so that the half space is located in the region $z \leq R$, and the charge is positioned on the $z$-axis at height $z = R + h$ (see figure).

In the lectures (or in the book), it was shown that the scalar potential for this system (valid for $z \geq R$) is given by

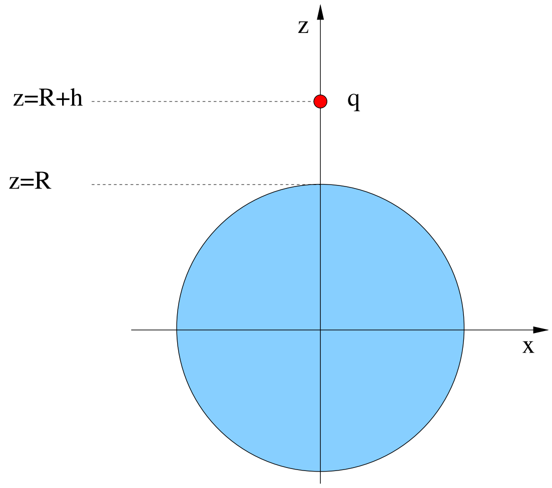
$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \frac{q}{|\boldsymbol{r} - (R+h)\hat{\boldsymbol{z}}|} - \frac{1}{4\pi\varepsilon_0} \frac{q}{|\boldsymbol{r} - (R-h)\hat{\boldsymbol{z}}|}, \tag{1}$$

where $\hat{\boldsymbol{z}}$ denotes the unit vector in direction $\boldsymbol{z}$.

**b)** Explain what is the physical meaning of the two terms in Eq. (1), and what role the vectors $\boldsymbol{\rho}_\pm = \boldsymbol{r} - (R \pm h)\hat{\boldsymbol{z}}$ have. What is the value of the potential along the plane $z = R$?

Now instead of the charge at $(R + h)\hat{\boldsymbol{z}}$, we place a charge $q > 0$ at position $(R + h)\hat{\boldsymbol{z}} + \boldsymbol{d}/2$ and a charge $-q$ at $(R+h)\hat{\boldsymbol{z}} - \boldsymbol{d}/2$, where $\boldsymbol{d} = d\hat{\boldsymbol{z}}$ is a vector for which $d/h \ll 1$ (and $d > 0$). Note that the grounded conducting half space is still present at $z \leq R$.

**c)** Make a sketch of the new configuration. Use Eq. (1) to obtain an expression for the scalar potential valid in the region $z \geq R$ under the assumption that $d/\rho_+ \ll 1$. Express your answer in terms of $\boldsymbol{p} = q\boldsymbol{d}$ and $\boldsymbol{\rho}_\pm$. Give a physical interpretation of the different terms of the resulting potential.




Another geometry will now be considered. Here a charge $q > 0$ is placed a distance $h > 0$ outside the surface of a grounded, conducting sphere of radius $R$ (see figure). We are interested in finding an expression of the scalar potential in the region outside the sphere. To this end, we will once more use the method of images.

**d)** *Derive* that an image charge

$$q' = -\frac{R}{R+h} q, \tag{2a}$$

placed a distance

$$z' = \frac{R^2}{R+h}, \tag{2b}$$

from the center of the sphere in the radial direction towards the charge $q$, can be used to solve the electrostatic problem with appropriate boundary conditions. [*Note*: You are not supposed to start from the expressions for $q'$ and $z'$ in showing this. Instead you are asked to derive these results using the boundary conditions.]

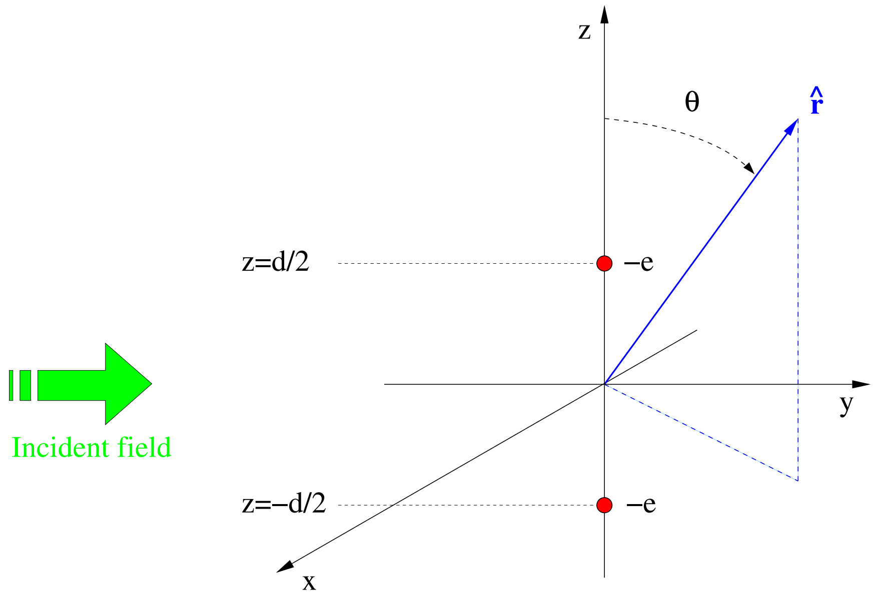
**e)** Using $q'$ and $z'$ as given by Eq. (2), write down the expression for the scalar potential $V(\boldsymbol{r})$ valid outside the sphere. Obtain the $R \gg h$ limit of this potential. Is the result as expected?

As above, we will now replace the charge at $(R+h)\hat{\boldsymbol{z}}$ by two opposite charges. One charge $q > 0$ is placed at $(R+h)\hat{\boldsymbol{z}} + \boldsymbol{d}/2$ and another charge $-q$ at $(R+h)\hat{\boldsymbol{z}} - \boldsymbol{d}/2$ where $\boldsymbol{d} = d\hat{\boldsymbol{z}}$.

**f)** Explain why, for finite $R$, there is *no* single image dipole that will solve this electrostatic problem (when $\boldsymbol{d} = d\hat{\boldsymbol{z}}$). As usual, it is assumed that one is far away from the dipole (and image dipole) and that $d \ll R, h$. [*Hint*: Use Eq. (2) and the physics of the problem. You are not encouraged to expand the potential (even if it works), since this easily may become technical and time consuming.]

There exist other vectors $\boldsymbol{d}$ (than the original choice $\boldsymbol{d} = d\hat{\boldsymbol{z}}$) so a (single) image dipole can be used to solve the corresponding electrostatic problem for our "two change outside a sphere" geometry. Specify one such choice for $\boldsymbol{d}$, and present your reasoning for arriving at it. For this choice of $\boldsymbol{d}$, obtain the corresponding image dipole moment, $\boldsymbol{p}'$, expressed in terms of $\boldsymbol{p} = q\boldsymbol{d}$, $R$ and $h$.

**Problem 2.**




Consider two electrons (of charges $q_1 = q_2 = -e$) placed a distance $d$ apart as shown in the figure above. A coordinate system is defined so that the electrons are located on the $z$-axis with its origin placed midway between them.

A plane electromagnetic wave is incident on the two electrons along the positive $y$-direction, and it is polarized along the $z$-axis. Mathematically we can write the (real valued) electric field component of the incident electromagnetic field as

$$\boldsymbol{E}(\boldsymbol{r}, t) = \mathrm{Re}\left\{\boldsymbol{E}_0 \exp\left(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r} - \mathrm{i}\omega t\right)\right\} = \hat{\boldsymbol{z}}\, E_0 \cos\left(ky - \omega t\right),$$

where $\boldsymbol{E}_0 = E_0\hat{\boldsymbol{z}}$ is a (real) constant vector and $\boldsymbol{k} = k\hat{\boldsymbol{y}}$ is the wave vector for which $k = \omega/c = 2\pi/\lambda$.

The field strength $E_0$ is assumed to be so weak that the velocities of the electrons always are much less then the speed of light $c$ (so that we can treat the system non-relativistically).

**a)** Under the assumption given above, show that the amplitude of the electron oscillations, $A_0$, is much smaller than the wavelength $\lambda$ of the incident wave. Use the result $A_0/\lambda \ll 1$ to argue why the radiation field is approximately harmonic (in time). [*Hint*: Show that $\omega t_r = \omega(t - R(t_r)/c)$ depends on time only *via* $\omega t$ (to a good approximation).]

**b)** The radiation field from the two oscillating electrons is now observed at a large distance from the origin, *i.e.* $r \gg d$ and $r \gg \lambda$. Explain how the radiation field is polarized, *i.e.* what is the direction of the electric field.

**c)** Show that the amplitude of the radiation field from the "upper" electron can be written in the form

$$E_{(1)}(\boldsymbol{r}, t) = -E_0 \frac{r_0}{r} \sin(\theta) \cos\left[\omega t - k\left(r - \frac{d}{2}\cos\theta\right)\right],$$

with $r_0$ the so-called classical electron radius given by

$$r_0 = \frac{e^2}{4\pi\varepsilon_0 m_e c^2},$$

where $m_e$ is the mass of the electron. Give a similar expression for the radiation field, $E_{(2)}(\boldsymbol{r}, t)$, from the lower electron.

**d)** Show that the *total* radiation field (from the two electrons), $E_{2e}(\boldsymbol{r}, t)$, can be written in the form

$$E_{2e}(\boldsymbol{r}, t) = \left[-E_0 \frac{r_0}{r} \sin(\theta) \cos(\omega t - kr)\right] f(\theta, \lambda, d), \tag{3}$$

where

$$f(\theta, \lambda, d) = 2\cos\left(\frac{\pi d}{\lambda}\cos\theta\right),$$

is called an *interference factor*, and the pre-factor in the square brackets $[\ldots]$ in Eq. (3) is identical to the radiation field from a single electron placed at the origin.

**e)** The intensity associated with the total radiation field is $I_{2e} \propto |E_{2e}(\boldsymbol{r}, t)|^2$, and we denote the radiated intensity from a single electron placed at the origin by $I_e$. Obtain an expression for the ratio

$$F(\theta, \lambda, d) = \frac{I_{2e}}{I_e},$$

and discuss the $\theta$-dependence of $F(\theta, \lambda, d)$ for various values of $d/\lambda$. How should one optimally choose the wavelength $\lambda$ if one wants to determine experimentally the distance $d$ between the electrons?

*Given formulae*

(you are supposed to know when these formulae apply and what the symbols mean)

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi} \int \mathrm{d}^3 r' \, \frac{\boldsymbol{J}(\boldsymbol{r}', t_r)}{R}$$

$$\boldsymbol{E}(\boldsymbol{r}, t) = \left[\frac{1}{4\pi\varepsilon_0} \frac{q}{c^2} \frac{\hat{\boldsymbol{R}} \times (\hat{\boldsymbol{R}} \times \boldsymbol{a})}{R} + \frac{q}{4\pi\varepsilon_0} \frac{\hat{\boldsymbol{R}}}{R^2}\right]_{\mathrm{Ret}}$$

## FUNDAMENTAL CONSTANTS

| | | | |
|---|---|---|---|
| $\epsilon_0$ | $=$ | $8.85 \times 10^{-12}\,\mathrm{C^2/Nm^2}$ | (permittivity of free space) |
| $\mu_0$ | $=$ | $4\pi \times 10^{-7}\,\mathrm{N/A^2}$ | (permeability of free space) |
| $c$ | $=$ | $3.00 \times 10^{8}\,\mathrm{m}/s$ | (speed of light) |
| $e$ | $=$ | $1.60 \times 10^{-19}\,\mathrm{C}$ | (charge of the electron) |
| $m$ | $=$ | $9.11 \times 10^{-31}\,\mathrm{kg}$ | (mass of the electron) |

## SPHERICAL AND CYLINDRICAL COORDINATES

**Spherical**

$$\left\{\begin{array}{lcl} x & = & r\sin\theta\cos\phi \\ y & = & r\sin\theta\sin\phi \\ z & = & r\cos\theta \end{array}\right. \qquad \left\{\begin{array}{lcl} \hat{\mathbf{x}} & = & \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} & = & \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} & = & \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{array}\right.$$

$$\left\{\begin{array}{lcl} r & = & \sqrt{x^2+y^2+z^2} \\ \theta & = & \tan^{-1}(\sqrt{x^2+y^2}/z) \\ \phi & = & \tan^{-1}(y/x) \end{array}\right. \qquad \left\{\begin{array}{lcl} \hat{\mathbf{r}} & = & \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} & = & \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} & = & -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{array}\right.$$

**Cylindrical**

$$\left\{\begin{array}{lcl} x & = & s\cos\phi \\ y & = & s\sin\phi \\ z & = & z \end{array}\right. \qquad \left\{\begin{array}{lcl} \hat{\mathbf{x}} & = & \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} & = & \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} & = & \hat{\mathbf{z}} \end{array}\right.$$

$$\left\{\begin{array}{lcl} s & = & \sqrt{x^2+y^2} \\ \phi & = & \tan^{-1}(y/x) \\ z & = & z \end{array}\right. \qquad \left\{\begin{array}{lcl} \hat{\mathbf{s}} & = & \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} & = & -\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} & = & \hat{\mathbf{z}} \end{array}\right.$$

# BASIC EQUATIONS OF ELECTRODYNAMICS

## Maxwell's Equations

*In general* :

$$\begin{cases} \nabla \cdot \mathbf{E} = \dfrac{1}{\epsilon_0}\rho \\[2ex] \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\[2ex] \nabla \cdot \mathbf{B} = 0 \\[2ex] \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} \end{cases}$$

*In matter* :

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho_f \\[2ex] \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\[2ex] \nabla \cdot \mathbf{B} = 0 \\[2ex] \nabla \times \mathbf{H} = \mathbf{J}_f + \dfrac{\partial \mathbf{D}}{\partial t} \end{cases}$$

## Auxiliary Fields

*Definitions* :

$$\begin{cases} \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \\[2ex] \mathbf{H} = \dfrac{1}{\mu_0}\mathbf{B} - \mathbf{M} \end{cases}$$

*Linear media* :

$$\begin{cases} \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \quad \mathbf{D} = \epsilon \mathbf{E} \\[2ex] \mathbf{M} = \chi_m \mathbf{H}, \quad \mathbf{H} = \dfrac{1}{\mu}\mathbf{B} \end{cases}$$

## Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

## Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

## Energy, Momentum, and Power

*Energy* : $U = \dfrac{1}{2}\displaystyle\int \left(\epsilon_0 E^2 + \dfrac{1}{\mu_0} B^2\right) d\tau$

*Momentum* : $\mathbf{P} = \epsilon_0 \int (\mathbf{E} \times \mathbf{B})\, d\tau$

*Poynting vector* : $\mathbf{S} = \dfrac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$

*Larmor formula* : $P = \dfrac{\mu_0}{6\pi c} q^2 a^2$

## VECTOR IDENTITIES

**Triple Products**

(1) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

(2) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

**Product Rules**

(3) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(4) $\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A}$

(5) $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f)$

(6) $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$

(7) $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$

(8) $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

**Second Derivatives**

(9) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

(10) $\nabla \times (\nabla f) = 0$

(11) $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$

## FUNDAMENTAL THEOREMS

**Gradient Theorem** : $\int_{\mathbf{a}}^{\mathbf{b}} (\nabla f) \cdot d\mathbf{l} = f(\mathbf{b}) - f(\mathbf{a})$

**Divergence Theorem** : $\int (\nabla \cdot \mathbf{A})\, d\tau = \oint \mathbf{A} \cdot d\mathbf{a}$

**Curl Theorem** : $\int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l}$

**Cartesian.** $d\mathbf{l} = dx\,\hat{\mathbf{x}} + dy\,\hat{\mathbf{y}} + dz\,\hat{\mathbf{z}}; \quad d\tau = dx\,dy\,dz$

*Gradient* : $\nabla t = \frac{\partial t}{\partial x}\hat{\mathbf{x}} + \frac{\partial t}{\partial y}\hat{\mathbf{y}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

*Divergence* : $\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$

*Curl* : $\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$

*Laplacian* : $\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$

**Spherical.** $d\mathbf{l} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}}; \quad d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

*Gradient* : $\nabla t = \frac{\partial t}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\boldsymbol{\phi}}$

*Divergence* : $\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$

*Curl* :

$$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

*Laplacian* : $\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial\phi^2}$

**Cylindrical.** $d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}; \quad d\tau = s\,ds\,d\phi\,dz$

*Gradient* : $\nabla t = \frac{\partial t}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial t}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial t}{\partial z}\hat{\mathbf{z}}$

*Divergence* : $\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z}$

*Curl* : $\nabla \times \mathbf{v} = \left[\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{\mathbf{s}} + \left[\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right]\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}}$

*Laplacian* : $\nabla^2 t = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial t}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 t}{\partial\phi^2} + \frac{\partial^2 t}{\partial z^2}$