TFY4240

Solution problemset 9 Autumn 2014

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Problem 1.

a) How to chose $\Lambda(\boldsymbol{r}, t)$ in the Lorentz gauge?

We want in the transformed system that

$$\nabla \cdot \mathbf{A}' + \epsilon\mu\partial_t V' = 0,$$

where (the gauge transformation)

$$\mathbf{A}' = \mathbf{A} + \nabla\Lambda(\mathbf{r}, t)$$
$$V' = V - \partial_t\Lambda(\mathbf{r}, t)$$

Plugging these latter results into the Lorentz gauge condition:

$$\nabla(\mathbf{A} + \nabla\Lambda) + \epsilon_0\mu_0\partial_t(V - \partial_t V) = \nabla\mathbf{A} + \nabla^2\Lambda + \epsilon_0\mu_0\partial_t V - \epsilon_0\mu_0\partial_t^2\Lambda$$
$$= 0$$

which leads to

$$\nabla^2\Lambda - \epsilon_0\mu_0\partial_t^2\Lambda = -\nabla\mathbf{A} - \epsilon_0\mu_0\partial_t V \quad (1)$$
$$\nabla^2\Lambda - \epsilon_0\mu_0\partial_t^2\Lambda = -\left[\nabla\mathbf{A} + \epsilon_0\mu_0\partial_t V\right] \quad (2)$$

For a given set (V,**A**) of allowed potentials, the Lorentz gauge can be obtained by choosing a gauge function $\Lambda(\mathbf{r}, t)$ that satisfies equation (2), i.e. a wave-equation with a right hand side given by the original potentials.

Problem 2.

a) We have

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\mathbf{J}(\mathbf{r}', t_r}{R})$$
$$t_r = t - \frac{R}{c}$$
$$R = |\mathbf{r} - \mathbf{r}'|$$
$$\mathbf{B} = \nabla \times \mathbf{A}$$

For the calculation

$$\mathbf{B} = \nabla \times \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int d^3r' \nabla \times \frac{\mathbf{J}(\mathbf{r}', t_r)}{R}$$
$$B_i = \frac{\mu_0}{4\pi}\int d^3r' \epsilon_{ijk}\partial_j \frac{J_k(\mathbf{r}', t_r)}{R}$$

$$\begin{aligned}
\partial_j \frac{J_k(\mathbf{r}', t_r)}{R} &= \frac{\dot{J}_k(\mathbf{r}', t_r)(-\frac{1}{c}\partial_j R)}{R} - \frac{J_k(\mathbf{r}', t_r)\partial_j R}{R^2} \\
&= -\left[\frac{J_k(\mathbf{r}', t_r}{R^2} + \frac{\dot{J}_k(\mathbf{r}', t_r}{Rc}\right]\partial_j R
\end{aligned}$$

$$\begin{aligned}
R = |\boldsymbol{R}| = |\mathbf{r} - \mathbf{r}'| &= \sqrt{(x_m - x'_m)(x_m - x'_m)} \\
\partial_j R &= \frac{2(x_m - x'_m)\partial_j x_m}{2R} \\
&= \frac{(x_m - x'_m)}{R}\delta_{jm} \\
&= \frac{x_j - x'_j}{R} \\
&= \frac{R_j}{R}
\end{aligned}$$

Collecting all the terms gives

$$B_i = \frac{\mu_0}{4\pi}\int d^3r' \epsilon_{ijk}\frac{R_i}{R}\left\{-\left[\frac{J_k(\mathbf{r}', t_r}{R^2} + \frac{\dot{J}_k(\mathbf{r}', t_r}{Rc}\right]\right\}$$

Hence

$$\begin{aligned}
\mathbf{B} &= \frac{\mu_0}{4\pi}\int d^3r' \hat{R} \times \left[\frac{\mathbf{J}(\mathbf{r}', t_r}{R^2} + \frac{\dot{\mathbf{J}}(\mathbf{r}', t_r}{Rc}\right](-1) \\
\mathbf{B} &= \frac{\mu_0}{4\pi}\int d^3r' \left[\frac{\mathbf{J}(\mathbf{r}', t_r}{R^2} + \frac{\dot{\mathbf{J}}(\mathbf{r}', t_r}{Rc}\right] \times \hat{R}
\end{aligned}$$