TFY4240

Problemset 10 and 11 Autumn 2014

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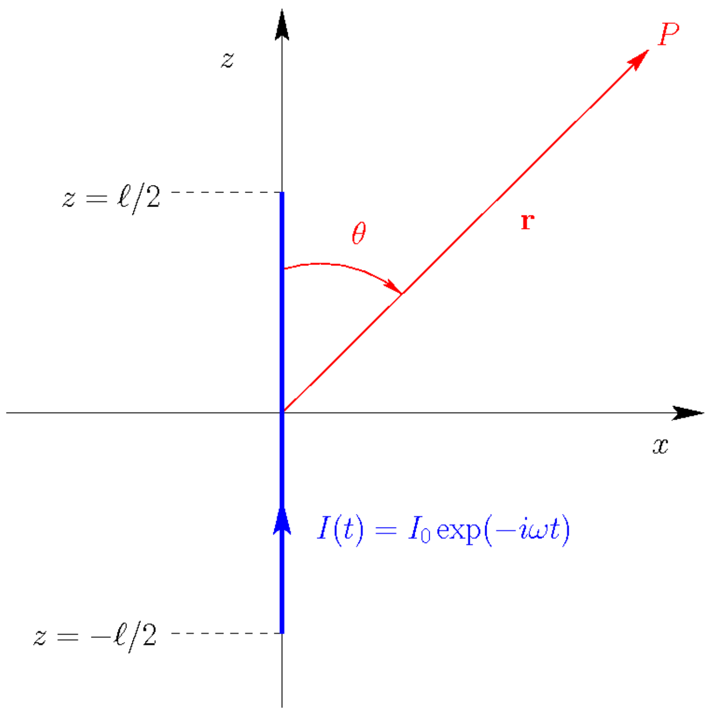
This exercise is kind of long so it will constitute both exercise 10 and 11. Hence, it will also be next weeks exercise.

Problem 1.



We consider a thin, straight, conducing wire of length ℓ that is oriented along the z-axis as shown in the figure above. The wire carries the time-varying current

$$I(t) = I_0 \exp\left(-i\omega t\right), \tag{1}$$

everywhere along its length ℓ. Here I_0 and ω are both positive constants. The system is a simple model for an antenna.

a) Charge will only build up at the endpoints of the wire. Explain why this is so. Find an expression for the time-dependent charge, $Q(t)$, building up at one endpoint.

b) Use your expression for $Q(t)$ to determine the dipole moment $\mathbf{p}(t)$ of the simple antenna.

c) Argue why the current density can be written as

$$\mathbf{J}(\mathbf{r}, t) = \hat{\mathbf{z}}\, I(t)\delta(x)\delta(y)\theta(\ell/2) - |z|), \tag{2}$$

where $\delta(\cdot)$ is the Dirac delta-function, $\theta(\cdot)$ is the (Heaviside) step-function, and $\hat{\mathbf{z}}$ is a unit vector in the positive z-direction.

We will now study the electromagnetic field radiated from the antenna. To this end, we will start by calculate the potentials. In the Lorentz-gauge the vector potential satisfies the equation

$$\nabla^2 \mathbf{A}(\mathbf{r},t) - \frac{1}{c^2}\partial_t^2 \mathbf{A}(\mathbf{r},t) = -\mu_0 \mathbf{J}(\mathbf{r},t), \tag{3}$$

where $c = 1/\sqrt{\varepsilon_0\mu_0}$ is the speed of light in vacuum. Here ε_0 and μ_0 are the electric permitivity and magnetic permeability of free space , respectively.

d) By using the Green's function for the wave-equation

$$g(\mathbf{r},t|\mathbf{r}',t') = -\frac{\delta\left(t - t' - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{4\pi|\mathbf{r}-\mathbf{r}'|}, \tag{4}$$

show that a solution to Eq. (3) is

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int d^3r'\, \frac{\mathbf{J}(\mathbf{r}',t_r)}{|\mathbf{r}-\mathbf{r}'|}. \tag{5}$$

What is the meaning of $\mathbf{r}$ and $\mathbf{r}'$? Moreover, what is the expression for t_r, and what is the physical interpretation of this quantity?

e) Calculate the vector potential $\mathbf{A}(\mathbf{r},t)$ under the assumption that $r \gg \ell$ and show that it can be written as ($k = \omega/c$)

$$\mathbf{A}(\mathbf{r},t) \approx \hat{\mathbf{z}}\,\frac{\mu_0}{4\pi}\frac{2I_0}{\cos\theta}\sin\left(\frac{k\ell}{2}\cos\theta\right)\frac{\exp\left(ikr - i\omega t\right)}{kr}. \tag{6}$$

It will now be assumed that the wavelength ($\lambda = 2\pi/k$) and distance to the observation point (r) are so that $kr \gg 1$.

f) Derive an expression for the magnetic field $\mathbf{H}(\mathbf{r},t)$ (under the condition $kr \gg 1$). [*Answer* : $\mathbf{H} \approx \frac{1}{\mu_0}\, i\mathbf{k}\times\mathbf{A}$.]

g) Calculate the corresponding electric field $\mathbf{E}(\mathbf{r},t)$ (under the same assumption as in the previous sub-problem). [*Hint* : Use Amperes law].

h) Obtain an expression for the time-averaged Poyntings vector, $\langle \mathbf{S}\rangle_t$ ($= \frac{1}{2}\mathbf{E}\times\mathbf{H}^*$), in terms of the known quantities of the problem.

The radiation pattern of the antenna is defined as

$$\frac{dP}{d\Omega} = |\langle \mathbf{S}\rangle_t|\, r^2, \tag{7}$$

where $P = \int d\Omega\, dP/d\Omega = \int \langle \mathbf{S}\rangle_t \cdot d\mathbf{A}$ is the total power radiated by the antenna.

i) Calculate $dP/d\Omega$ and show that

$$\frac{dP}{d\Omega} \propto \sin^2\left(\frac{k\ell}{2}\cos\theta\right)\tan^2\theta. \tag{8}$$

j) Assume that the antenna is small compared to the wavelength, *i.e.* $k\ell \ll 1$, and obtain the expression for $dP/d\Omega$ in this limit.

k) Argue why your expression for $dP/d\Omega$ in the limit $k\ell \ll 1$ is reasonable. Make a sketch of the radiation pattern in this case. What is this pattern called?