NTNU

Institutt for fysikk

TFY4240

Problemset 3 Autumn 2014

Problem 1.
Let $f(x)$ be some function defined on the interval $[-1, 1]$.

a) Show that $f(x)$ can be expanded in terms of the Legendre polynomials $P_\ell(x)$ as

$$f(x) = \sum_{\ell=0}^{\infty} A_\ell P_\ell(x), \tag{1a}$$

where

$$A_\ell = \frac{2\ell+1}{2} \int_{-1}^{1} dx\, f(x) P_\ell(x). \tag{1b}$$

Hint: Use the orthogonality of the Legendre polynomials.

b) For the particular case that

$$f(x) = \begin{cases} -1 & x < 0 \\ +1 & x > 0 \end{cases} \tag{2}$$

argue without doing any calculation that the expansion (1a) only will contain contributions from the polynomials $P_{2n+1}(x)$ with $n = 0, 1, 2, \ldots$.

c) For $f(x)$ given by Eq. (2) show that the first few terms of the expansion read

$$f(x) = \frac{3}{2}P_1(x) - \frac{7}{8}P_3(x) + \frac{11}{16}P_5(x) + \ldots \tag{3}$$

Problem 2.
The (electrical) potential on the surface of a sphere of radius R is given by

$$V(R, \theta) \quad = \quad V_0 \cos^2\theta,$$

where V_0 is some positive constant.

a) Show that the potential outside the sphere is given by

$$V(r, \theta) \quad = \quad \frac{V_0}{3}\left[\frac{R}{r} + 2\left(\frac{R}{r}\right)^3 P_2(\cos\theta)\right].$$

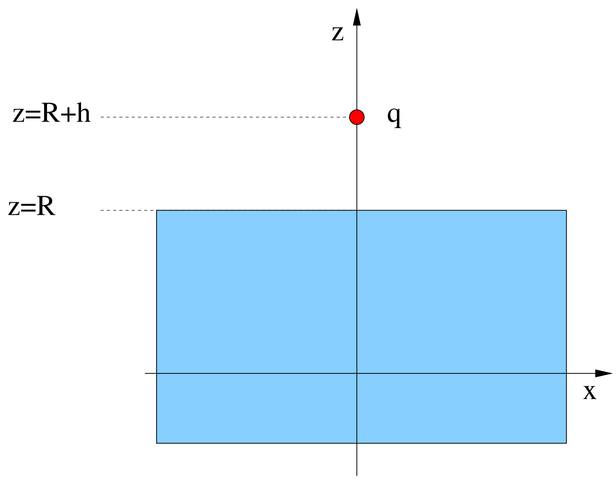
b) Determine the surface charge distribution on the sphere.

c) Integrate over the surface charge to find the quadrupole moment of the sphere. Check to see that this agrees with the $l = 2$ term in the potential outside the sphere.

Problem 3.

This problem is devoted to the study of electrostatics for various geometries, and we will apply the method of images.

a) Describe in words what is meant by the method of images and for what region the scalar potential obtained by this method can be used.



A change $q > 0$ is situated a distance $\Delta z = h > 0$ above a *grounded* conducting half space. A coordinate system is defined so that the half space is located in the region $z \leq R$, and the charge is positioned on the z-axis at height $z = R + h$ (see figure).

In the lectures (or in the book), it was shown that the scalar potential for this system (valid for $z \geq R$) is given by

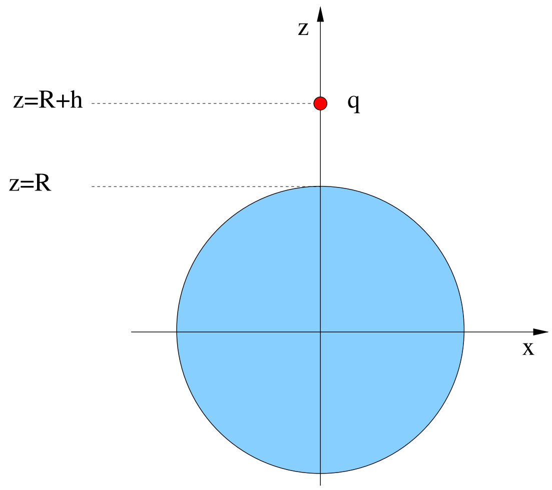
$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\boldsymbol{r} - (R+h)\hat{\boldsymbol{z}}|} - \frac{1}{4\pi\varepsilon_0}\frac{q}{|\boldsymbol{r} - (R-h)\hat{\boldsymbol{z}}|}, \tag{4}$$

where $\hat{\boldsymbol{z}}$ denotes the unit vector in direction $\boldsymbol{z}$.

b) Explain what is the physical meaning of the two terms in Eq. (4), and what role the vectors $\boldsymbol{\rho}_\pm = \boldsymbol{r} - (R \pm h)\hat{\boldsymbol{z}}$ have. What is the value of the potential along the plane $z = R$?

Now instead of the charge at $(R + h)\hat{\boldsymbol{z}}$, we place a charge $q > 0$ at position $(R + h)\hat{\boldsymbol{z}} + \boldsymbol{d}/2$ and a charge $-q$ at $(R+h)\hat{\boldsymbol{z}} - \boldsymbol{d}/2$, where $\boldsymbol{d} = d\hat{\boldsymbol{z}}$ is a vector for which $d/h \ll 1$ (and $d > 0$). Note that the grounded conducting half space is still present at $z \leq R$.

c) Make a sketch of the new configuration. Use Eq. (4) to obtain an expression for the scalar potential valid in the region $z \geq R$ under the assumption that $d/\rho_+ \ll 1$. Express your answer in terms of $\boldsymbol{p} = q\boldsymbol{d}$ and $\boldsymbol{\rho}_\pm$. Give a physical interpretation of the different terms of the resulting potential.



Another geometry will now be considered. Here a charge $q > 0$ is placed a distance $h > 0$ outside the surface of a grounded, conducting sphere of radius R (see figure). We are interested in finding an expression of the scalar potential in the region outside the sphere. To this end, we will once more use the method of images.

d) *Derive* that an image charge

$$q' = -\frac{R}{R+h}q, \tag{5a}$$

placed a distance

$$z' = \frac{R^2}{R+h}, \tag{5b}$$

from the center of the sphere in the radial direction towards the charge q, can be used to solve the electrostatic problem with appropriate boundary conditions. [*Note*: You are not supposed to start from the expressions for q' and z' in showing this. Instead you are asked to derive these results using the boundary conditions.]

e) Using q' and z' as given by Eq. (5), write down the expression for the scalar potential $V(\boldsymbol{r})$ valid outside the sphere. Obtain the $R \gg h$ limit of this potential. Is the result as expected?

As above, we will now replace the charge at $(R+h)\hat{\boldsymbol{z}}$ by two opposite charges. One charge $q > 0$ is placed at $(R+h)\hat{\boldsymbol{z}} + \boldsymbol{d}/2$ and another charge $-q$ at $(R+h)\hat{\boldsymbol{z}} - \boldsymbol{d}/2$ where $\boldsymbol{d} = d\hat{\boldsymbol{z}}$.

f) Explain why, for finite R, there is *no* single image dipole that will solve this electrostatic problem (when $\boldsymbol{d} = d\hat{\boldsymbol{z}}$). As usual, it is assumed that one is far away from the dipole (and image dipole) and that $d \ll R, h$. [*Hint*: Use Eq. (5) and the physics of the problem. You are not encouraged to expand the potential (even if it works), since this easily may become technical and time consuming.]

There exist other vectors $\boldsymbol{d}$ (than the original choice $\boldsymbol{d} = d\hat{\boldsymbol{z}}$) so a (single) image dipole can be used to solve the corresponding electrostatic problem for our "two change outside a sphere" geometry. Specify one such choice for $\boldsymbol{d}$, and present your reasoning for arriving at it. For this choice of $\boldsymbol{d}$, obtain the corresponding image dipole moment, $\boldsymbol{p}'$, expressed in terms of $\boldsymbol{p} = q\boldsymbol{d}$, R and h.