TFY4275 Classical Transport Theory
Problemset 9 Spring 2013

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In this problem set, we will consider the mathematical solution of both *initial value* and *boundary value* diffusion problems.

Problem 1.
Consider the following *initial value problem* in one-dimension (*i.e.* $u \equiv u(x,t)$)

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}, \qquad x \in R, t > 0,$$

$$u(x,0) = \begin{cases} u_0, & \ell_- < x < \ell_+ \\ 0, & \text{otherwise} \end{cases},$$

where u_0 is a positive constant, and $\ell_\pm$ are arbitrary constants satisfying $\ell_- < \ell_+$.

a) Give examples of physical situations being modeled by the above equation. What is the physical meaning of D and $u(x,t)$ in your examples?

b) Use the Laplace-Fourier technique to find the general solution to the above initial value problem.

c) Show that for this particular problem (and also for the one considered when introducing the Laplace-Fourier transform in the lectures) the combined transform is strictly speaking not needed. Derive the solution to the above initial value problem without the use of the both combined transform.

d) In the lectures we derived an expression for the "propagator" $p(x,t|x_0,t_0)$. Show that in terms of the propagator, the above solution can be easily obtained.

e) Demonstrate that in the long-time limit, the solution behaves as $u(x,t) \sim 1/\sqrt{t}$, and derive the full expression for $u(x,t)$ in this limit. Give an estimate for when this long-time limit is appropriate (in terms of the physical parameters of the problem).

f) Plot the (spatial dependence of the) solution $u(x,t)$ for a few fixed times to indicate the time development of the solution from its initial value to it long-time limit (found above).

Problem 2.
Consider the following *boundary value problem* in one-dimension:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}, \qquad x > 0,\, t > 0,$$

$$u(x,0) = 0, \qquad x > 0,$$

$$u(0,t) = f(t) \qquad t > 0,$$

$$\lim_{x\to\infty} u(x,t) = 0,$$

where $f(t)$ is some well-behaved function.

a) What physical situation is described by this problem?

b) Find the general solution to this problem for general $f(t)$. *Comment* : Your final answer will (hopefully) contain an integral over time which you can not perform before we have specified $f(t)$. [Hint : Use the propagator]

c) Now set $f(t) = f_0\delta(t)$ and obtain the solution $u(x,t)$ in closed form. What is the physical meaning of this form of $f(t)$?

d) Instead we now consider the "time-extended" source of the form

$$f(t) = \left(\frac{t}{\tau_1}\right)^2 \exp\left(-\frac{t}{\tau_2}\right),$$

where τ_1 and τ_2 are positive constants. Obtain the solution $u(x,t)$ in this case.

e) Make a (3D or contour) plot of the solution $u(x,t)$ over the xt-plane assuming the latter form for $f(t)$ with (for simplicity) $\tau_1 = \tau_2 = 1$.