TFY4240 ELECTROMAGNETIC THEORY

Exam Dec. 2010

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Problem 1

a) The total potential is the sum of the potentials from the individual charges:

$$V(\vec{r}) = \sum_{i=1}^{4} V_i(\vec{r}),$$

$$= \sum_{i=1}^{4} \frac{1}{4\pi\varepsilon_0} \frac{q_i}{|\vec{r}-\vec{r}_i|} \tag{1}$$

b) By a multi-pole expansion of the scalar potential one means an expansion in powers of $(r'/r)^n$. Here r is the distance to the observer from a reference point inside the charge distribution, while r' is the distance to a point inside the charge distribution.
Far away from the charge distribution only the first few terms will contribute significantly.

In general the expansion takes the form

$$V(\vec{r}) =$$

where

c) Since we are far away from the charge distribution, we may make the following approximation

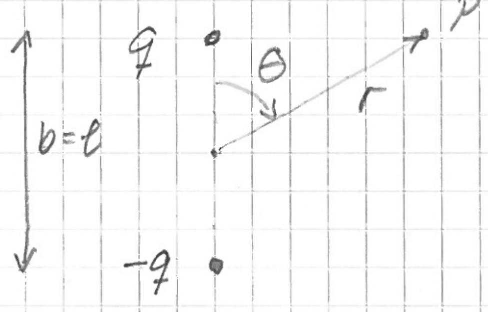
$$\frac{1}{|\vec{r}-\vec{r}_i|} \simeq \frac{1}{r}$$

which when used in Eq (1) gives

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\,\frac{1}{r}\sum_{i=1}^{4} q_i = \frac{1}{4\pi\varepsilon_0}\,\frac{Q}{r}. \qquad (2)$$

where $Q = \sum_i q_i \neq 0$.

d)



The scalar potential in this case is (using $q_2 = q_4 = 0$)

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\,\frac{q}{|\vec{r}-\vec{r}_1|} - \frac{1}{4\pi\varepsilon_0}\,\frac{q}{|\vec{r}-\vec{r}_3|} \qquad (3)$$

In general one has:

$$|\vec{r}-\vec{r}_i| = [(\vec{r}-\vec{r}_i)(\vec{r}-\vec{r}_i)]^{1/2}$$
$$= [r^2 - 2\vec{r}\cdot\vec{r}_i + r_i^2]^{1/2}$$
$$= r\left[1 - 2\frac{\vec{r}\cdot\vec{r}_i}{r^2} + \left(\frac{r_i}{r}\right)^2\right]^{1/2}$$
$$= r\left[1 - 2\frac{\hat{r}\cdot\vec{r}_i}{r} + \left(\frac{r_i}{r}\right)^2\right]^{1/2}$$

Hence it follows by expanding that

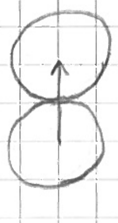
$$\frac{1}{|\vec{r}-\vec{r}_i|} = \frac{1}{r}\left[1 - 2\hat{r}\cdot\frac{\vec{r}_i}{r} + \left(\frac{r_i}{r}\right)^2\right]^{-1/2}$$
$$\simeq \frac{1}{r}\left[1 + \frac{\hat{r}\cdot\vec{r}_i}{r}\right]$$

Therefore, it follows from Eq. (3)

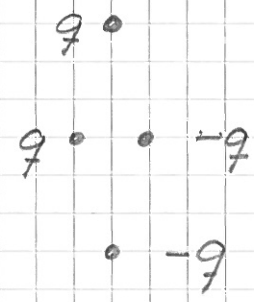
$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} q\left[\frac{1}{|\vec{r}-\vec{r}_1|} - \frac{1}{|\vec{r}-\vec{r}_3|}\right]$$
$$\simeq \frac{1}{4\pi\varepsilon_0}\frac{q}{r}\left[\frac{\hat{r}\cdot\vec{r}_1}{r} - \frac{\hat{r}\cdot\vec{r}_3}{r}\right]; \qquad \vec{r}_1 = \frac{b}{2}\hat{z}, \quad \vec{r}_3 = -\frac{b}{2}\hat{z}$$
$$= \frac{1}{4\pi\varepsilon_0}\frac{q b\,\hat{z}\cdot\hat{r}}{r^2}$$
$$\equiv \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} \qquad (4)$$

Where $\vec{p} = qb\hat{z} = q\ell\hat{z}$ is the so-called dipole moment.

Plot:



e)



Since $b = c > 0$ and $a/c \ll 1$ it follows that the potential for this case is similar to Eq (4). The physical argument for this is that the charge distribution can be seen as consisting of two electrical dipoles:

$$\vec{p} = q c \hat{z}$$
$$\vec{p}\,' = -q a \hat{x}$$

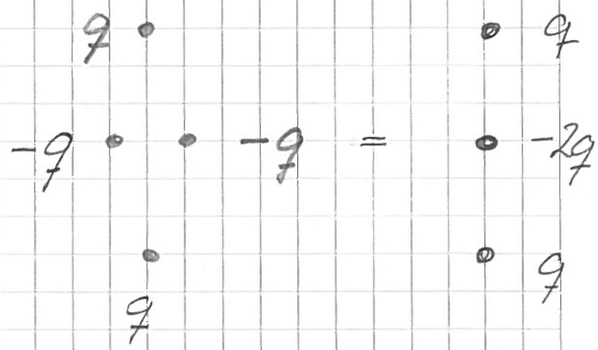
However, since $|p'|/|p| \ll 1$ it follows that

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{\vec{p}\cdot\hat{r}}{r^2} + \frac{1}{4\pi\varepsilon_0} \frac{\vec{p}\,'\cdot\hat{r}}{r^2}$$
$$\simeq \frac{1}{4\pi\varepsilon_0} \frac{\vec{p}\cdot\hat{r}}{r^2}$$

Plot of the angular distribution is as in previous sub-problem.

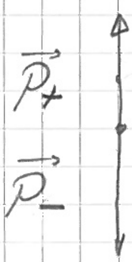
[You may also look at the charge dist as the sum of two almost parallel dipoles of dipole moment $q \frac{c}{2} \hat{z}$ Note, their sum is non-zero]

f) $q_1 = q_3 = q$
$q_2 = q_4 = -q$



Also in this case the charge distribution can be looked upon as two almost paralell dipoles. However, in this case their sum is zero (like the total charge). Hence, a higher order term in the multipole expansion will contribute.

g) Method 1



Two oposite pointing dipoles located at $\vec{R}_\pm = \pm \frac{\ell}{4} \hat{z}$

The potential from one such dipole is

$$V_\pm(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{\vec{p}_\pm \cdot (\vec{r} - \vec{R}_\pm)}{|\vec{r} - \vec{R}_\pm|^3} \qquad \vec{p}_\pm = \pm q \frac{\ell}{2} \hat{z}$$

Note : The solution to problem #2
is not complete

(My original solutions were
stolen in Paris last year (Dec. 2010)
together with my laptop.....)

Nov. 2011

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Problem 2

a) Since the current is going from one end to the other of the wire, only the ends will be charged. The charge can be determined from the differential equation that follows from the definition of current.

$$I(t) = \frac{dQ(t)}{dt}$$

$$\Rightarrow \quad Q(t) = \int^{t} dt' \, I(t') = \frac{i}{\omega} I(t) \qquad (6)$$

[As usual it is the real part of this which is the physical charge, that is
$\mathrm{Re}\, Q(t) = (I_0/\omega) \sin\omega t$]

b) The (complex) time-dependent dipole moment is

$$\vec{p}(t) = Q(t)\,\ell\,\hat{z}$$
$$= i\,\frac{I_0 \ell}{\omega} \exp(-i\omega t)\,\hat{z}$$

c) Since wire is coinciding with the z-axis and has length ℓ it follows directly that

$$\vec{J}(\vec{r},t) = \hat{z}\, I(t)\, \delta(x)\, \delta(y)\, \theta(|z| - \ell/2)$$

The δ-functions place the wire on the z-axis, and the θ-function gives it the correct length.

d) The PDF for the vector potential (in the Lorentz gauge reads)

$$\nabla^2 \vec{A}(\vec{r},t) - \frac{1}{c^2}\partial_t^2 \vec{A}(\vec{r},t) = -\mu_0 \vec{J}(\vec{r},t) \quad (7)$$

A particular solution to this equation can be constructed as

$$\vec{A}_p(\vec{r},t) = \int d^3r'dt' \, g(\vec{r},t/\vec{r}',t')[-\mu_0 \vec{J}(\vec{r}',t')]$$

$$= \mu_0 \int d^3r'dt' \, \frac{\delta\left(t-t'-\frac{|\vec{r}-\vec{r}'|}{c}\right)}{4\pi|\vec{r}-\vec{r}'|} \vec{J}(\vec{r}',t')$$

$$= \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\vec{J}(\vec{r}',t_r)}{|\vec{r}-\vec{r}'|} \quad (8)$$

where the retarded time is given as.

$$t_r = t - \frac{|\vec{r}-\vec{r}'|}{c}. \quad (9)$$

This is the time it takes for a signal to propagate from $\vec{r}'$ (the source point) to $\vec{r}$ (the observation point).

e) We start by looking at the term $I(t_r)$ contained in the expression for $\vec{J}(\vec{r}',t_r)$

$$I(t_r) = I_0 \exp(-i\omega t_r)$$

$$= I_0 \exp\left(-i\omega t + i\omega \frac{|\vec{r}-\vec{r}'|}{c}\right)$$

$$= I_0 \, e^{-i\omega t} \, e^{ik|\vec{r}-\vec{r}'|}$$

Now expanding $|\vec{r}-\vec{r}\,'|$ as

$$|\vec{r}-\vec{r}\,'| = r\left[1 - 2\frac{\hat{r}\cdot\vec{r}\,'}{r} + \left(\frac{r'}{r}\right)^2\right]^{1/2}$$

$$\simeq r\left[1 - \frac{\hat{r}\cdot\vec{r}\,'}{r}\right]$$

$$= r - \hat{r}\cdot\vec{r}\,'$$

so that

$$I(t_r) \simeq I_0\, e^{-i\omega t}\, e^{ikr}\, e^{-ik\,\hat{r}\cdot\vec{r}\,'}. \quad (9)$$

Now the expression for the vector potential becomes from Eqs (8) and (9)

$$\vec{A}(\vec{r},t) = \hat{z}\,\frac{\mu_0}{4\pi}\int_{-\ell/2}^{\ell/2} dz'\,\frac{I(t_r)}{|\vec{r}-\vec{z}\,'|} \qquad \vec{r}\,' = \vec{z}\,'$$

$$\simeq \hat{z}\,\frac{\mu_0 I_0}{4\pi}\,\frac{e^{ikr-i\omega t}}{r}\int_{-\ell/2}^{\ell/2} dz'\, e^{-ikz'\cos\theta}$$

$$= \hat{z}\,\frac{\mu_0 I_0}{4\pi}\,\frac{e^{ikr-i\omega t}}{r}\left[\frac{e^{-ikz'\cos\theta}}{-ik\cos\theta}\right]_{-\ell/2}^{\ell/2}$$

$$= \hat{z}\,\frac{\mu_0 I_0}{4\pi}\,\frac{e^{ikr-i\omega t}}{r}$$

$$\times \frac{1}{-ik\cos\theta}\left[\exp\left(-ik\frac{\ell}{2}\cos\theta\right) - \exp\left(+ik\frac{\ell}{2}\cos\theta\right)\right]$$

$$= \hat{z}\,\frac{\mu_0 I_0}{2\pi}\,\frac{\exp(ikr-i\omega t)}{kr}\,\frac{1}{\cos\theta}$$

$$\times \sin\left(\frac{k\ell}{2}\cos\theta\right)$$

$$= \hat{z} \frac{\mu_0 I_0}{2\pi \cos\theta} \sin\left(\frac{kl}{2}\cos\theta\right) \frac{e^{ikr-i\omega t}}{kr}$$

f) The scalar potential is given as

$$\nabla \vec{A}(\vec{r},t) + \frac{1}{c^2} \partial_t V(\vec{r},t) = 0$$

$$\Rightarrow V(\vec{r},t) = - c^2 \nabla \int^t dt' \vec{A}(\vec{r},t')$$

$$= + c^2 \frac{\mu_0 I_0}{i 2\pi \omega} \sin\left(\frac{kl}{2}\cos\theta\right) \partial_z \left(\frac{e^{ikr-i\omega t}}{kr}\right)$$

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g) By definition it follows that

$$\vec{H}(\vec{r},t) = \frac{1}{\mu_0} \nabla \times \vec{A}(\vec{r},t)$$

g] $\vec{H} \simeq \frac{1}{\mu_0} i\vec{k} \times \vec{A}$

h] $-i\omega \vec{E} = \nabla \times \vec{H} \simeq \underline{i\vec{k} \times \vec{H}}$

$$\Rightarrow \vec{E} \simeq -\frac{\vec{k}}{\omega} \times \vec{H} = -\frac{\vec{k}}{\omega\mu_0} \times (i\vec{k} \times \vec{A})$$

$$= \frac{-1}{\omega\mu_0} \vec{k} \times (\vec{k} \times \vec{A})$$

$$\vec{E} = \frac{i}{\omega\varepsilon_0\mu_0} \vec{k} \times (\vec{k} \times \vec{A})$$

i] $\langle \vec{S} \rangle_t = \frac{1}{2} \vec{E} \times \vec{H}^*$

$$= \frac{-1}{\omega} (\vec{k} \times \vec{H}) \times \vec{H}^*$$

$$= +\frac{1}{\omega} \vec{H}^* \times (\vec{k} \times \vec{H})$$

$$= \frac{1}{\omega} \left\{ \vec{k} |\vec{H}|^2 - \vec{H} \underbrace{(\vec{H}^* \cdot \vec{k})}_{0} \right\}$$

$$= |\vec{H}|^2 \frac{\vec{k}}{\omega}$$

$$= \frac{1}{c} |\vec{H}|^2 \hat{k}$$

$$= \frac{1}{c} \frac{1}{\mu_0^2} k^2 |\vec{A}|^2 \sin^2\theta \; \hat{k}$$

$$= \frac{\varepsilon_0}{\mu_0} k^2 |\vec{A}|^2 \sin^2\theta \; \hat{k}$$

$$\frac{dP}{d\Omega} = |\langle \vec{S} \rangle_t| \cdot r^2$$

$$= \frac{\varepsilon_0}{\mu_0} k^2 \frac{\mu_0^2}{(2\pi)^2} \frac{I_0^2}{\cos^2\theta} \sin^2\left(\frac{kl}{2}\cos\theta\right) \frac{r^2}{kr^2}$$

$$= \varepsilon_0 \mu_0 k \frac{I_0^2}{(2\pi)^2} \sin^2\left(\frac{kl}{2}\cos\theta\right) \tan^2\theta$$