TFY4275 Classical Transport Theory
Problemset 8 Spring 2013

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Problem 1.
Let us consider the rather generic (first order) process $A \to B$ where a species A is converted into B with a rate constant that we will call k.
Physically one may think of radioactive decay, a chemical reaction, or the de-excitation of exited atom from the excited state A to the de-excited state B.
Anyhow, whatever physical realization, the kinetic equation for such a system is

$$\frac{d\langle A\rangle}{dt} = -k\langle A\rangle\,,$$

with the well-known solution

$$\langle A\rangle(t=0)e^{-kt}.$$

However, if one does not average over the state A, say, the number of radioactive nuclei, this number will constitute be a stochastic process, since the time when a given nucleus is converted into state B is undetermined. This is easily realized by *e.g.* listening to the "clicks" of a Geiger counter.
In this exercise we will investigate this stochastic process.

a) Let $P(n,t)$ be the probability that the number of nuclei of type A in the system is n at time t. Show that the Master equation for this system is:

$$\frac{\partial P(n,t)}{\partial t} = k(n+1)\,P(n+1,t) - kn\,P(n,t). \tag{1}$$

Notice that unlike the random walk problem, the transition rate out of a given state n depends explicitly on n.

We will now solve the Master equation (1). Moreover, we will explicitly show that the solution to the kinetic equation given above follows.

This is most easily done by using generating functions. Since the random variable n can *only* take on non-negative *integer* values, the generating function is[1]

$$G(s,t) = \sum_{n=0}^{\infty} s^n P(n,t), \qquad 0 < |s| < 1.$$

a) The generating function is useful due to the relation

$$\left\langle n^k\right\rangle = \left[\left(s\frac{\partial}{\partial s}\right)^k G(s,t)\right]\Bigg|_{s=1}. \tag{2}$$

Demonstrate that this relation holds.

[1] Compare this to what was presented in the lectures.

b) Show that Eq. (1) is equivalent to

$$\frac{\partial G(s,t)}{\partial t} = k(1-s)\frac{\partial G}{\partial s}. \tag{3}$$

[Hint: Multiply Eq. (1) by s^n and sum over all n.]

c) If the initial condition is $P(n, t=0) = \delta_{n,n_0}$, what is then the expression for $G(s, t=0)$?

d) Show that with this initial condition the solution for the generating function is

$$G(s,t) = \left[1 + (s-1)e^{-kt}\right]^{n_0} \tag{4}$$

e) Find an expression for $\langle n \rangle$ and compare this solution to the solution of the kinetic equation.

f) Determine the fluctuations around the mean $\left\langle n^2 - \langle n \rangle^2 \right\rangle$.

g) Make a plot of $\left\langle n^2 - \langle n \rangle^2 \right\rangle$. It should be zero for $t = 0$ and $t = \infty$ and go through a maximum at some intermediate time. Find this time of maximum fluctuations t_m.

h) Try to argue from physical arguments why the above behavior for $\left\langle n^2 - \langle n \rangle^2 \right\rangle$ is reasonable.