TFY4240

Solution problemset 3 Autumn 2014

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Problem 1.

a) We have a function $f(x)$ on the interval $[-1, 1]$. There is possible to write it as a combination of Legendre polynomials as following

$$f(x) = \sum_{\ell=0}^{\infty} A_\ell P_\ell(x).$$

Multiply by P_m on both sides and integrate from -1 to 1. This gives

$$\int_{-1}^{1} dx f(x) P_m(x) = \int_{-1}^{1} dx \sum_{\ell=0}^{\infty} A_\ell P_\ell(x) P_m(x). \tag{1}$$

Using the orthogonality of the Legendre polynomials, *i.e.*

$$\int_{-1}^{1} dx\, P_m(x) P_n(x) = \frac{2}{2n+1}\delta_{mn},$$

and interchanging the order of summation and integration in Eq. (1) leads to the relation

$$\int_{-1}^{1} dx\, f(x) P_m(x) = A_m \frac{2}{2m+1},$$

which is readily solved for the coefficients A_ℓ to give (after renaming m to ℓ)

$$A_\ell = \frac{2\ell+1}{2} \int_{-1}^{1} dx\, f(x) P_\ell(x). \tag{2}$$

Equation (2) is the relation that we should show.

b)

$$f(x) = \begin{cases} -1 & x < 0 \\ +1 & x > 0 \end{cases}$$

$f(x)$ is an odd function, hence *only* Legendre polynomials of odd order are needed, *i.e.*

$$f(x) = \sum_{n=0}^{\infty} P_{2n+1}(x)$$

c) Do the integral in equation (2)

$$\begin{aligned}
A_0 &= \tfrac{2\cdot 0+1}{2} \textstyle\int_{-1}^{1} dx f(x) P_0(x) \\
A_0 &= \tfrac{1}{2}\left(\textstyle\int_0^1 dx \cdot 1 + \int_{-1}^{0} dx \cdot (-1)\right) \\
A_0 &= \tfrac{1}{2}\left(1 + (-1)\right) \\
A_0 &= 0
\end{aligned}$$

We get the same result for all even n. For odd n, the integral from -1 to 0 is equal the integral from 0 to 1

$$A_1 = \tfrac{2\cdot 1+1}{2}\int_{-1}^{1} dx f(x)P_1(x)$$
$$A_1 = \tfrac{3}{2}\left(2\int_0^1 dx\cdot x\right)$$
$$A_1 = \tfrac{3}{2}\left(2\tfrac{1}{2}\right)$$
$$A_1 = \tfrac{3}{2}$$

$$A_3 = \tfrac{2\cdot 3+1}{2}\int_{-1}^{1} dx f(x)P_3(x)$$
$$A_3 = \tfrac{7}{2}\left(2\int_0^1 dx\cdot\tfrac{1}{2}(5x^3-3x)\right)$$
$$A_3 = \tfrac{7}{2}\left(2\cdot\tfrac{1}{2}\left(\tfrac{5}{4}-\tfrac{3}{2}\right)\right)$$
$$A_3 = -\tfrac{7}{8}$$

$$A_5 = \tfrac{2\cdot 5+1}{2}\int_{-1}^{1} dx f(x)P_5(x)$$
$$A_5 = \tfrac{11}{2}\left(2\int_0^1 dx\cdot\tfrac{1}{8}(63x^5-70x^3+15x)\right)$$
$$A_5 = \tfrac{11}{2}\left(2\cdot\tfrac{1}{2}\left(\tfrac{63}{6}-\tfrac{70}{4}+\tfrac{15}{2}\right)\right)$$
$$A_5 = \tfrac{11}{16}$$

Problem 2.

a) For the boundary condition $V(R,\theta) = V_0\cos^2\theta$ on a sphere of radius R, the potential outside the sphere can be written in the form (since the charge distribution has a azimuthal symmetry)

$$V(r,\theta) = \sum_l (R/r)^{l+1} A_l P_l(\cos\theta), \qquad r \geq R$$

with the coefficients A_l given by

$$\begin{aligned} A_l &= \tfrac{2l+1}{2}\int_{-1}^{1} d(cos\theta)V(\theta)P_l(\cos\theta) \\ &= \tfrac{(2l+1)V_0}{2}\int_{-1}^{1} d(x)x^2P_l(x) \end{aligned} \tag{3}$$

with $x = \cos\theta$. To do this integral, we recognize that

$$x^2 = \frac{3x^2-1}{3} + \frac{1}{3} = \frac{2}{3}P_2(x) + \frac{1}{3}P_0(x).$$

Then we can use the orthogonality of the Legendre polynomials together with equation (3) to get

$$A_0 = \frac{V_0}{2}\int_{-1}^{1} dx\frac{1}{3}[P_0(x)]^2 = \frac{1}{3}V_0$$

$$A_2 = \frac{5V_0}{2}\int_{-1}^{1} dx\frac{2}{3}[P_2(x)]^2 = \frac{2}{3}V_0$$

$$A_l = 0, \qquad l \neq 0,2$$

Hence, we conclude that the scalar potential can be written as

$$V(r,\theta) = \frac{V_0}{3}\left[\frac{R}{r} + 2\left(\frac{R}{r}\right)^3 P_2(\cos\theta)\right], \tag{4}$$

in the region outside the sphere, $r \geq R$.

b) The surface charge distribution on the sphere of the sphere is given by

$$\sigma(\theta) = -\epsilon_0 \left.\frac{\partial V}{\partial r}\right|_{r=R},$$

which gives

$$\sigma(\theta) = -\frac{\epsilon_0 V_0}{3}\left.\left[\frac{R}{r^2}(-1)P_0(\cos\theta) + 2\left(\frac{R^3}{r^4}\right)(-3)P_2(\cos\theta)\right]\right|_{r=R}$$

or

$$\sigma(\theta) = \frac{\epsilon_0 V_0}{3R}\left[P_0(\cos\theta) + 6P_2(\cos\theta)\right]$$

c) To appear later....!

Problem 3.

a)

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{|\boldsymbol{r} - (R+h)\hat{\boldsymbol{z}}|} - \frac{1}{4\pi\epsilon_0}\frac{q}{|\boldsymbol{r} - (R-h)\hat{\boldsymbol{z}}|} \tag{5}$$

The method of images is a technique used to solve electrostatic problems (i.e. solving Laplace eq.). It consists of placing so-called image charges outside the domain of interest such that the boundary conditions (cont. of V and $\epsilon\partial_n V$) are satisfied, then the total potential in the domain of interest is the sum of the potential from the charge and image-charge since the solution of Laplace equation is unique.

b) The two terms in the potential in equation (5) corresponds to the potential from the charge q (first term) and image charge -q (second term). They are located at $(R \pm h)\hat{\boldsymbol{z}}$ so the discance from each one of them to an observation point $\boldsymbol{r}$ (with $z > R$) is $\boldsymbol{r} - (R \pm h)\hat{\boldsymbol{z}}$. On the surface $z = R$ we have

$$\begin{aligned} |\boldsymbol{r} - (R \pm h)\hat{\boldsymbol{z}}| &= \left|\boldsymbol{r}_{\parallel} + R\hat{\boldsymbol{z}} - (R \pm h)\hat{\boldsymbol{z}}\right| \\ &= \left|\boldsymbol{r}_{\parallel} \pm h\hat{\boldsymbol{z}}\right| \end{aligned}$$

($\boldsymbol{r}_{\parallel}$ is a vector in the xy-plane) Since $\left|\boldsymbol{r}_{\parallel} \pm h\hat{\boldsymbol{z}}\right|$ is independent of sign it follows that $V(\boldsymbol{r} = 0$ when $\boldsymbol{r}$ is in the plane $z = R$.

c) The total potential for the system consist of two potentials of the form of equation (5) but with the chartes located at

$$\boldsymbol{r}_{\pm \boldsymbol{q}} = \left(R + h \pm \frac{d}{2}\right)\hat{\boldsymbol{z}}$$

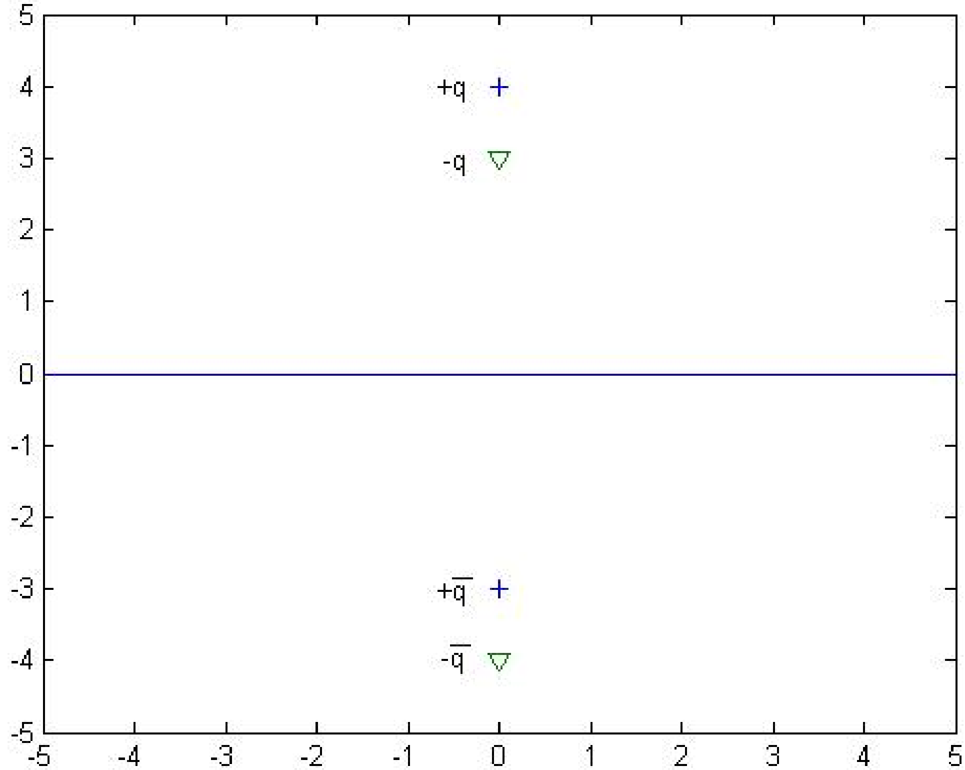

Figure 1: Position of image charges

and image charges at

$$\begin{aligned} \boldsymbol{r}_{\pm \boldsymbol{q}} &= \left(R - \left[h \mp \tfrac{d}{2}\right]\right) \hat{\boldsymbol{z}} \quad \text{note reversed sign} \\ &= \left(R - h \pm \tfrac{d}{2}\right) \hat{\boldsymbol{z}} \end{aligned}$$

Hence the total potential becomes

$$V(\boldsymbol{r}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_q|} - \frac{1}{|\boldsymbol{r}-\boldsymbol{r}_{-\bar{q}}|} + \frac{-1}{|\boldsymbol{r}-\boldsymbol{r}_{-q}|} - \frac{-1}{|\boldsymbol{r}-\boldsymbol{r}_{+\bar{q}}|}\right]$$

Now doing an expansion around $z = (R \pm h)$

$$\begin{aligned}
\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_{\pm q}|} &= \frac{1}{|\underbrace{\boldsymbol{r}-(R+h)\hat{\boldsymbol{z}}}_{\boldsymbol{\rho}_+}\mp\frac{\boldsymbol{d}}{2}|} \\
&= \frac{1}{\left(\rho_+^2\mp 2\boldsymbol{\rho}_+\frac{\boldsymbol{d}}{2}+\frac{d^2}{4}\right)^{1/2}} \\
&= \frac{1}{\rho_+\left[1\mp\frac{\hat{\boldsymbol{\rho}}_+\cdot\boldsymbol{d}}{\rho_+}+\left(\frac{d}{2\rho_+}\right)^2\right]^{1/2}} \\
&= \frac{1}{\rho_+}\left[1\pm\frac{1}{2}\frac{\hat{\boldsymbol{\rho}}_+\cdot\boldsymbol{d}}{\rho_+}+\dots\right] \\
&= \frac{1}{\rho_+}\pm\frac{1}{2}\frac{\hat{\boldsymbol{\rho}}_+\cdot\boldsymbol{d}}{\rho_+^2}+\dots
\end{aligned}$$

Similary one obtains

$$\begin{aligned}
\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_{\pm\bar{q}}|} &= \frac{1}{|\underbrace{\boldsymbol{r}-(R-h)\hat{\boldsymbol{z}}}_{\boldsymbol{\rho}_-}\pm\frac{\boldsymbol{d}}{2}|} \\
&= \frac{1}{\rho_-}\mp\frac{1}{2}\frac{\hat{\boldsymbol{\rho}}_-\cdot\boldsymbol{d}}{\rho_-^2}+\dots
\end{aligned}$$

Hence, to leading order

$$\begin{aligned}
V(\boldsymbol{r}) &= \frac{q}{4\pi\epsilon_0}\left[\frac{\hat{\boldsymbol{\rho}}_+\cdot\boldsymbol{d}}{\rho_+^2}-\frac{\hat{\boldsymbol{\rho}}_-\cdot\boldsymbol{d}}{\rho_-^2}\right] \\
&= \frac{1}{4\pi\epsilon_0}\frac{\hat{\boldsymbol{\rho}}_+\cdot\boldsymbol{p}}{\rho_+^2}-\frac{1}{4\pi\epsilon_0}\frac{\hat{\boldsymbol{\rho}}_-\cdot\boldsymbol{p}}{\rho_-^2}
\end{aligned}$$

Hence the potential is the sum of an electric dipole and an oppositly directet image dipole.

d) Since the sphere in Figure 2 is grounded we choose the potential at the surface to be zero. We try to solve the problem by the method of images by placing an image charge q' on the z-axis at position z'. The total potential outside the sphere becomes

$$V(\boldsymbol{r})=\frac{1}{4\pi\epsilon_0}\left[\frac{q}{|\boldsymbol{r}-(R+h)\hat{\boldsymbol{z}}|}+\frac{q'}{|\boldsymbol{r}-z'\hat{\boldsymbol{z}}|}\right]$$

Now we have two unknown, q' and z'. To determine them, we choose the points $\boldsymbol{r}=\pm R\hat{\boldsymbol{z}}$ and impose the boundary condition on V

i] $V(\boldsymbol{r}=R\hat{\boldsymbol{z}})=0$

$$\frac{q}{|R-R-h|}+\frac{q'}{|R-z'|}=0,(z'<R),h\neq 0$$

$$q(R-z')+q'h=0$$

$$qR-qz'+q'h=0 \tag{6}$$

ii] $V(\boldsymbol{r}=-R\hat{\boldsymbol{z}})=0$

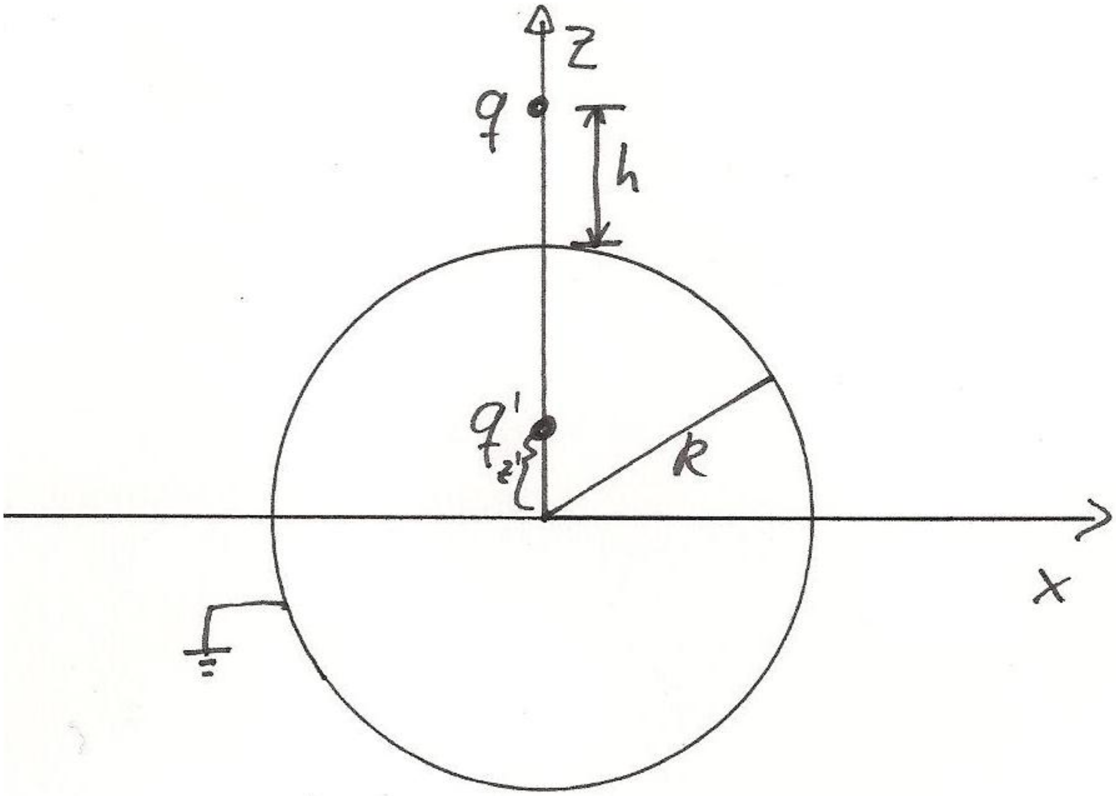

Figure 2: A charge above a grounded sphere

$$\frac{q}{|-R-R-h|} + \frac{q'}{|-R-z'|} = 0, (z' < R), h \neq 0$$
$$q(R+z') + q'(2R+h) = 0, z' > 0$$
$$qR + qz' + (2R+h)q' = 0 \tag{7}$$

Adding equation (6) and equation (7) gives

$$2Rq + (h + 2R + h)q' = 0$$

$$q' = -\frac{R}{R+h}q$$

From equation (6) it follows that

$$z' = R + \frac{q'}{q}h$$

$$z' = \frac{R(R+h) - Rh}{R+h} = \frac{R^2}{R+h}$$

e) Hence the total scalar potential becomes

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|\boldsymbol{r} - (R+h)\hat{\boldsymbol{z}}|} + \frac{\frac{R}{R+h}q}{\left|\boldsymbol{r} - \frac{R^2}{R+h}\hat{\boldsymbol{z}}\right|}\right]$$

When $R \gg h$ one has

$$\frac{R}{R+h} = \frac{R}{R(1+h/R} \simeq 1 - \frac{h}{R} + \cdots$$

$$\frac{R^2}{R+h} = \frac{R^2}{R(1+h/R} \simeq R\left(1 - \frac{h}{R} + \cdots\right) = R - h$$

Hence, in the limit $R \gg h$ one gets to lowest order

$$V(\boldsymbol{r}) \simeq \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|\boldsymbol{r} - (R+h)\hat{\boldsymbol{z}}|} - \frac{q}{|\boldsymbol{r} - (R-h)\hat{\boldsymbol{z}}|}\right]$$

This is the potential for a charge q above a flat grounded plate. This is a reasonable result!

f) Let the image charges corresponding to $\pm q$ be denoted $q_\pm$. These image charges are given by

$$q'_\pm = \mp \frac{R}{R+h \pm \frac{d}{2}} q$$

Since the image charge is depending on the distance from the center of the sphere it follows that

$$q'_+ + q'_- \neq 0$$

Hence, there will be a monopole contribution to the potential coming from the image charges. Since this term will be dominating it is not possible to find an image dipole so that the potential on r=R vanishes.

However, if $\boldsymbol{d}$ is chosen to be parallel with the xy-plane, i.e.

$$\boldsymbol{d} = d\hat{\boldsymbol{r}}_{\|}$$

then the distances from the center of the sphere to $q'_\pm$ are the same so $q'_+ + q'_- = 0$. Therefore, the mono-pole term coming from the image charges vanishes, and the leading order term is an image dipole. The diploe moment of the image charges is

$$p' = |\boldsymbol{p}'| = |q'|d'$$

where d' is the distance given in Figure 3.

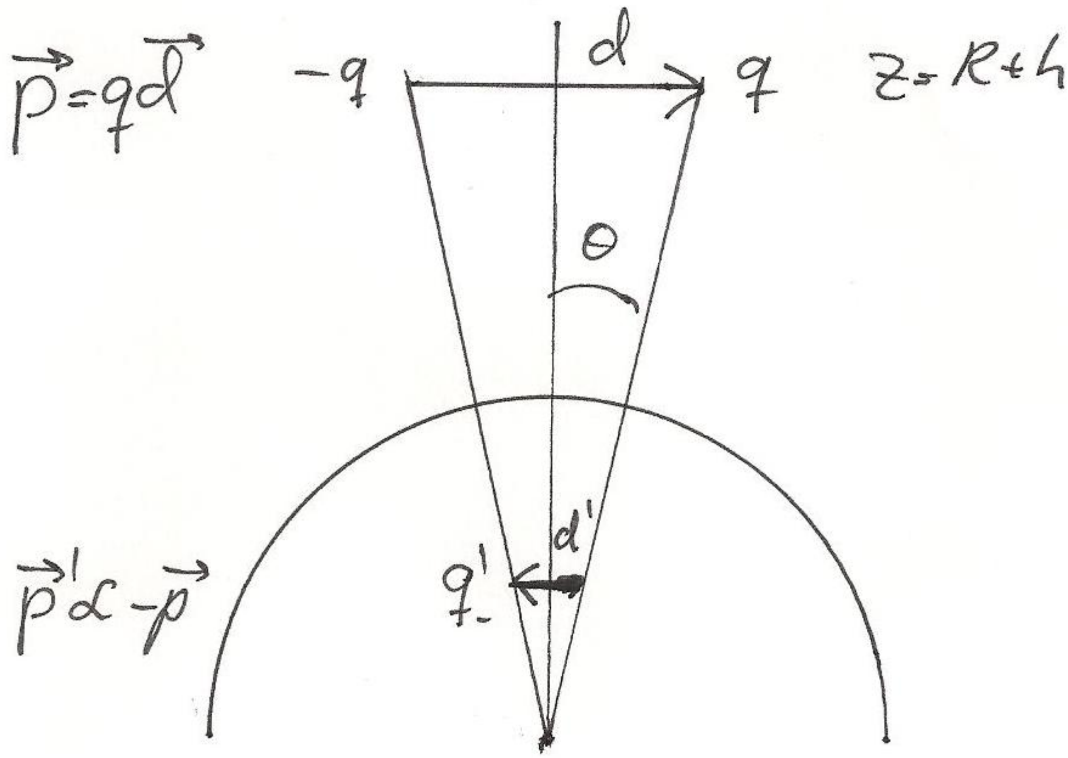

Figure 3: Dipole image

Let R+H be the distance to, say, q. From geometry it follows

$$\frac{d}{R+H} = \frac{d'}{\frac{R^2}{R+H}}$$

$$d' = \frac{R^2}{(R+H)^2} d$$

Now

$$q' = -\frac{R}{R+H} q$$

so that (direction follows from Figure 3)

$$\boldsymbol{p}' = \frac{R^3}{(R+H)^3} \boldsymbol{p}$$

However, we have assumed that $d/h \ll 1$ so that one my write

$$\begin{aligned}
\boldsymbol{p}' &= \frac{R^3}{\left[(R+h)^2+\left(\frac{d}{2}\right)^2\right]^{3/2}}\boldsymbol{p} \\
&= \frac{R^3}{(R+h)^3}\left[1+\left(\frac{d}{2(R+h)}\right)^2\right]^{-3/2}\boldsymbol{p} \\
&= \frac{R^3}{(R+h)^3}\boldsymbol{p} + O(d^3/(R/h)^5)
\end{aligned}$$

Thus, one my safly conclude that

$$\boldsymbol{p}' \simeq \frac{R^3}{(R+h)^3}\boldsymbol{p}$$

Alt: We could have made the approximation $R+H \simeq R+h$ from the very beginning. (To simplify the calcluation)