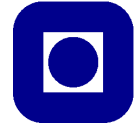


TFY4275 Classical Transport Theory

Problemset 9 Spring 2013

NTNU

Institutt for
fysikk

In this problem set, we will consider the mathematical solution of both *initial value* and *boundary value* diffusion problems.

Problem 1.

Consider the following *initial value problem* in one-dimension (*i.e.* $u \equiv u(x, t)$)

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}, \quad x \in R, t > 0,$$

$$u(x, 0) = \begin{cases} u_0, & \ell_- < x < \ell_+ \\ 0, & \text{otherwise} \end{cases},$$

where u_0 is a positive constant, and ℓ_{\pm} are arbitrary constants satisfying $\ell_- < \ell_+$.

- a) Give examples of physical situations being modeled by the above equation. What is the physical meaning of D and $u(x, t)$ in your examples?
- b) Use the Laplace-Fourier technique to find the general solution to the above initial value problem.
- c) Show that for this particular problem (and also for the one considered when introducing the Laplace-Fourier transform in the lectures) the combined transform is strictly speaking not needed. Derive the solution to the above initial value problem without the use of the both combined transform.
- d) In the lectures we derived an expression for the “propagator” $p(x, t|x_0, t_0)$. Show that in terms of the propagator, the above solution can be easily obtained.
- e) Demonstrate that in the long-time limit, the solution behaves as $u(x, t) \sim 1/\sqrt{t}$, and derive the full expression for $u(x, t)$ in this limit. Give an estimate for when this long-time limit is appropriate (in terms of the physical parameters of the problem).
- f) Plot the (spatial dependence of the) solution $u(x, t)$ for a few fixed times to indicate the time development of the solution from its initial value to its long-time limit (found above).

Problem 2.

Consider the following *boundary value problem* in one-dimension:

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}, \quad x > 0, t > 0,$$

$$u(x, 0) = 0, \quad x > 0,$$

$$u(0, t) = f(t) \quad t > 0,$$

$$\lim_{x \rightarrow \infty} u(x, t) = 0,$$

where $f(t)$ is some well-behaved function.

- a) What physical situation is described by this problem?
- b) Find the general solution to this problem for general $f(t)$. *Comment*: Your final answer will (hopefully) contain an integral over time which you can not perform before we have specified $f(t)$. [Hint : Use the propagator]
- c) Now set $f(t) = f_0\delta(t)$ and obtain the solution $u(x,t)$ in closed form. What is the physical meaning of this form of $f(t)$?
- d) Instead we now consider the “time-extended” source of the form

$$f(t) = \left(\frac{t}{\tau_1}\right)^2 \exp\left(-\frac{t}{\tau_2}\right),$$

where τ_1 and τ_2 are positive constants. Obtain the solution $u(x,t)$ in this case.

- e) Make a (3D or contour) plot of the solution $u(x,t)$ over the xt -plane assuming the latter form for $f(t)$ with (for simplicity) $\tau_1 = \tau_2 = 1$.