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# Exam in TFY4275/FY8907 CLASSICAL TRANSPORT THEORY

May 16, 2013
09:00–13:00

Allowed help: Alternativ **C**
Authorized calculator and mathematical formula book

This problem set consists of 4 pages, plus an Appendix of one page.

This exam consists of three problems each containing several sub-problems. Each of the sub-problems will be given approximately equal weight during grading (if nothing else is said to indicate otherwise). It is estimated that problem three is the most time consuming to answer. Problem two may seem long, but there is actually not so much calculations required to answer it.

I (or a substitute) will be available for questions related to the problems themselves (though not the answers!). The first round (of two), I plan to do a round 10am, and the other one, about two hours later.

The problems are given in English only. Should you have any language problems related to the exam set, do not hesitate to ask. For your answers, you are free to use either English or Norwegian.

Good luck to all of you!

**Problem 1.**

In this problem, we study the *continuous time random walk* (CTRW) model characterized by the (joint) jump-distribution

$$\psi(x,t) = N \exp\left(-\frac{|x|}{a} - \frac{t}{\tau}\right). \tag{1}$$

In Eq. (1), $x$ denotes the *jump-size* and $t$ the *waiting time* of the CTRW process, and $N$, $a$ and $\tau$ are all positive constants.

**a)** Determine the constant $N$, when assuming $a$ and $\tau$ to be given. Obtain expressions for the marginal distributions for jump-size and waiting time, denoted $p_x(x)$ and $p_t(t)$, respectively.

**b)** Explain why the CTRW process characterized by Eq. (1) is an ordinary diffusive process. Determine the corresponding diffusion constant, $D$.

**Problem 2.**

Here we will study a surface growth model where atoms from some external source absorb onto a surface and absorbed atoms can evaporate. Depending on the nature of the incident flux, the mobility of the adatoms, and relaxation mechanisms, a wide variety of surface morphologies can arise.

A surface is characterized by its height from a reference plane $x_3 = H(\mathbf{x}_\parallel, t)$, as a function of transverse (in-plane) coordinate $\mathbf{x}_\parallel = (x_1, x_2, 0)$ and time $t$. Here we have defined a coordinate system where the $x_1x_2$ plane is parallel to the mean (plane) surface, and the $x_3$-axis is perpendicular to this plane and pointing upward.

It is more convenient to consider deviations of the surface from its average height ($\langle H(\mathbf{x}_\parallel, t)\rangle$), so we define

$$h(\mathbf{x}_\parallel, t) = H(\mathbf{x}_\parallel, t) - \langle H(\mathbf{x}_\parallel, t)\rangle . \tag{2}$$

The particular surface growth model that will be consider here is described by the so-called Edwards-Wilkinson equation

$$\frac{\partial h(\mathbf{x}_\parallel, t)}{\partial t} = D\nabla^2 h(\mathbf{x}_\parallel, t) + \eta(\mathbf{x}_\parallel, t), \tag{3}$$

where $D$ is a positive constant. In Eq. (3), $\eta(\mathbf{x}_\parallel, t)$ denotes a noise term that is assumed to be Gaussian with zero mean, independent of $h$, and spatially and temporally uncorrelated, *i.e.*

$$\langle \eta(\mathbf{x}_\parallel, t)\rangle = 0 \tag{4a}$$

$$\left\langle \eta(\mathbf{x}_\parallel, t)\eta(\mathbf{x}'_\parallel, t')\right\rangle = 2\Gamma\delta(\mathbf{x}_\parallel - \mathbf{x}'_\parallel)\delta(t-t'), \tag{4b}$$

where $\Gamma > 0$ is a constant. Hence, Eq. (3) represents a *diffusion equation with noise*.

*Comment*: Viewed as a surface evolution model, the Laplace operator in Eq. (3) is positive near local surface minima (positive curvature) and negative near local maxima (negative curvature). Thus the Laplacian tends to smooth a surface and mimics the influence of the surface tension.

The purpose of this problem, is to study some of the statistical properties of the growing surface. For simplicity we will restrict ourselves to one-dimensions in space and let $x$ denote the lateral (in-plane) spatial coordinate[1] In particular, we will be interested in the time-evolution of the *surface width* that we defined as

$$w(t) = \sqrt{\langle h^2(x,t)\rangle}. \tag{5}$$

Initially it is assumed that the surface is flat, *i.e.*, $h(x, t = 0) = 0$.

**a)** Use dimensional analysis to show that the non-trivial scaling relation for the width (squared) is

$$w^2(t) = Dt\, F(\kappa), \tag{6}$$

where $F(\cdot)$ is an unknown, *dimensionless* function. Identify the function argument $\kappa$ in terms of the parameters of the problem.

**b)** Use the linearity of the Edwards-Wilkinson equation (3) to determine the functional form of $F(\cdot)$ up to a constant.

Hence, dimensional analysis and linearity are alone capable of determining the form of the surface width up to a constant. We now aim at, by direct calculations of the surface width, to verify this result and to obtain the unknown constant.

**c)** Use the propagator, $p(x, t|x_0, t_0)$, of the diffusion equation, also called the fundamental solution (see the appendix and/or Eq. (9)), to write down a general solution, $h(x, t)$, of the Edwards-Wilkinson equation (3). State the physical reasons for the expression you give. Note that your answer will contains some integrals, but you are *not* expected to evaluate them in this sub-problem.

**d)** Use your expression for $h(x, t)$ to explicitly calculate the surface width $w^2(t)$. Is your result consistent with what was found above by dimensional analysis?

**Problem 3.**

In this problem, we will be concerned with diffusion in one-dimension, but the diffusion "constant" may potentially be spatially dependent.

**a)** State Fick's two laws of diffusion, and use them to obtain the diffusion equation for a system of a general, potentially spatially dependent, diffusion constant $D(x)$.

In the rest of this problem, we will limit ourselves to the following special form for the diffusion constant:

$$D(x) = \begin{cases} D_-, & x < 0 \\ D_+, & x \geq 0 \end{cases}, \tag{7}$$

where $D_\pm$ are positive constants ($D_\pm > 0$). Furthermore, we will assume (the initial condition) that at $t = 0$ the particles are at $x = x_0 > 0$.

[1]Strictly speaking, the amplitude of the noise (see Eq. (4)), have different units in one and two-dimensions. Since it will cause no confusion, we will still denote the noise amplitude in one-dimension by $\Gamma$.

**b)** State the boundary conditions that the solution of the diffusion problem, $p(x,t|x_0,0)$, should satisfy at (*i*) $x = \pm\infty$ and at (*ii*) $x = 0$. Give reasons for your answers.

We will below use these boundary conditions to analytically solve this diffusion problem. To this end, we will assume a solution of the form

$$p(x,t|x_0,0) = \begin{cases} A_- p_-(x,t|x_0,0), & x < 0 \\ A_+ p_+(x,t|x_0,0), & x \geq 0 \end{cases}, \tag{8}$$

where $A_\pm$ are unknown functions independent of space (but potentially dependent on time), and $p_\pm(x,t|x_0,0)$ are the fundamental solutions of the (one-dimensional) diffusion equation with diffusion constants $D_\pm$ everywhere in space, i.e.

$$p_\pm(x,t|x_0,0) = \frac{1}{\sqrt{4\pi D_\pm t}} \exp\left(-\frac{(x-x_0)^2}{4D_\pm t}\right), \tag{9}$$

where $p_\pm(x,0|x_0,0) = \delta(x-x_0)$. Since the form (8) automatically satisfies the boundary conditions at $x = \pm\infty$, the remaining boundary condition is the one at $x = 0$.

Before we embark on the analytic solution of this problem, we will consider a property that the solution should satisfy.

**c)** Since the initial condition is, $p(x,0|x_0,0) = \delta(x-x_0)$, with $x_0 > 0$, one should expect that $p(x,t|x_0,0) \approx p_+(x,t|x_0,0)$ for sufficiently short time $t \ll t_*$ (for some constant $t_*$ to be determined). From general principles, obtain an expression for $t_*$ and express your answer in terms of $x_0$ and $D_+$.

Now we will try to obtain an analytic expression for the solution $p(x,t|x_0,0)$ of our diffusion problem.

**d)** Use the boundary condition at $x = 0$ to obtain a relation between $A_+$ and $A_-$, and express $A_-$ in terms of $A_+$. Check explicitly your answer in the limit $D_+ = D_-$. Is the answer as expected?

[*Answer* :

$$A_- = A_+ \sqrt{\frac{D_-}{D_+}} \exp\left\{\frac{(D_+ - D_-)x_0^2}{4D_+D_-t}\right\}.$$

This expression you should derive, but we give you the answer here so that the remaining sub-problems can still be completed even without answering this one correctly.]

**e)** With the relation for $A_-$ in terms of $A_+$, write down the expression for the solution $p(x,t|x_0,0)$. Explicitly demonstrate that for time $t \ll t_*$ (with your expression for $t_*$), the solution satisfies $p(x,t|x_0,0) \approx p_+(x,t|x_0,0)$.

**f)** (*Double weight*) Finally, obtain an expression for $A_+$. Express you answer in terms of the error-function (see appendix), erf(·), the constants $x_0$ and $D_\pm$ as well as $t$. Check you answer in the special case where $D_+ = D_-$.

[Are you not able to obtain $A_+$, or should you not have time to do so, explain, for partial credit, how you in principle would proceed with the calculation.]

## Appendix

- The error function ($x \geq 0$):

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{d}\chi \, \exp\left(-\chi^2\right)$$

- Properties of erf($x$): $\mathrm{erf}(-x) = -\mathrm{erf}(x)$ (odd function); $\mathrm{erf}(0) = 0$; $\mathrm{erf}(\pm\infty) = \pm 1$;
- The complementary error function

$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \mathrm{d}\chi \, \exp\left(-\chi^2\right)$$

- Gaussian integrals

$$\int_{-\infty}^{\infty} \mathrm{d}x \, \exp\left(-ax^2 - 2bx\right) = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{a}\right), \qquad a > 0$$

- Integrals of integrands containing exponential functions

$$\int_0^\infty \mathrm{d}x \, x^n \exp(-ax) = \frac{n!}{a^{n+1}}; \qquad a > 0; \; n = 0, 1, 2 \ldots$$

- The fundamental solution of the diffusion equation:

$$p(x, t | x_0, t_0) = \frac{1}{\sqrt{4\pi D(t - t_0)}} \exp\left(-\frac{(x - x_0)^2}{4D(t - t_0)}\right),$$