TFY4275 Classical Transport Theory
Problemset 7 Spring 2013

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Problem 1.
In this exercises we will revisit the fuel spill problem of Exercise 6. Hence, we consider a straight (one-dimensional) water canal. This canal is part of a larger dike system that is designed to bring boats from sea level to a large lake located inland above sea level.
We consider a segment of this canal bounded by a dike in each end. Some time after both *both* dikes are closed, there is no water current in the canal. However, when one of the dikes is opened, so that the water is flowing into the considered stretch of the canal at a rate depending on the dike opening, water current v can be considered practically constant. (We assume that we can neglect the more realistic two-dimension complex flow pattern, and simply assume for the same of simplicity that the water current can be approximated to be constant and one-dimensional.) Somewhere along the stretch of the canal, say at position $x = x_0$, a fuel spill occurs at time $t = t_0$.

We will now study the fuel spread problem of the previous exercise, but now in addition including the water current.

a) Let $c(x, t|x_0, t_0)$ denote the concentration of the fuel at position x along the canal at time t, given that the initial spill occurred at $x = x_0$ at time $t = t_0$. Argue why the relevant "transport equation" for the fuel is ($t > t_0$)

$$\partial_t c(x, t|x_0, t_0) + v\partial_x c(x, t|x_0, t_0) = D\partial_x^2 c(x, t|x_0, t_0),$$

i.e. the *diffusion-advection* equation. The amount of fuel spill is so that the initial condition can be written $c(x, t_0|x_0, t_0) = \rho_0\delta(x - x_0)$.

b) Use a change of variable $x' = x - vt$ and $t' = t$ to show that the solution of the diffusion-advection equation given above (for constant velocity v) is

$$c(x, t|x_0, t_0) = \frac{\rho_0}{\sqrt{4\pi D(t - t_0)}} \exp\left(-\frac{(x - x_0 - v(t - t_0))^2}{4D(t - t_0)}\right),$$

where $\rho_0 > 0$ is some arbitrary mass (line) density. This solution is known as the fundamental solution to the problem. Show explicitly that it satisfies the correct initial condition.

c) Derive the same result using the Fourier-Laplace transform technique discussed in the lectures.

d) In the previous exercise (where $v = 0$), we defined the time for the concentration to "reach" a distance $x - x_0 = \ell$ away from its source as equal to the time it takes the "2σ-point" of the diffusion process to move the same distance, *i.e.* the exponent of the exponential function in the expression for the concentration is minus two. We will here use an analogous definition of reaching. Use this to calculate these times for a general ℓ and v. Compare the answer for two points located symmetrically around the spill point, *i.e.* at point $x - x_0 = \pm|\ell_0|$ where $\ell_0 > 0$.

e) What is the *critical* velocity (smallest velocity) needed for the fuel to never reach a location located a distance ℓ_0 up stream? What is this speed in the case of reaching Prof. Baltazar's cabin of exercise 6?

f) Try to derive this critical velocity from general arguments, *i.e.* without using the explicit solution of the diffusion-advection equation.