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Solution to the exam in
TFY4275/FY8907 CLASSICAL TRANSPORT THEORY

Jun 08, 2010

This solution consists of 7 pages.

Problem 1. "Icelandic" ash problem

a) The main transport mechanisms are advection and convection. Diffusion plays only a minor role.

b) The diffusion-advection equation reads

$$\frac{\partial c(\mathbf{r},t)}{\partial t} + \boldsymbol{\nabla}\left[c(\mathbf{r},t)\mathbf{v}\right] = D\nabla^2 c(\mathbf{r},t) \tag{1}$$

c) Here it is simplest to make the following change of variables

$$\begin{aligned}\mathbf{R} &= \mathbf{r} - \mathbf{r}_s - \mathbf{v}(t-t_s),\\ T &= t - t_s.\end{aligned}$$

To simplify the notation let $\mathbf{R} = (X, Y)$. Hence it follows that $\frac{\partial}{\partial x} = \frac{\partial X}{\partial x}\frac{\partial}{\partial X}$ so that

$$\boldsymbol{\nabla} = \boldsymbol{\nabla}_R.$$

Moreover, it readily follows that since both T and $\mathbf{R}$ depend on time t

$$\begin{aligned}\frac{\partial}{\partial t} &= \frac{\partial T}{\partial t}\frac{\partial}{\partial T} + (\boldsymbol{\nabla}\cdot\mathbf{R})\,\boldsymbol{\nabla}_R,\\ &= \frac{\partial}{\partial T} - \mathbf{v}\cdot\boldsymbol{\nabla}_R.\end{aligned}$$

Substituting these results into the diffusion-advection equation and using that $P(\mathbf{R},T) \equiv p(\mathbf{r},t|\mathbf{r}_s,t_s)$ it follows that Eq. (1) is converted into an ordinary diffusion equation in $\mathbf{R}$ and T which has the solution

$$P(\mathbf{R},T) = \frac{1}{4\pi DT}\exp\left\{-\frac{\mathbf{R}^2}{4DT}\right\}. \tag{2}$$

From this expression it is straight forward to show that Eq. (2) fulfills the two-dimensional ordinary diffusion equation.

The meaning of $\mathbf{r}_s$ and t_s in the expression for $p(\mathbf{r},t|\mathbf{r}_s,t_s)$ is the source location and time when the (δ-function) source was active. The initial condition that this function satisfies is

$$\lim_{t\to t_s} p(\mathbf{r},t|\mathbf{r}_s,t_s) = \delta(\mathbf{r}-\mathbf{r}_s), \tag{3}$$

while the boundary condition is

$$\lim_{|\mathbf{r}-\mathbf{r}_s|\to\infty} p(\mathbf{r},t|\mathbf{r}_s,t_s) = 0. \tag{4}$$

d) The propagator, $p(\mathbf{r},t|\mathbf{r}_s,t_s)$, gives the response of the system at an arbitrary position $\mathbf{r}$ and time $t > t_s$ to a delta function source located at the source point $\mathbf{r}_s$ that was turned on at time t_s and turned off shortly thereafter. Due to *causality* one must have that $p(\mathbf{r},t|\mathbf{r}_s,t_s) = 0$ for $t < t_s$ —that is, "no effect before the cause".

e) Let us assume that time t satisfies $t_0 \le t < t_1$. During this period the (dust) source term coming from the volcano (located at $\mathbf{r}_s$) will be time independent and can be written as

$$\begin{aligned}\mathcal{A}(\mathbf{r},t) &= \mathcal{A}_0\delta^2(\mathbf{r}-\mathbf{r}_s)\\ &= \mathcal{A}_0\delta^2(\mathbf{r}-\mathbf{r}_s)\int dt'\,\delta(t-t'),\end{aligned} \tag{5}$$

that is, a sum of delta-function sources of the same strength. Therefore, the dust concentration $c(\mathbf{r},t)$ will be given by the following integral

$$\begin{aligned}c(\mathbf{r},t) &= \int_{\mathbb{R}^2} d^2r' \int_{t_0}^{t} dt'\, p_0(\mathbf{r},t|\mathbf{r}',t')\mathcal{A}(\mathbf{r}',t')\\ &= \mathcal{A}_0\int_{t_0}^{t} dt'\, p_0(\mathbf{r},t|\mathbf{r}_s,t'), \qquad t_0 \le t < t_1,\end{aligned} \tag{6}$$

where $p_0(\mathbf{r},t|\mathbf{r}_s,t_s)$ denotes the propagator, Eq. (1) of the problems set, but with $\mathbf{v}=0$. Note that the spatial integration here is trivial since the source is localized at the position of the volcano. Eq. (6) can be viewed as a sum, *i.e.* superposition, of fundamental solutions corresponding to different times.

f) For time $t > t_1$ the wind starts to play a role ($\mathbf{v} \neq 0$). Hence, an advaction term will appear, and as a result, the propagator will be different from $p_0(\mathbf{r},t|\mathbf{r}_s,t_s)$ (since $\mathbf{v}\neq 0$). We will in this case denote it by $p_1(\mathbf{r},t|\mathbf{r}_s,t_s)$ and the mathematical form is that of Eq. (1) of the problems set.

In this case the source term will consist of *two* terms; One term is due to the ash that is emitted from the volcano after $t > t_1$. This terms will spread out in a way similar to Eq. (6), but with the important difference that the propagator now is $p_1(\mathbf{r},t|\mathbf{r}_s,t_s)$ instead of $p_0(\mathbf{r},t|\mathbf{r}_s,t_s)$. The second term, however, will result from the spread of the now *spatially* distributed ash as it appeared at time $t=t_1$, that is, $c(\mathbf{r},t=t_1)$ as given by Eq. (6). Due to the spatially distribution of $c(\mathbf{r},t=t_1)$, the source term in this case, we will now have a non-trivial spatial integral. Since for this latter case, the source term does not inject more ash, no time-integration will be needed.

Hence the ash concentration valid for $t > t_1$ results from adding the two terms mentioned above in order to obtain

$$\begin{aligned} c(\mathbf{r},t) &= \mathcal{A}_0 \int_{t_1}^{t} dt'\, p_1(\mathbf{r},t|\mathbf{r}_s,t') + \int_{\mathbb{R}^2} d^2r'\, p_1(\mathbf{r},t|\mathbf{r}',t_1) c(\mathbf{r}',t_1) \\ &= \mathcal{A}_0 \left[\int_{t_1}^{t} dt'\, p_1(\mathbf{r},t|\mathbf{r}_s,t') + \int_{\mathbb{R}^2} d^2r'\, p_1(\mathbf{r},t|\mathbf{r}',t_1) \int_{t_0}^{t_1} dt'\, p_0(\mathbf{r}',t_1|\mathbf{r}_s,t') \right]. \end{aligned} \tag{7}$$

Problem 2. Langevin equation

a) The Langevin equation is the equation of motion (Newtons law) for a small particle where the result of the many interactions with even smaller particles of the surrounding medium is given in terms of a stochastic force. The Langevin equation is used for stochastic systems.

b) Applying Newton's 2nd law to the particle results in

$$m\ddot{\mathbf{x}} = -\mathbf{F}(t) + \mathbf{S}(t),$$

and after dividing this equation through by m, using the given expression for the friction force $\mathbf{F}(t)$, and finally introducing the velocity, it reads

$$\dot{\mathbf{v}}(t) + \gamma \mathbf{v}(t) = \boldsymbol{\xi}(t), \tag{8}$$

where $\boldsymbol{\xi}(t) = \mathbf{S}(t)/m$ is the scaled stochastic force. Equation (8) is the final expression for the Langevin equation for the particle.

c) The formal solution of Eq. (8) consists of the sum of a homogeneous solution, $\mathbf{v}_H(t)$, and a particular solution, $\mathbf{v}_P(t)$. The homogeneous solution is easily obtained as

$$\mathbf{v}_H(t) = \mathbf{v}_0\, e^{-\gamma t}.$$

However, one particular solution is given by

$$\mathbf{v}_P(t) = e^{-\gamma t} \int_0^t dt'\, e^{\gamma t'} \boldsymbol{\xi}(t'),$$

as can be shown by substituting this expression into the Langevin equation. [Alternatively you may assume a form of the solution $\mathbf{v}(t) = \mathbf{u}(t)\, e^{-\gamma t}$ where $\mathbf{u}(t)$ is an unknown function to be determined.] Hence, the general solution becomes

$$\begin{aligned} \mathbf{v}(t) &= \mathbf{v}_H(t) + \mathbf{v}_P(t) \\ &= \mathbf{v}_0\, e^{-\gamma t} + e^{-\gamma t} \int_0^t dt'\, e^{\gamma t'} \boldsymbol{\xi}(t'), \end{aligned} \tag{9}$$

which was the expression that one was asked to derive.

d) On physical grounds it is reasonable to expect that the system is isotropic. This has the consequence that $\langle \mathbf{S}(t) \rangle = \langle \boldsymbol{\xi}(t) \rangle = 0$.

e) From Eq. (9) it follows that

$$\langle \mathbf{v}(t)\rangle = \mathbf{v}_0\, e^{-\gamma t} + e^{-\gamma t}\int_0^t dt'\, e^{\gamma t'}\, \langle \boldsymbol{\xi}(t')\rangle = \mathbf{v}_0\, e^{-\gamma t},$$

where we have used that $\langle \boldsymbol{\xi}(t)\rangle = 0$.

Moreover, one has that

$$\begin{aligned}\delta\mathbf{v}(t) &= \mathbf{v}(t) - \langle \mathbf{v}(t)\rangle \\ &= e^{-\gamma t}\int_0^t dt'\, e^{\gamma t'}\boldsymbol{\xi}(t'),\end{aligned}$$

so that

$$\begin{aligned}\left\langle [\delta\mathbf{v}(t)]^2\right\rangle &= \left\langle e^{-2\gamma t}\int_0^t dt'\, e^{\gamma t'}\boldsymbol{\xi}(t')\int_0^t dt''\, e^{\gamma t''}\boldsymbol{\xi}(t'')\right\rangle \\ &= e^{-2\gamma t}\int_0^t dt'\, e^{\gamma t'}\int_0^t dt''\, e^{\gamma t''}\,\langle \boldsymbol{\xi}(t')\boldsymbol{\xi}(t'')\rangle \\ &= e^{-2\gamma t}\int_0^t dt'\int_0^t dt''\, e^{\gamma(t''+t')}\frac{W(t''-t')}{m^2}.\end{aligned}$$

Now making a change of variable $\tau = t'' - t'$ and using that the stochastic force is assumed to be stationary, one arrives at the final result

$$\left\langle [\delta\mathbf{v}(t)]^2\right\rangle = \frac{e^{-2\gamma t}}{m^2}\int_0^t dt'\, e^{2\gamma t'}\int_{-t'}^{t-t'} d\tau\, e^{\gamma\tau}W(\tau). \tag{10}$$

f) The equipartition theorem states that every quadratic term in the Hamiltonian of the system will contribute a factor $k_BT/2$ to energy of the system when it is in thermal equilibrium. When $t\to\infty$, the system should be in thermal equilibrium, so that the equipartition theorem should apply.

The quadratic terms in the Hamiltonian in this case corresponds to the kinetic energy, $K = m[\delta\mathbf{v}(t)]^2/2$, so one should have with Eq. (10) that

$$\begin{aligned}\frac{3}{2}k_BT &= \lim_{t\to\infty}\langle K\rangle \\ &= \lim_{t\to\infty}\left\langle \frac{1}{2}m\,[\delta\mathbf{v}(t)]^2\right\rangle \\ &= \lim_{t\to\infty}\frac{e^{-2\gamma t}}{2m}\int_0^t dt'\, e^{2\gamma t'}\int_{-t'}^{t-t'} d\tau\, e^{\gamma\tau}W(\tau).\end{aligned}$$

[Note that $\lim_{t\to\infty}\langle \mathbf{v}^2(t)\rangle = \lim_{t\to\infty}\langle [\delta\mathbf{v}(t)]^2\rangle$ since $\lim_{t\to\infty}\langle \mathbf{v}(t)\rangle = 0$.] Rearranging this equation results in the expression that should be derived, *i.e.*

$$3k_BT\, m = \lim_{t\to\infty} e^{-2\gamma t}\int_0^t dt'\, e^{2\gamma t'}\int_{-t'}^{t-t'} d\tau\, e^{\gamma\tau}W(\tau). \tag{11}$$

g) Due to the finite correlation time of the correlation function $W(\tau)$ the integrand of the second integral will vanish whenever $|\tau| \geq \tau_0$. Therefore, without loss of generality, the integration limits of the second integral can be set to $\pm\infty$ so that it becomes

$$I_2(t) = \int_{-\infty}^{\infty} d\tau\, e^{\gamma\tau} W(\tau),$$

which is independent of t', and the two integrals, as a result, have been *decoupled*. Under the assumption that $\gamma\tau_0 \ll 1$ the exponential function appearing in $I_2(t)$ can be set equal to one with the result that

$$I_2(t) = \int_{-\infty}^{\infty} d\tau\, W(\tau).$$

The first integral can now be calculated analytically to produce

$$I_1(t) = \int_0^t dt'\, e^{2\gamma t'} = \frac{1}{2\gamma}\left[e^{2\gamma t} - 1\right].$$

Hence, by substituting these results for $I_1(t)$ and $I_2(t)$ back into Eq. (11) one finds

$$\begin{aligned} 3k_B T\, m &= \lim_{t\to\infty} e^{-2\gamma t} I_1(t) I_2(t) \\ &= \frac{1}{2\gamma} \int_{-\infty}^{\infty} d\tau\, W(\tau), \end{aligned}$$

and after solving this expression for γ one obtains

$$\gamma = \frac{1}{6k_B T\, m} \int_{-\infty}^{\infty} d\tau\, W(\tau), \tag{12}$$

which is the final results.

Equation (12) determines the friction coefficient, γ, in terms of the stochastic properties of the fluctuating force $\mathbf{S}(t)$. So, by studying the stochastic force, one may obtain information on the friction coefficient.

At equilibrium, the energy that the particle looses via the friction term as heat (dissipation) to the bath is exactly balanced by that being gained by the particle as kinetic energy taken up from the bath via the stochastic force. This is a feature of the so-called fluctuation-dissipation theorem.

Problem 3. Student random walk

a) The characteristic function, $\phi(k)$, of a general pdf, $p(x)$, is defined *via* the Fourier transform as

$$\phi(k) = \left\langle e^{-ikx} \right\rangle = \int_{-\infty}^{\infty} dx\, p(x) e^{-ikx}. \tag{13}$$

b) According to the definition of the characteristic function, Eq. (13), it follows that we have to calculate the integral

$$\phi(k) = \int_{-\infty}^{\infty} dx \, \frac{A\, e^{-ikx}}{(1+x^2)^2}. \tag{14}$$

The simplest way to evaluate this integral is to note that the integrand can be written as

$$\frac{e^{-ikx}}{(1+x^2)^2} = \frac{e^{-ikx}}{(x-i)^2(x+i)^2}, \tag{15}$$

which means that it has poles of order *two* at $x = \pm i$. Thus, we will calculate the integral using the residue theorem.

Let us start by considering the case where $k < 0$. By closing the integration contour by a half-circle in the upper half-plane, the contour will encompass the poles at $x = i$ and make the integration along the semi-circle vanish so that only the integration along the real axis survives. Hence, one has that

$$\begin{aligned}
\phi(k) &= A \int_{-\infty}^{\infty} dx \, \frac{e^{-ikx}}{(x-i)^2(x+i)^2}, \\
&= 2\pi i A \, \mathrm{Res}\left(\frac{e^{-ikx}}{(x-i)^2(x+i)^2}; x = i \right) \\
&= 2\pi i A \, \frac{d}{dx} \left(\frac{e^{-ikx}}{(x+i)^2} \right) \Bigg|_{x=i} \\
&= \frac{A\pi}{2} e^{k} \,(1-k) \\
&= \frac{A\pi}{2} e^{-|k|} \,(1+|k|) \,.
\end{aligned} \tag{16}$$

For the case when $k > 0$ the integration contour has to be closed in the lower half-plane, and the calculation is analogous to what was just shown above. However, note that an extra minus sign has to be included that steams from the direction of integration (negatively oriented closed path) . Anyhow, the result is that also for $k > 0$ the characteristic function is given by Eq. (16). Hence the form of the characteristic function is proven. In passing we note that alternatively one for the case $k > 0$ could have made a change of variable $y = -x$ in the integral and mapped it onto the that of the case $k < 0$.

c) From Eq. (13) it follows that for $k = 0$ one has

$$\phi(0) = \int_{-\infty}^{\infty} dx \, p(x) = 1,$$

where the last relation follows from the fact that $p(x)$ is a pdf and therefore has to be normalized. Thus, the constant A becomes

$$A = \frac{2}{\pi}.$$

To get the average $\langle x \rangle$ we simply argue that it should be zero since $p(x)$ is symmetrically distributed around zero. [The integrand is a product of a symmetric and an antisymmetric term integrated over a symmetric interval].

An alternative way of arriving at the same result is to Taylor expand $\phi(k)$ to obtain

$$\begin{aligned}\phi(k) &= \left[1 - |k| + \frac{k^2}{2} - \ldots\right][1 + |k|] \\ &= 1 - \frac{k^2}{2} + \ldots.\end{aligned}$$

Thus, one may conclude that $\langle x \rangle = 0$, but also that $\sigma^2 = 1$. This latter result follows from taking successive derivatives with respect to k of the characteristic function as outlined in the lecture notes, *i.e.*

$$\langle x^n \rangle = \frac{1}{(-i)^n} \frac{d^n}{dk^n} \phi(k) \bigg|_{k=0}.$$

d) Since the variance for the step size distribution is finite, the random walk process is ordinary diffusion.

e) Since the characteristic functions of independent random increments multiply when adding the corresponding random variables, one has that

$$\begin{aligned}p_N(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, \phi^N(k) e^{ikx} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{-N|k|} [1 + |k|]^N e^{ikx}.\end{aligned}$$

f) In the limit $|x| \to \infty$ we have that

$$p(x) \sim \frac{2}{\pi x^4}, \qquad |x| \to \infty.$$

Hence, from the addition theorem of power-laws one has that the asymptotic limit of $p_N(x)$ should be

$$p_N(x) \sim \frac{2N}{\pi x^4}, \qquad |x| \to \infty. \tag{17}$$

To realize that this addition law is correct follows from the small k dependence of the characteristic function $\phi^N(k)$ (which governs the tail of the pdf $p_N(x)$).

When $|k| \ll 1$ one has that

$$\begin{aligned}\phi^N(k) &= \left[1 - \frac{k^2}{2} + \ldots\right]^N \\ &\simeq 1 - N\frac{k^2}{2} + \ldots,\end{aligned} \tag{18}$$

which should be compared with the small $|k|$ expansion for $\phi(k)$ that is similar to that of Eq. (18) except for the pre-factor N of the second term. Hence, the power law tail (for large $|x|$) of $p_N(x)$ will be that of $p(x)$, *but* multiplied by N.

This result can be derived more rigorously for any power-law tail behavior, and the result is known as the addition theorem of power-law tails.