TFY4240

Solution problemset 10-11 Autumn 2014

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Problem 1.

a) Since the current is going from one end to the other of the wire, only the ends will be charged. The charge can be determined from the differential equation that follows from the definition of current.

$$I(t) = \frac{\mathrm{d}Q(t)}{\mathrm{d}t}$$
$$\Rightarrow \quad Q(t) = \int_t \mathrm{d}t' I(t') = \frac{i}{\omega} I(t)$$

[As usual, it is the real part of this which is the physical charge, that is $Re\{Q(t)\} = (I_0/\omega)\sin\omega t$].

b) The (complex) time-dependent dipole moment is

$$\boldsymbol{p}(t) = Q(t)\ell\hat{\boldsymbol{z}} = i\frac{I_0\ell}{\omega}\exp(-i\omega t)\hat{\boldsymbol{z}}$$

c) Since the wire is coinciding with the z-axis and has length ℓ, it follows directly that

$$\boldsymbol{J}(\boldsymbol{r},t) = \hat{\boldsymbol{z}} I(t)\delta(x)\delta(y)\theta(|z| - \ell/2)$$

The δ-functions place the wire along the z-axis, and the θ-function gives it the correct length.

d) The PDE for the vector potential (in the Lorentz gauge reads)

$$\nabla^2 \boldsymbol{A}(\boldsymbol{r},t) - \frac{1}{c^2}\frac{\partial}{\partial t}^2 \boldsymbol{A}(\boldsymbol{r},t) = -\mu_0 \boldsymbol{J}(\boldsymbol{r},t)$$

A particular solution to this equation can be constructed as

$$\begin{aligned}
\boldsymbol{A}_p(\boldsymbol{r},t) &= \int \mathrm{d}^3r'\mathrm{d}t' g(\boldsymbol{r},t|\boldsymbol{r'},t')\Big[-\mu_0\boldsymbol{J}(\boldsymbol{r'},t)\Big] \\
&= \mu_0 \int \mathrm{d}^3r'\mathrm{d}t' \frac{\delta\Big(t - t' - \frac{|\boldsymbol{r}-\boldsymbol{r'}|}{c}\Big)}{4\pi|\boldsymbol{r}-\boldsymbol{r'}|}\boldsymbol{J}(\boldsymbol{r'},t) \\
&= \frac{\mu_0}{4\pi}\int \mathrm{d}^3r' \frac{\boldsymbol{J}(\boldsymbol{r'},t_r)}{|\boldsymbol{r}-\boldsymbol{r'}|}
\end{aligned}$$

where the retarded tiem is given as

$$t_r = t - \frac{|\boldsymbol{r} - \boldsymbol{r'}|}{c}.$$

This is the time it takes for a signal to propagate from $\boldsymbol{r'}$ (the source point) to $\boldsymbol{r}$ (the observation point).

e) We start by looking at the term $I(t_r)$ contained in the expression for $\boldsymbol{J}(\boldsymbol{r'}, t_r)$

$$\begin{aligned} I(t_r) &= I_0 \exp(-i\omega t_r) \\ &= I_0 \exp(-i\omega t + i\omega \frac{|\boldsymbol{r} - \boldsymbol{r'}|}{c}) \\ &= I_0 \exp(-i\omega t) \exp(i\omega \frac{|\boldsymbol{r} - \boldsymbol{r'}|}{c}) \end{aligned}$$

Now expanding $|\boldsymbol{r} - \boldsymbol{r'}|$ as

$$\begin{aligned} |\boldsymbol{r} - \boldsymbol{r'}| &= r\Big[1 - 2\frac{\hat{\boldsymbol{r}} \cdot \boldsymbol{r'}}{r} + \Big(\frac{r'}{r}\Big)^2\Big]^{1/2} \\ &\approx r\Big[1 - \frac{\hat{\boldsymbol{r}} \cdot \boldsymbol{r'}}{r}\Big] \\ &= r - \hat{\boldsymbol{r}} \cdot \boldsymbol{r'} \end{aligned}$$

so that

$$I(t_r) \approx I_0 \exp(-i\omega t) \exp(ikr) \exp(-ik\hat{\boldsymbol{r}} \cdot \boldsymbol{r'}).$$

Now the expression for the vector potential becomes using the result from d)

$$\begin{aligned} \boldsymbol{A}(\boldsymbol{r}, t) &= \hat{\boldsymbol{z}} \frac{\mu_0}{4\pi} \int_{-\ell/2}^{\ell/2} \mathrm{d}z' \frac{I(t_r)}{|\boldsymbol{r} - z'\hat{\boldsymbol{z}}|} \quad \boldsymbol{r'} = z'\hat{\boldsymbol{z}} \\ &\approx \hat{\boldsymbol{z}} \frac{\mu_0 I_0}{4\pi} \frac{\exp(ikr - i\omega t)}{r} \int_{-\ell/2}^{\ell/2} \mathrm{d}z' \exp(-ikz' \cos\theta) \\ &= \hat{\boldsymbol{z}} \frac{\mu_0 I_0}{4\pi} \frac{\exp(ikr - i\omega t)}{r} \Big[\frac{\exp(-ikz' \cos\theta)}{-ik\cos\theta}\Big]_{-\ell/2}^{\ell/2} \\ &= \hat{\boldsymbol{z}} \frac{\mu_0 I_0}{4\pi} \frac{\exp(ikr - i\omega t)}{r} \frac{1}{-ik\cos\theta} \Big[\exp(-ik\ell\cos\theta/2) - \exp(ik\ell\cos\theta/2)\Big] \\ &= \hat{\boldsymbol{z}} \frac{\mu_0 I_0}{2\pi} \frac{\exp(ikr - i\omega t)}{kr} \frac{1}{\cos\theta} \sin(k\ell\cos\theta/2) \\ &= \hat{\boldsymbol{z}} \frac{\mu_0 I_0}{2\pi\cos\theta} \sin(k\ell\cos\theta/2) \frac{\exp(ikr - i\omega t)}{kr} \end{aligned}$$

f) By definition, it follows that

$$\boldsymbol{H}(\boldsymbol{r}, t) = \frac{1}{\mu_0} \nabla \times \boldsymbol{A}(\boldsymbol{r}, t).$$

In spherical coordinates is $\mathbf{A} = A(\cos\theta\hat{r} - \sin\theta\hat{\theta})$ and $\nabla \times \mathbf{A}$ are given by

$$
\begin{aligned}
\nabla \times \mathbf{A} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\sin\theta\underbrace{A_\phi}_{=0} - \underbrace{\frac{\partial A_\theta}{\partial\phi}}_{=0}\right]\hat{\mathbf{r}} \\
&+ \frac{1}{r}\left[\frac{1}{\sin\theta}\underbrace{\frac{\partial A_r}{\partial\phi}}_{=0} - \frac{\partial}{\partial r}(r\underbrace{A_\phi}_{=0}\right]\hat{\boldsymbol{\theta}} \\
&+ \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial\theta}A_r\right]\hat{\boldsymbol{\phi}} \\
&= \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial\theta}A_r\right]\hat{\boldsymbol{\phi}}
\end{aligned}
$$

$\mathbf{A}$ might be written as $\mathbf{A} = R(r)\Theta(\theta)\hat{z}$ which leads to

$$
\begin{aligned}
\frac{1}{r}\frac{\partial(rR(r))}{\partial r} &= \frac{1}{r}\frac{\partial}{\partial r}\frac{r\exp(ikr - i\omega t)}{kr} \\
&= \frac{ik\exp(ikr - i\omega t)}{kr} \\
&= ikR(r) \\
&\propto \frac{1}{r}
\end{aligned}
$$

for large r

$$
\begin{aligned}
\nabla \times \mathbf{A} &= \left(-\sin\theta\frac{\Theta(\theta)}{r}\frac{\partial(rR(r))}{\partial r} - \cos\theta\frac{R(r)}{r}\frac{\partial\Theta(\theta)}{\partial\theta}\right)\hat{\boldsymbol{\phi}} \\
&\propto \left(\frac{1}{r} - \frac{1}{r^2}\right)\hat{\boldsymbol{\phi}} \\
&\approx -ik\sin\theta R(r)\Theta(\theta)\hat{\boldsymbol{\phi}} \\
&= -ik\sin\theta A\hat{\phi}, \\
\hat{k} \times \hat{z} &= -\sin\theta\hat{\boldsymbol{\phi}}
\end{aligned}
$$

And finally

$$
\begin{aligned}
\mathbf{H}(\mathbf{r}, t) &= \frac{1}{\mu_0}\nabla \times \mathbf{A} \\
&= -\frac{1}{\mu_0}ikA\underbrace{\sin\theta\hat{\boldsymbol{\phi}}}_{=-\hat{\boldsymbol{k}}\times\hat{\boldsymbol{z}}} \\
&= \frac{i}{\mu_0}k\hat{\boldsymbol{k}} \times A\hat{\boldsymbol{z}} \\
&= \frac{i}{\mu_0}\boldsymbol{k} \times \mathbf{A}
\end{aligned}
$$

g) From Amperes law it follows

$$-i\omega\epsilon_0 \boldsymbol{E}(\boldsymbol{r},t) = \nabla \times \boldsymbol{H}(\boldsymbol{r},t),$$

so that the electric field becomes

$$\boldsymbol{E}(\boldsymbol{r},t) = \frac{1}{-i\omega\epsilon_0}\nabla \times \boldsymbol{H}(\boldsymbol{r},t).$$

Now to calculate $\nabla \times \boldsymbol{H}(\boldsymbol{r},t)$ can be done in the "quick" or not so quick manner. Since we are interested in the behavior in the far field, the magnetic field is locally essentially a place wave for which it follows that $\nabla \times \mathbf{H} \approx i\boldsymbol{k} \times \mathbf{H}(\mathbf{r},t)$. This expression we will now derive by doing an explicit calculation were we will only keep radiation field terms, i.e., terms that decay like $1/r$. The details of the calculation are like follows :

$$\begin{aligned}
[\nabla \times \boldsymbol{H}(\boldsymbol{r},t)]_i &= \varepsilon_{ijk}\partial_j H_k \\
&\approx \varepsilon_{ijk}\partial_j \left(\frac{i}{\mu_0}\varepsilon_{klm}k_l A_m\right) \\
&= \frac{i}{\mu_0}\varepsilon_{ijk}\varepsilon_{klm}k_l\partial_j A_m.
\end{aligned}$$

From this expression it follows that we have to calculate $\partial_j A_m$, and we note that $A_m = \delta_{m3}A_3$. By keeping only terms that are of order $1/r$ (radiation fields) we get

$$\begin{aligned}
\partial_j A_m &= \delta_{m3}\partial_j A_3 \\
&= \delta_{m3}\frac{\mu_0 I_0}{2\pi\cos\theta}\sin\left(\frac{k\ell}{2}\cos\theta\right)\partial_j\left(\frac{\exp(ikr - i\omega t)}{kr}\right) + \mathcal{O}(1/r^2).
\end{aligned}$$

Here we have used that derivatives of the angular part will not contribute to $1/r$ terms in $\partial_j A_m$ (show this!). In fact there is only one term that contributes to the $1/r$ terms and that comes from the derivatives of the exponential function. This is seen as follows:

$$\begin{aligned}
\partial_j A_m &= \delta_{m3}\partial_j A_3 \\
&= \delta_{m3}\frac{\mu_0 I_0}{2\pi\cos\theta}\sin\left(\frac{k\ell}{2}\cos\theta\right)\frac{ik_j\,kr - k_j}{(kr)^2}\exp(ikr - i\omega t) + \mathcal{O}(1/r^2) \\
&= \delta_{m3}\frac{\mu_0 I_0}{2\pi\cos\theta}\sin\left(\frac{k\ell}{2}\cos\theta\right)ik_j\frac{\exp(ikr - i\omega t)}{kr} + \mathcal{O}(1/r^2) \\
&= ik_j A_m + \mathcal{O}(1/r^2).
\end{aligned}$$

Hence, it follows that

$$\begin{aligned}
[\nabla \times \boldsymbol{H}(\boldsymbol{r},t)]_i &= \frac{i}{\mu_0}\varepsilon_{ijk}\varepsilon_{klm}k_l\partial_j A_m + \mathcal{O}(1/r^2) \\
&= \varepsilon_{ijk}ik_j\left(\frac{i}{\mu_0}\varepsilon_{klm}k_l A_m\right) + \mathcal{O}(1/r^2) \\
&= \varepsilon_{ijk}ik_j H_k + \mathcal{O}(1/r^2) \\
&= [i\boldsymbol{k} \times \mathbf{H}(\mathbf{r},t)]_i + \mathcal{O}(1/r^2),
\end{aligned}$$

so that in the far field one has (i.e. to order $1/r$)

$$\nabla \times \mathbf{H} \approx i\boldsymbol{k} \times \mathbf{H}(\mathbf{r}, t). \tag{1}$$

With this results it follows directly from Amperes law that the electric field in the far field (the electric radiation field) is:

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{r}, t) &= -\frac{\boldsymbol{k}}{\omega\epsilon_0} \times \boldsymbol{H}(\boldsymbol{r}, t) \\ &= -\frac{\boldsymbol{k}}{\omega\mu_0\epsilon_0} \times (i\boldsymbol{k} \times \boldsymbol{A}(\boldsymbol{r}, t)) \\ &= \frac{-i}{\omega\mu_0\epsilon_0}\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{A}(\boldsymbol{r}, t)). \end{aligned}$$

h)

$$\begin{aligned} \langle \boldsymbol{S} \rangle_t &= \frac{1}{2}\boldsymbol{E} \times \boldsymbol{H}^*, \qquad \boldsymbol{E} = \frac{-1}{\omega\epsilon_0}\boldsymbol{k} \times \mathbf{H} \\ &= \frac{-1}{2\omega\epsilon_0}(\boldsymbol{k} \times \boldsymbol{H}) \times \boldsymbol{H}^* \\ &= \frac{1}{2\omega\epsilon_0}\boldsymbol{H}^* \times (\boldsymbol{k} \times \boldsymbol{H}) \\ &= \frac{1}{2\omega\epsilon_0}\left\{\boldsymbol{k}|\boldsymbol{H}|^2 - \boldsymbol{H}(\underbrace{\boldsymbol{H}^* \cdot \boldsymbol{k}}_{=0})\right\} \\ &= |\boldsymbol{H}|^2 \frac{\boldsymbol{k}}{2\omega\epsilon_0} \\ &= \frac{1}{2c\epsilon_0}|\boldsymbol{H}|^2\hat{\boldsymbol{k}} \\ &= \frac{c\mu_0}{2}|\boldsymbol{H}|^2\hat{\boldsymbol{k}} \\ &= \frac{c\mu_0}{2}\frac{1}{\mu_0^2}k^2|\boldsymbol{A}|^2\sin^2(\theta)\hat{\boldsymbol{k}} \\ &= \frac{c}{2\mu_0}k^2|\boldsymbol{A}|^2\sin^2(\theta)\hat{\boldsymbol{k}} \end{aligned}$$

i)

$$\begin{aligned} \frac{\mathrm{d}P}{\mathrm{d}\Omega} &= |\langle \boldsymbol{S} \rangle_t| \cdot r^2 \\ &= \frac{c}{2\mu_0}k^2\frac{\mu_0^2}{2^2\pi^2}\frac{I_0^2}{\cos^2(\theta)}\sin^2(k\ell\cos\theta/2)\frac{r^2}{k^2r^2}\sin^2(\theta) \\ &= \frac{c\mu_0}{8\pi^2}I_0^2\sin^2(k\ell\cos\theta/2)\tan^2(\theta) \end{aligned}$$

j) For small x we have $\sin x \approx x$.

$$\begin{aligned} \frac{\mathrm{d}P}{\mathrm{d}\Omega} &= \frac{c\mu_0}{8\pi^2}I_0^2\sin^2(k\ell\cos\theta/2)\tan^2(\theta) \\ &\approx \frac{c\mu_0}{8\pi^2}I_0^2(\frac{k\ell}{2}\cos\theta)^2\tan^2(\theta) \\ &= \frac{c\mu_0}{8\pi^2}I_0^2\frac{k^2\ell^2}{4}\sin^2\theta \end{aligned}$$

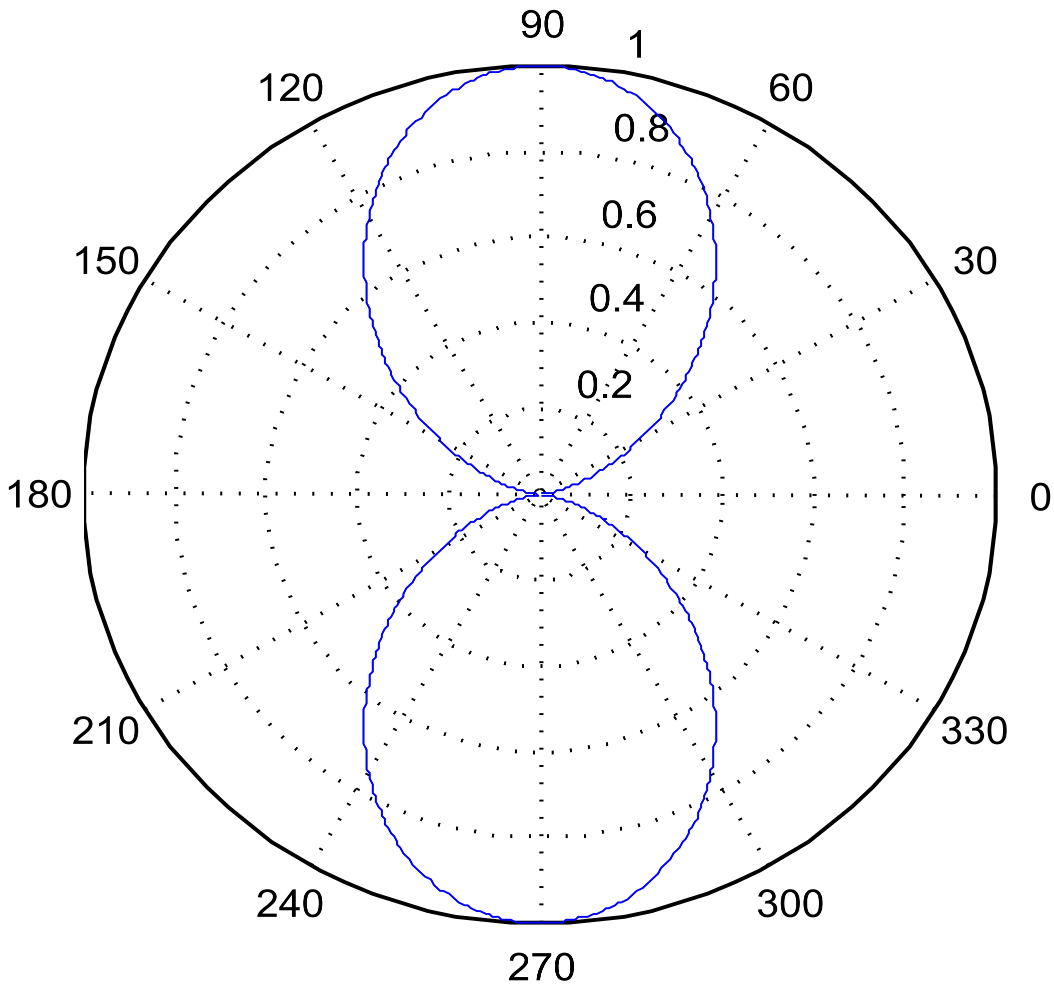

Figure 1: Radiation pattern, NOTE: z-axis along the line $0^\circ - 180^\circ$

k) This radiation pattern is the same as for a small dipole, Figure 1.