SOLUTION TO EXAM IN

TFY4240 – Dec 2011

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Problem 1

a) The method of images is a technique used to solve electrostatic problems (i.e. solving Laplace eq.). It consists of placing so-called image charges <u>outside</u> the domain of interest such that the boundary conditions (cont. of V and $\epsilon\partial_n V$) are satisfied. If all appropriate BC are satisfied, then the total potential in the domain of interest is the sum of the potential from the charge and image-charge since the solution of Laplace eq. is unique.

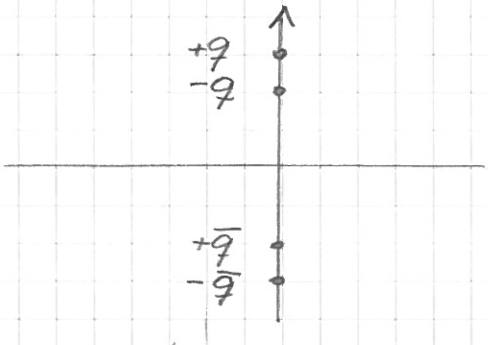
b) The two terms in the potential in Eq (1) corresponds to the potential from the charge q (first term) and image charge $-q$ (second term). They are located at at $(R\pm h)\hat{z}$ so the distance from each one of them to an observation point $\vec{r}$ (with $z>R$) is $\vec{r}-(R\pm h)\hat{z}$.

On the surface $z=R$ we have

$$|\vec{r} - (R \pm h)\hat{z}| = |\vec{r}_{\parallel} + R\hat{z} - (R \pm h)\hat{z}|$$
$$= |\vec{r}_{\parallel} \pm h\hat{z}| \qquad (1.1)$$

($\vec{r}_{\parallel}$ is a vector in the xy-plane)
Since $|\vec{r}_{\parallel} \pm h\hat{z}|$ is independent of sign it follows that $V(\vec{r})=0$ when $\vec{r}$ is in the plane $z=R$.

c)



The total potential for the system consists of two potentials of the form of Eq (1) but with the charges located at

$$\vec{r}_{\pm q} = \left(R + h \pm \frac{d}{2}\right)\hat{z}$$

and image charges at

$$\vec{r}_{\pm\bar{q}} = \left(R - \left[h \mp \frac{d}{2}\right]\right)\hat{z} \quad \text{(note reversed sign)}$$
$$= \left(R - h \pm \frac{d}{2}\right)\hat{z}$$

Hence the total potential becomes

$$V(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|\vec{r}-\vec{r}_q|} - \frac{1}{|\vec{r}-\vec{r}_{-q}|}\right.$$

$$+\frac{-1}{|\vec{r}-\vec{r}_{-q}|}-\frac{-1}{|\vec{r}-\vec{r}_{+q}|}\Big] \tag{1.2}$$

Now doing an expansion around $z=(R\pm h)$:

$$\frac{1}{|\vec{r}-\vec{r}_{\pm q}|}=\frac{1}{|\underbrace{\vec{r}-(R+h)\hat{z}}_{\vec{\rho}_+}\mp\frac{\vec{d}}{2}|}$$

$$=\frac{1}{\left(\rho_+^2\mp 2\vec{\rho}_+\frac{\vec{d}}{2}+\frac{d^2}{4}\right)^{1/2}}$$

$$=\frac{1}{\rho_+\left[\left(1\mp\frac{\hat{\rho}_+\vec{d}}{\rho_+}+\left(\frac{d}{2\rho_+}\right)^2\right)\right]^{1/2}}$$

$$=\frac{1}{\rho_+}\left[1\pm\frac{1}{2}\frac{\hat{\rho}_+\vec{d}}{\rho_+}+\cdots\right]$$

$$=\frac{1}{\rho_+}\pm\frac{1}{2}\frac{\hat{\rho}_+\vec{d}}{\rho_+^2}+\cdots$$

Similarly one obtains:

$$\frac{1}{|\vec{r}-\vec{r}_{\pm q}|}=\frac{1}{|\underbrace{\vec{r}-(R-h)\hat{z}}_{\vec{\rho}_-}\pm\frac{\vec{d}}{2}|}$$

$$=\frac{1}{\rho_-}\mp\frac{1}{2}\frac{\hat{\rho}_-\cdot\vec{d}}{\rho_-^2}+\cdots$$

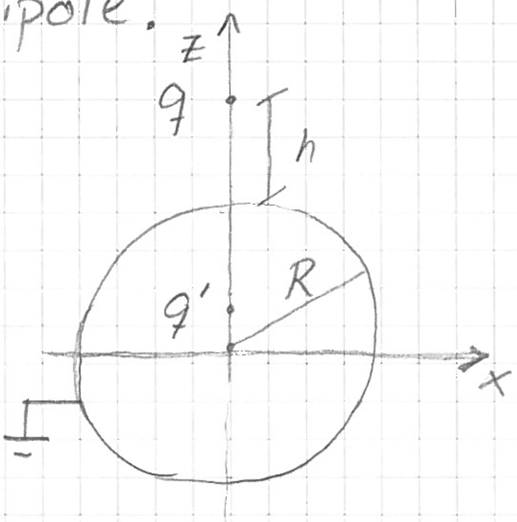
Hence, to leading order

$$V(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{\hat{\rho}_+ \cdot \vec{d}}{\rho_+^2} - \frac{\hat{\rho}_- \cdot \vec{d}}{\rho_-^2} \right]$$

$$= \underline{\underline{\frac{1}{4\pi\varepsilon_0} \frac{\hat{\rho}_+ \cdot \vec{p}}{\rho_+^2} - \frac{1}{4\pi\varepsilon_0} \frac{\hat{\rho}_- \cdot \vec{p}}{\rho_-^2}}} \quad (1.3)$$

Hence the potential is the sum of an electric dipole and an oppositly directed image dipole.

d)



Since the sphere is grounded we choose the potential at the surface to be zero. We try to solve the problem by the method of images by placing an image charge q' on the z-axis at position z'.

The total potential outside the sphere becomes

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{|\vec{r} - (R+h)\hat{z}|} + \frac{q'}{|\vec{r} - z'\hat{z}|} \right] \quad (1.4)$$

Now we have two unknown, q' and z'.
To determine them, we choose the points $\vec{r} = \pm R\hat{z}$ and impose the BC on V

i) At $\vec{r} = R\hat{z}$

$$V(\vec{r} = R\hat{z}) = 0$$

$$\frac{q}{|R-R-h|} + \frac{q'}{|R-z'|} = 0 \qquad (z' < R), \quad h \neq 0$$

$$q(R-z') + q'h = 0$$

$$qR - qz' + q'h = 0 \qquad (1.5a)$$

ii) $V(\vec{r} = -R\hat{z}) = 0$

$$\frac{q}{|-2R-h|} + \frac{q'}{|-R-z'|} = 0$$

$$q(R+z') + q'(2R+h) = 0, \qquad z' > 0$$

$$qR + qz' + (2R+h)q' = 0 \qquad (1.5b)$$

Adding (1.5a) and (1.5b) gives

$$2Rq + (h + 2R + h)q' = 0$$

$$\Rightarrow \underline{\underline{q' = -\frac{R}{R+h}q}}$$

From Eq. (1.5a) it follows that

$$z' = R + \frac{q'}{q}h =$$

$$z' = \frac{R(R+h) - Rh}{R+h}$$

$$z' = \frac{R^2}{R+h}$$

e) Hence the total scalar potential becomes

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{|\vec{r}-(R+h)\hat{z}|} - \frac{\frac{R}{R+h}\,q}{\left|\vec{r}-\frac{R^2}{R+h}\hat{z}\right|}\right]$$

When $R \gg h$ one has

$$\frac{R}{R+h} = \frac{R}{R(1+h/R)} \simeq 1 - \frac{h}{R} + \dots$$

$$\frac{R^2}{R+h} = \frac{R^2}{R(1+h/R)} \simeq R\left(1-\frac{h}{R}+\dots\right) = R-h \quad \dots$$

Hence, in the limit $R \gg h$ one gets to lowest order:

$$V(\vec{r}) \simeq \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{|\vec{r}-(R+h)\hat{z}|} - \frac{q}{|\vec{r}-(R-h)\hat{z}|}\right]$$

This is the potential for a charge q above a flat grounded plate.

This is a reasonable result!

f)

Let the image charges corresponding to $\pm q$ be denoted $q'_\pm$. These image charges are given by

$$q'_\pm = \mp \frac{R}{R+h\pm\frac{d}{2}}\, q$$

Since the image charge is depending on the distance from the center of the sphere it follows that

$$q'_+ + q'_- \neq 0.$$

Hence, there will be a monopole contribution to the potential coming from the image charges. Since this term will be dominating it is not possible to find an image dipole so that the potential on $r=R$ vanishes.

However, if $\vec{d}$ is chosen to be parallel with the xy-plane, i.e.

$$\vec{d} = d\,\hat{r}_{\parallel}$$

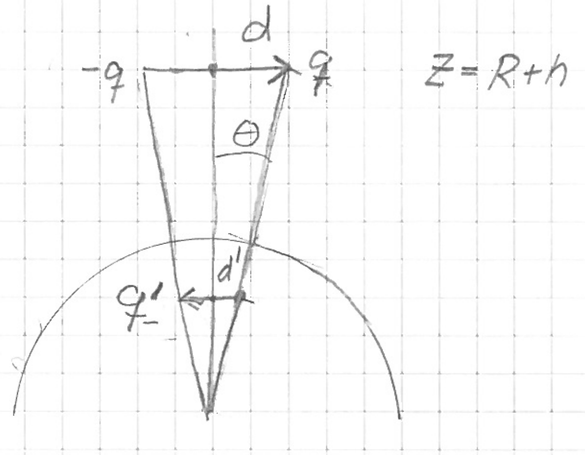
then the distances from the center of the sphere to $q'_\pm$ are the same so $q'_+ + q'_- = 0$. Therefore, the mono-pole term coming from the image charges vanishes, and the leading order term is a image dipole.

The dipole moment of the image charges is:

$$p' = |\vec{p}\,'| = |q'|\,d'$$

where d' is the distance given in the figure below:

$\vec{p} = q\vec{d}$



$\vec{p}\,' \propto -\vec{p}$

Let $R+H$ be the distance to, say, q. From the geometry it follows:

$$\frac{d}{R+H} = \frac{d'}{\frac{R^2}{(R+H)}}$$

$$\Rightarrow d' = \frac{R^2}{(R+H)^2}\, d$$

Now
$$q' = -\frac{R}{R+H}\, q$$

so that (direction follows from figure)

$$\vec{p}\,' = \frac{R^3}{(R+H)^3}\,\vec{p}$$

However, we have assumed that $d/h \ll 1$ so that one may write

$$\vec{p}\,' = \frac{R^3}{\left[(R+h)^2 + \left(\frac{d}{2}\right)^2\right]^{3/2}}\,\vec{p}$$

$$= \frac{R^3}{(R+h)^3}\left[1 + \left(\frac{d}{2(R+h)}\right)^2\right]^{-3/2}\vec{p}$$

$$= \frac{R^3}{(R+h)^3}\,\vec{p} + O\left(d^3/(R+h)^5\right)$$

Thus, one may safly conclude that

$$\underline{\underline{\vec{p}\,' \simeq \frac{R^3}{(R+h)^3}\,\vec{p}}}$$

Alt: we could have made the approximation $R+H \simeq R+h$ from the very beginning. (to simplify the calculation)

Not to be expected as answers to the exam, but the following calculation may still be of interest. We assum here that $\vec{d} = d\hat{z}$

Charges are located at $z_\pm = R + h \pm \frac{d}{2}$.
The corresponding image charces are located at

$$z'_\pm = \frac{R^2}{R+h \pm \frac{d}{2}}$$

$$\simeq \frac{R^2}{R+h}\left[1 \mp \frac{d}{2(R+h)} + \cdots\right]$$

and they have charges:

$$q'_\pm = \mp \frac{R}{R+h \pm \frac{d}{2}} q$$

$$\simeq \mp q \frac{R}{R+h}\left[1 \mp \frac{d}{2(R+h)} + \cdots\right]$$

The distance vector from the image charges is:

$$|\vec{r} - z'_\pm \hat{z}| = \Big|\underbrace{\vec{r} - \frac{R^2}{R+h}\hat{z}}_{\vec{\rho}} \mp \frac{R^2 d}{(R+h)^2}\hat{z}\Big|$$

$$= \left[\rho^2 \mp 2\vec{\rho}\cdot\hat{z}\frac{R^2 d}{(R+h)^2} + \left(\frac{R^2 d}{(R+h)^2}\right)^2\right]^{1/2}$$

$$= \rho\left[1 \mp \frac{\hat{\rho}\cdot\hat{z}}{\rho}\frac{R^2 d}{(R+h)^2} + \cdots\right]^{1/2}$$

For the potential from $q_\pm'$ we need:

$$\frac{1}{|\vec{r}-z_\pm'\hat{z}|} \simeq \frac{1}{\rho}\left[1 \pm \frac{1}{2}\frac{\hat{\rho}\cdot\hat{z}}{\rho}\frac{R^2 d}{(R+h)^2} + \dots\right]$$

$$= \frac{1}{\rho} \pm \frac{1}{2}\frac{\hat{\rho}\cdot\hat{z}}{\rho^2}\frac{R^2 d}{(R+h)^2} + \dots$$

The contribution to the potential from the image charges is:

$$\frac{1}{4\pi\varepsilon_0}\left[\frac{q_+'}{|\vec{r}-z_+'\hat{z}|} + \frac{q_-'}{|\vec{r}-z_-'\hat{z}|}\right]$$

$$= \frac{1}{4\pi\varepsilon_0} q_+'\left[\frac{1}{\rho} + \frac{\hat{\rho}\cdot\hat{z}}{2\rho^2}\frac{R^2 d}{(R+h)^2} + \dots\right]$$

$$+ \frac{1}{4\pi\varepsilon_0} q_-'\left[\frac{1}{\rho} - \frac{\hat{\rho}\cdot\hat{z}}{2\rho^2}\frac{R^2 d}{(R+h)^2} + \dots\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q_+'+q_-'}{\rho} + \frac{q_+'-q_-'}{2}\frac{\hat{\rho}\cdot\hat{z}}{\rho^2}\frac{R^2 d}{(R+h)^2} + \dots\right]$$

$$q_+'+q_-' \simeq \frac{R}{(R+h)^2} qd$$

$$q_+'-q_-' \simeq -2\frac{R}{R+h} q$$

$$\simeq \frac{1}{4\pi\varepsilon_0}\left[\underbrace{\frac{R}{(R+h)^2}\frac{qd}{\rho}}_{\neq 0} - \frac{R^3}{(R+h)^3}\frac{qd}{\rho^2}\hat{\rho}\cdot\hat{z} + \dots\right]$$

Problem 2

a) The harmonic force $\vec{F} = -e\,\hat{z}\cos(\omega t)$ will cause both electrons to have the same harmonic dependence:

$$z(t) = \pm\frac{d}{2} + A_0\cos(\omega t).$$

Hence the electron velocity becomes

$$\dot{z}(t) = -\omega A_0 \sin(\omega t)$$

Since it is assumed that $|\dot{z}(t)| \ll c$ it follows that

$$\omega A_0 \ll c$$

$$A_0 \ll \frac{c}{\omega} = \frac{\lambda}{2\pi} \Rightarrow \underline{\underline{A_0 \ll \lambda}}$$

The time dependence is via retarded time

$$t_r = t - \frac{R(t_r)}{c}$$

$$R(t_r) = \left|\vec{r} - \left[\pm\frac{d}{2} + A_0\cos(\omega t_r)\right]\hat{z}\right|$$

$$\simeq r - \left[\pm\frac{d}{2} + A_0\cos(\omega t_r)\right]\hat{r}\cdot\hat{z}$$

$$\omega t_r = \omega t - \frac{\omega}{c}R(t_r)$$

$$\simeq \omega t - k\left\{r - \left[\pm\frac{d}{2} + A_0\cos(\omega t_r)\right]\hat{r}\cdot\hat{z}\right\}$$

$$= \omega t - \underbrace{kA_0}_{\ll 1}\cos(\omega t_r)\,\hat{r}\cdot\hat{z} - k\left(r \mp \frac{d}{2}\,\hat{r}\cdot\hat{z}\right)$$

$$\simeq \omega t - k\left(r \mp \frac{d}{2}\,\hat{r}\cdot\hat{z}\right)$$

Hence the radiation fields are time-harmonic.

b) The radiation field from an accelerated particle is (given formulae)

$$\vec{E}(\vec{r},t) = \left[\frac{1}{4\pi\varepsilon_0}\frac{-e}{c^2}\frac{\hat{R}\times(\hat{R}\times\vec{a})}{R}\right]_{Ret}$$

$$= \left[\frac{1}{4\pi\varepsilon_0}\frac{e}{c^2}\frac{\vec{a}_\perp}{R}\right]_{Ret} \qquad (2.1)$$

since $\hat{R}\times(\hat{R}\times\vec{a}) = \hat{R}(\hat{R}\vec{a}) - \vec{a} = -\vec{a}_\perp$.
Thus, the electric field is polarized along the apparent acceleration.

c) From Newtons 2nd law applied to the upper electron follows:

$$m_e\vec{a}^{(1)} = -e\vec{E}|_{y=0} = -eE_0\cos(\omega t)\,\hat{z}$$

$$\vec{a}^{(1)}(t) = -\frac{eE_0}{m_e}\cos(\omega t)\,\hat{z}$$

Therefore, the apparent acceleration becomes

$$\vec{a}_\perp^{(1)}(t) = \vec{a}^{(1)}\sin\theta$$

$$= -\frac{eE_0}{m_e}\cos(\omega t)\sin\theta\;\hat{z} \qquad (2.2)$$

so that the electric field amp. can be written

$$E_{(1)}(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\frac{e}{c^2}\frac{a_\perp(t-R_1/c)}{R_1}$$

Now using Eq. (2.1) one gets:

$$E_{(1)}(\vec{r},t) \simeq \frac{1}{4\pi\varepsilon_0}\frac{+e}{c^2}\left(\frac{1}{r}\right) \sin\theta$$

$$\times \frac{-eE_0}{m_e} \cos\left(\omega t - k\left(r - \frac{d}{2}\cos\theta\right)\right)$$

$$= -E_0 \frac{e^2}{4\pi\varepsilon_0 m_e c^2}\left(\frac{1}{r}\right) \sin\theta \cos\left[\omega t - k\left(r - \frac{d}{2}\cos\theta\right)\right]$$

$$= -E_0 \frac{r_0}{r} \sin\theta \cos\left[\omega t - k\left(r - \frac{d}{2}\cos\theta\right)\right]$$

For the "lower" electron:

$$E_{(2)}(\vec{r},t) \simeq -E_0 \frac{r_0}{r} \sin\theta \cos\left[\omega t - k\left(r + \frac{d}{2}\cos\theta\right)\right]$$

d) The total field becomes

$$E_{2e}(\vec{r},t) = E_{(1)}(\vec{r},t) + E_{(2)}(\vec{r},t)$$

$$= \left[-E_0 \frac{r_0}{r} \sin\theta \cos(\omega t - kr)\right] 2\cos\left(\frac{kd}{2}\cos\theta\right)$$

Thus

$$f(\theta,\lambda,d) = 2\cos\left(\frac{kd}{2}\cos\theta\right)$$

$$= 2\cos\left(\frac{\pi d}{\lambda}\cos\theta\right)$$

e) $$F(\theta,d,\lambda) = \frac{I_{2e}}{I_e} = |f(\theta,\lambda,d)|^2$$

$$= 4\cos^2\left(\frac{\pi d}{\lambda}\cos\theta\right)$$

$$= 2\left[1 + \cos\left(\frac{2\pi d}{\lambda}\cos\theta\right)\right]$$

- $\lambda \gg d$

  Here $F \approx 4$ independent of θ and the fields from the two electrons interfers constructively for all directions. A measurment $I_{2e} \simeq 4 I_e$, over a wide range of angles θ, tells only that we have two particles seperated by a distance $d \ll \lambda$.

- $\lambda \sim d$

  In this case F will depend on θ. Hence a measurment of the angular dependece of F can be used to determine d. The same can be achieved by varying λ.

  Therefore, to be able to determine d from measurments one must have $d \sim \lambda$.