Diffusion and Networks: *From simple models to applications*

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Part II

Part I:

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Acknowledgments

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What are we going to discuss?

- Topic of the school:

Simple Models + Complex Systems

- What we will do is:

Random Walk + Complex Networks
=> Some Application

Outline

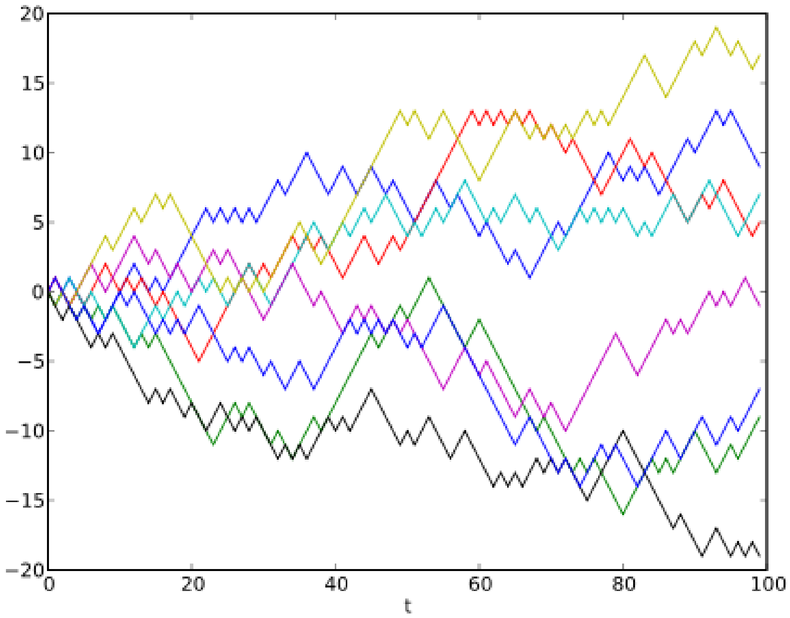
- A simple model:
  - The Random Walk Model
- Applications
  - Structures of Complex Networks
  - Cascading Failures on Networks
  - Epidemology on Networks
- Conclusions

Random Walks on Networks

The Model

The Classic Random (RW) Walk Model

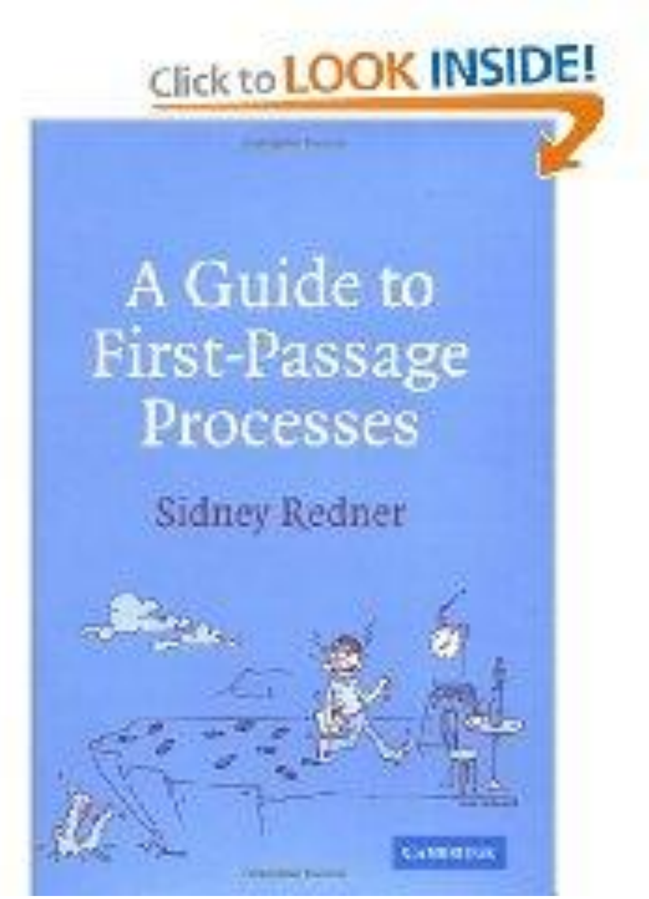
- The term RM was coined by Karl Pearson (1905)
- RW is useful in many branches of sciences
- The walker goes up or down with equal probability



$$\sigma(t) \propto \sqrt{t}$$

For FOFYs……..

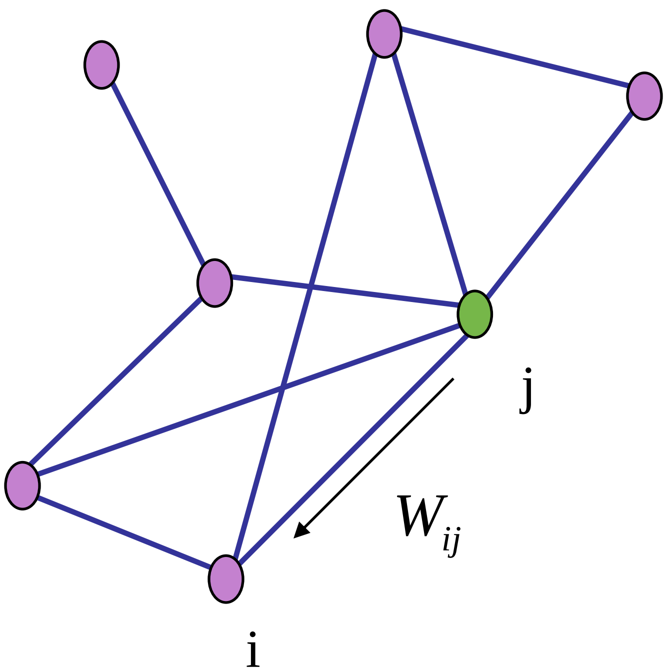
- Recommend book:



A Primer on Complex Networks?

- A network consists of a collection of
  - Nodes (vertices)
  - Links (edges)

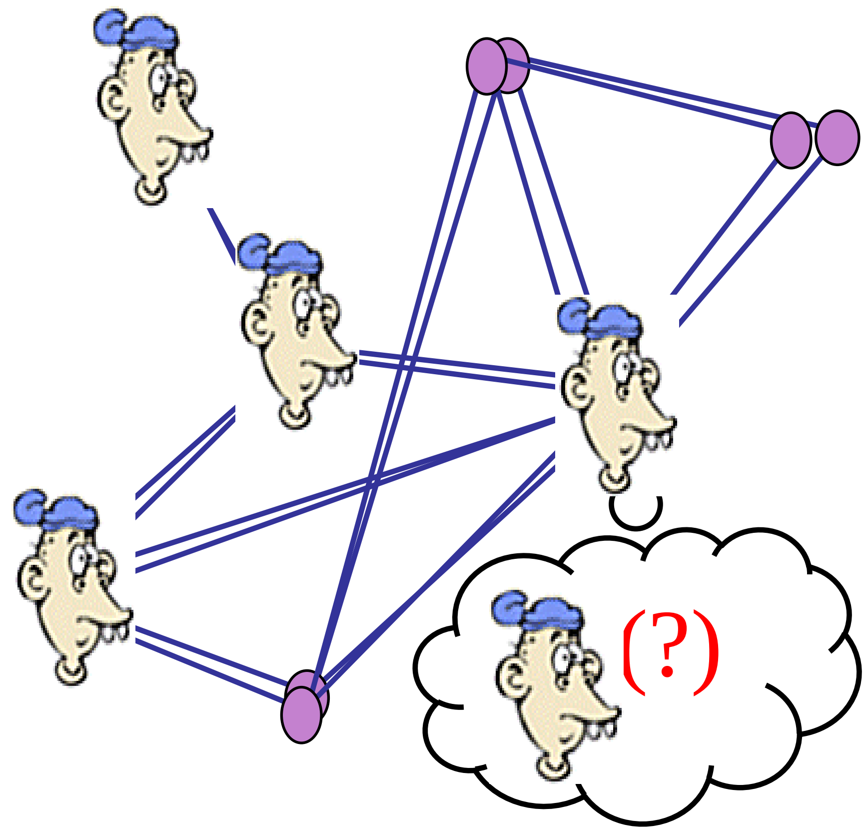
- Adjacency matrix : **W**=[W_{ij}]
  - W_{ij} is the weight on link from node j to i (our convention)
  - W_{ij} =0 means no link



Random Walks on Complex Networks

- Random walkers (i.e. particles) “live” on the nodes
- They are moving (flowing) around on the network!
- In each time step, a walker moves towards one of the neighboring nodes chosen by random
- This process is repeated over and over again…….

- Note: The number of walkers is constant in time



$t=t_0+3$

The Master Equation for No. of Particle

- Convention : W_{ij} refers to the link from node j to *i*;
- Define the outgoing link weight from node j :

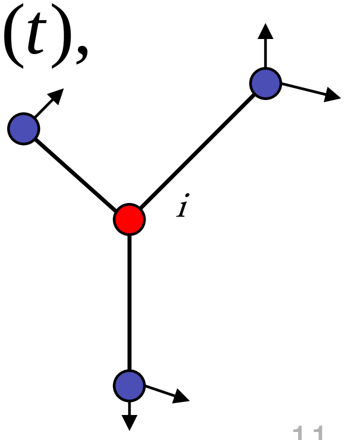
$$w_j = \sum_i W_{ij} \qquad \left(k_j = \sum_i A_{ij} \right)$$

- The change in no. of particle at node *i* from ***t*** to ***t+1***

$$n_i(t+1) - n_i(t) = \sum_j W_{ij} \frac{n_j(t)}{w_j} - \sum_j W_{ji} \frac{n_i(t)}{w_i} + n_i^{\pm}(t),$$

- Or….

$$n_i(t+1) = \sum_j \frac{W_{ij}}{w_j} n_j(t) + n_i^{\pm}(t)$$



The Master Equation for Node Density

- Node Density of walkers at node i

$$\rho_i(t) = \frac{n_i(t)}{N}$$

- The (master) equation

$$\rho_i(t+1) = \sum_j T_{ij}\rho_j(t) + \rho_i^{\pm}(t), \qquad T_{ij} = \frac{W_{ij}}{w_j}$$

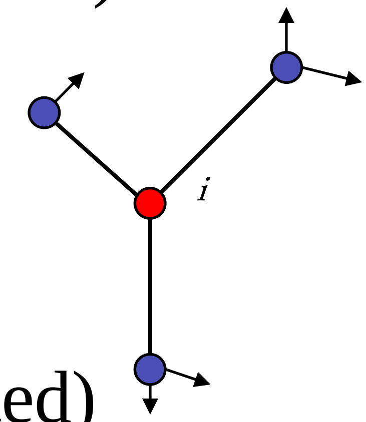
- …or using matrix-multiplication

$$\vec{\rho}(t+1) = T\vec{\rho}(t) + \vec{\rho}^{\pm}(t)$$

The Master Equation for Link Current

- The outgoing link current (per link/weight unit)

$$c_i(t) = \frac{\rho_i(t)}{w_i} = \frac{n_i(t)}{w_i N}$$



- Hence, it follows (if the network is undirected)

$$\vec{c}(t+1) = T^T \vec{c}(t) + \vec{j}^{\pm}(t);$$

$$T_{ij} = \frac{W_{ij}}{w_j} \qquad j_i^{\pm}(t) = \frac{\rho_i(t)}{w_i}$$

Solution of the Master Equation (assuming $\vec{\rho}^{\pm}(t)=0$)

- We start from some arbitrary walker dist. : $\rho_i(t=0)$
- Question : What will happen when $t\to\infty$?
- If T was symmetric, we know the answer:

$$\vec{\rho}(t) = T^t \vec{\rho}(0) = V \Lambda^t V^{-1} \vec{\rho}(0)$$

- but T is NOT symmetric…so what do we do…
- However, T is similar to a symmetric matrix

$$S = KTK^{-1} \qquad \text{with } K_{ij} = \frac{\delta_{ij}}{\sqrt{k_i}} \quad \Rightarrow \quad S_{ij} = \frac{\delta_{ij}}{\sqrt{k_i k_j}}$$

Solution of the Master Equation (cont…)

- Since S is symmetric, the transfer matrix T has
  - real eigenvalues, $\lambda^{(\alpha)}$ (sorted in decreasing order)
  - due to the conservation of walkers $|\lambda^{(\alpha)}| \leq 1$ $(\lambda^{(1)}=1)$
  - These eigenvalues will control the walker dynamics via the factor : $[\lambda^{(\alpha)}]^t$
    - Principal eigenvalue : $\lambda^{(1)}=1$ (non-decaying)
    - Slowest decaying mode : $\lambda^{(2)}$ *etc.*

- What does the stationary state look like?

Solution of the Master Equation (cont…)

- What does the stationary state look like?
  - Solve : $\rho_{st} = T\rho_{st}$

$$[\rho_{st}]_i = \frac{k_i}{\sum_n k_n} \propto k_i$$

$$\rho_{st} = \rho(\infty) = \rho^{(\alpha=1)}$$

  - Vertices of high degree will have more walkers in the st. state
- Introduce the outgoing walker current from vertex *i* :

$$c_i^{(\alpha)}(t) = \frac{\rho_i^{(\alpha)}(t)}{k_i}$$

$$c_i^{(1)} = \frac{1}{\sqrt{N}}$$

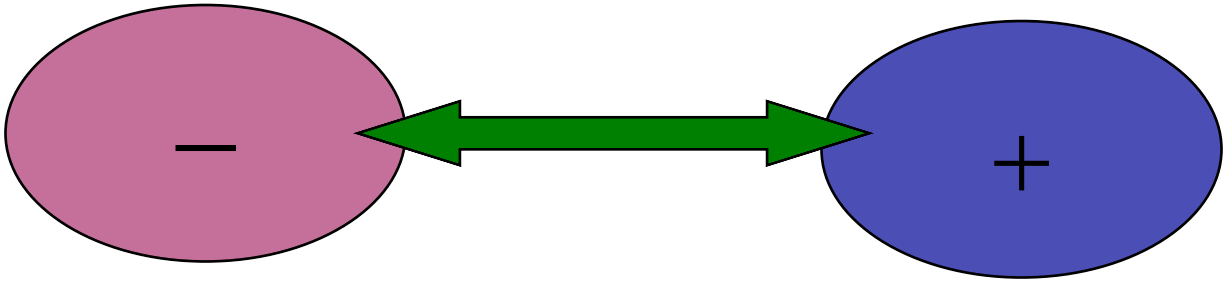
Structure of Complex Networks

Application I

K. Eriksen et al., Phys. Rev. Lett. **90**, 148701 (2003).

Motivations

- Idea: The slowest decaying mode will probe the large scale structure of the network



Current Projection Technique

- In the stationary state the currents are all the same
- **Idea** : For the slowly decaying modes, the currents are similar for nodes belonging to the same module!
- Plot *all* nodes in *current space*
- Eg. : d=2 (dimension of the projection)

$$P_i^{(d=2)} = (c_i^{(2)}, c_i^{(3)})$$

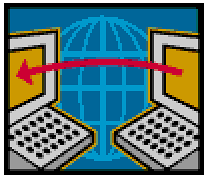
Results for Real-World Networks

Two examples:

- Zachary's Karate club Network (friendship network)
  - The university karate club breaks up due to an internal conflict
  - A small network (N=34; L=72)
  - *The modular structure is known!*
  - Reference :
    - W.W. Zachary, J. Antropol. Res. **33**, 452473 (1977).

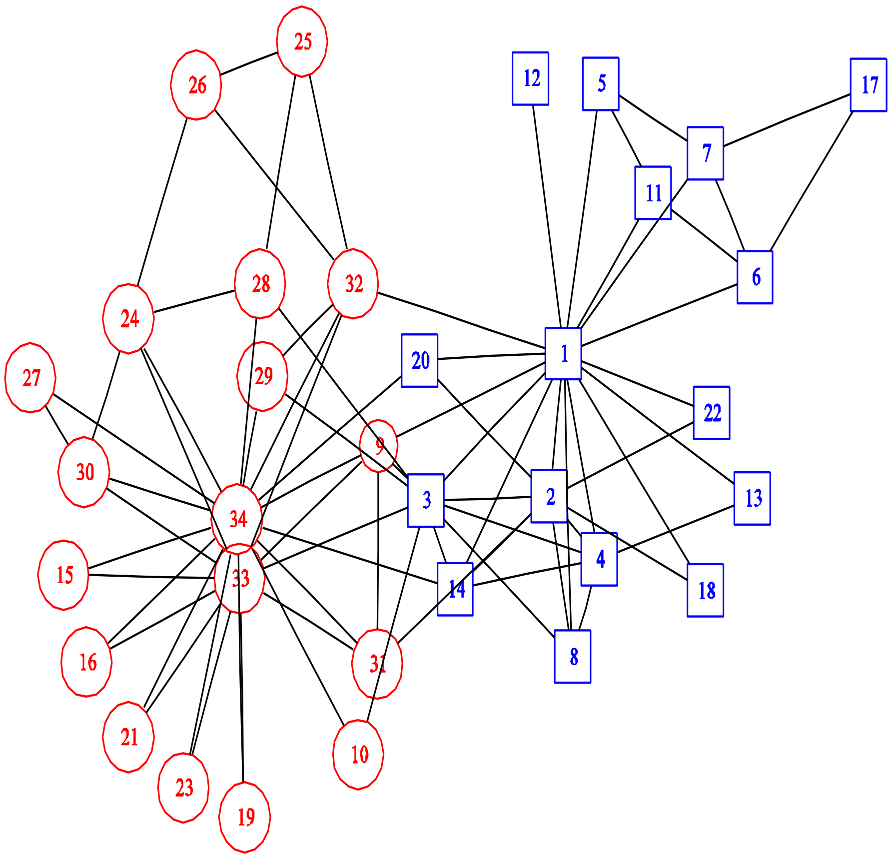
Results for Real-World Networks (cont..)

- Course-grained Internet Network (Autonomous Systems)
  - Medium sized network (N=6,474; L=12,572)
  - *The modular structure is not well-known in advance*
  - Reference :
    - National Laboratory of Applied Network Research
    - http://moat.nlanr.net/AS/



The Zachary Friendship Network

- It is a workbench for community-finding algorithms
- W.W. Zachary (1977) studied a university karate club, and weighted the friendship among its members
  - Nodes : 34
  - Links : 78
- Known community structure:
  - Trainer (node 1)
  - Administrator (node 34)



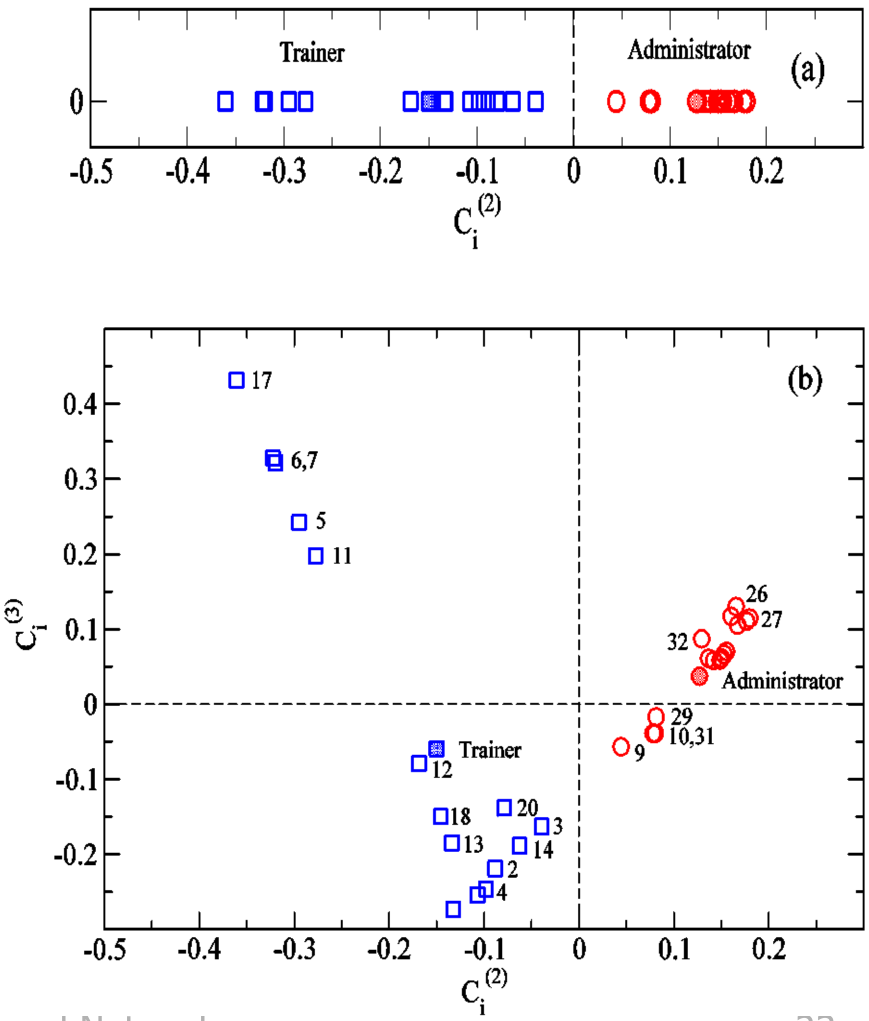
Source : PNAS 99, 7821 (2002)

The Zachary Friendship Network (cont..)

- **Current projection**

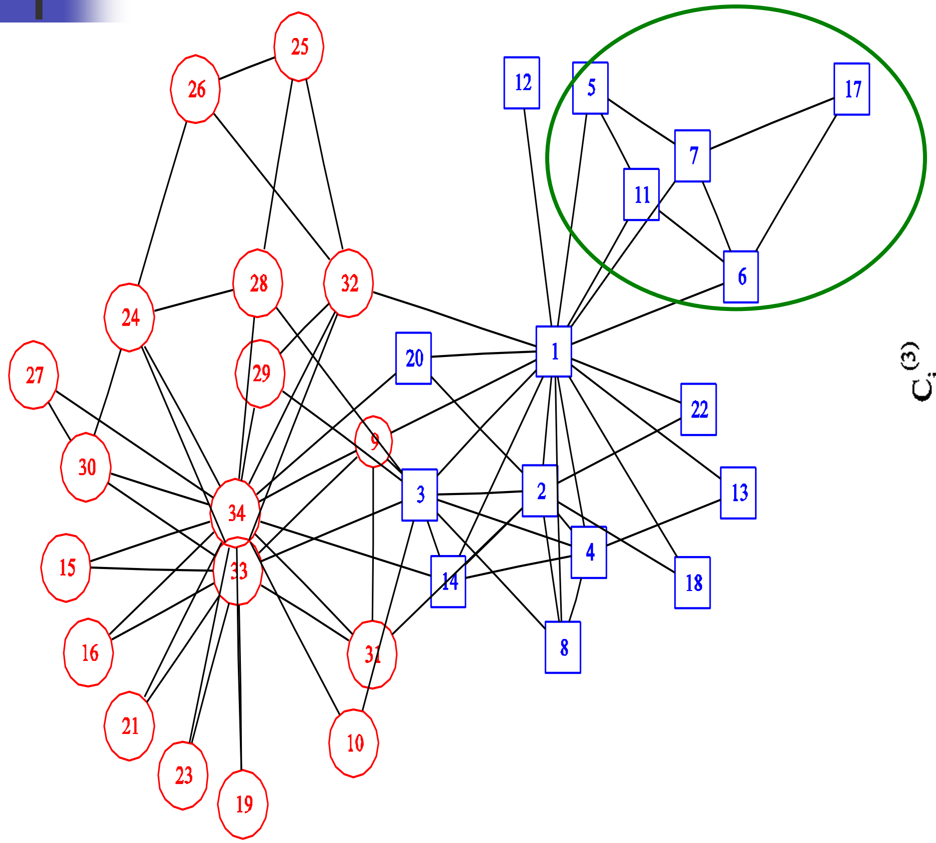
- d=1 projection
  - α=2 : Trainer-Administrator

- d=2 projection
  - α=2 : Trainer-Administrator
  - α=3 : Sub-clusters are mapped out

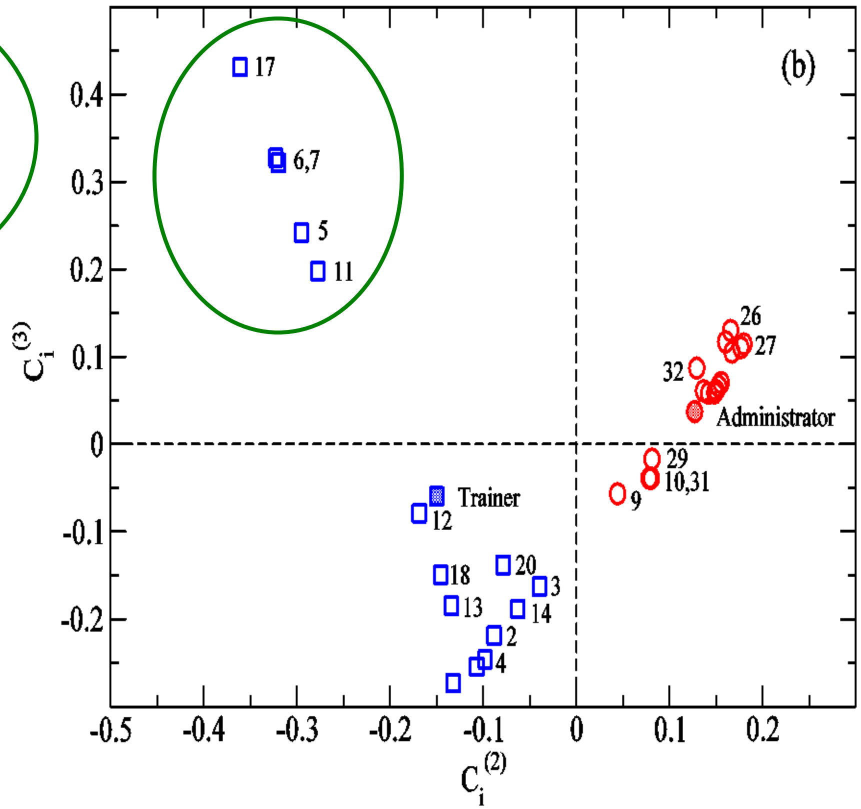


Q : How does this fit with the known community structure?

Zachary's friendship Network, *cont.*

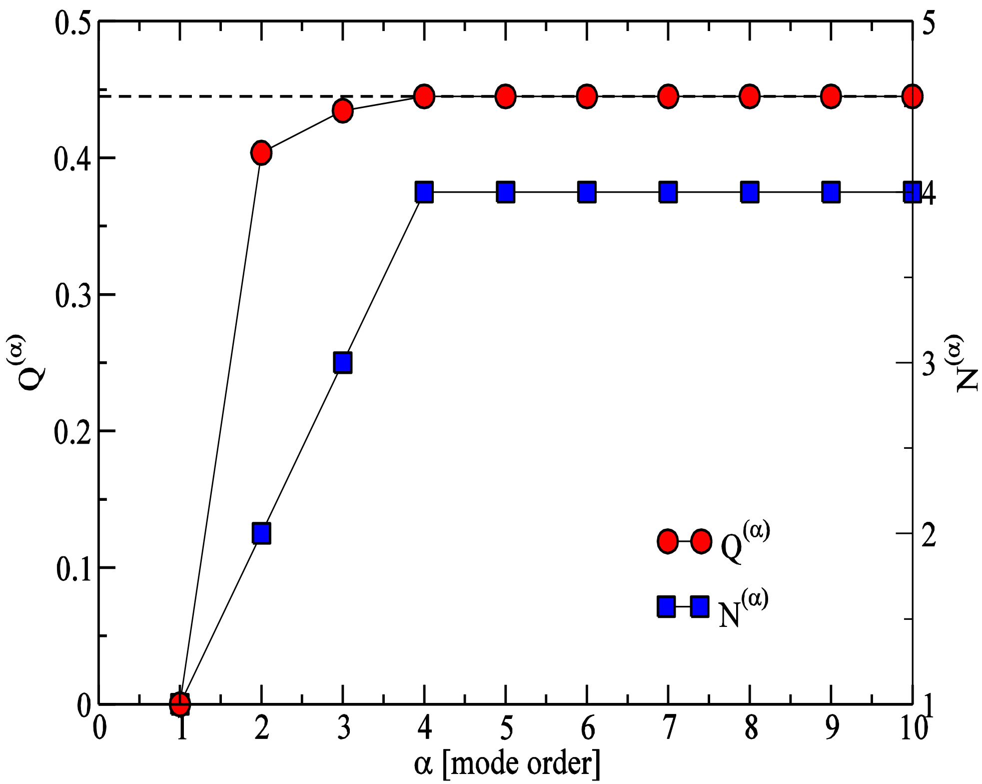


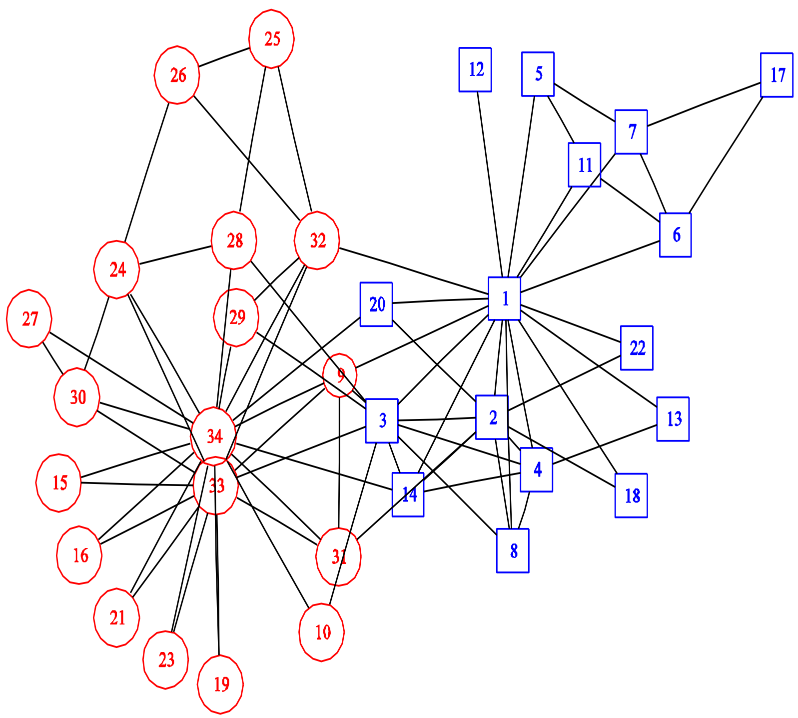
[From PNAS 99, 7821 (2002)]



The known modular structure is well reproduced by the diffusion model !

Projection in higher dimensions





The known modular structure is well reproduced by the diffusion model !

Network Scientists

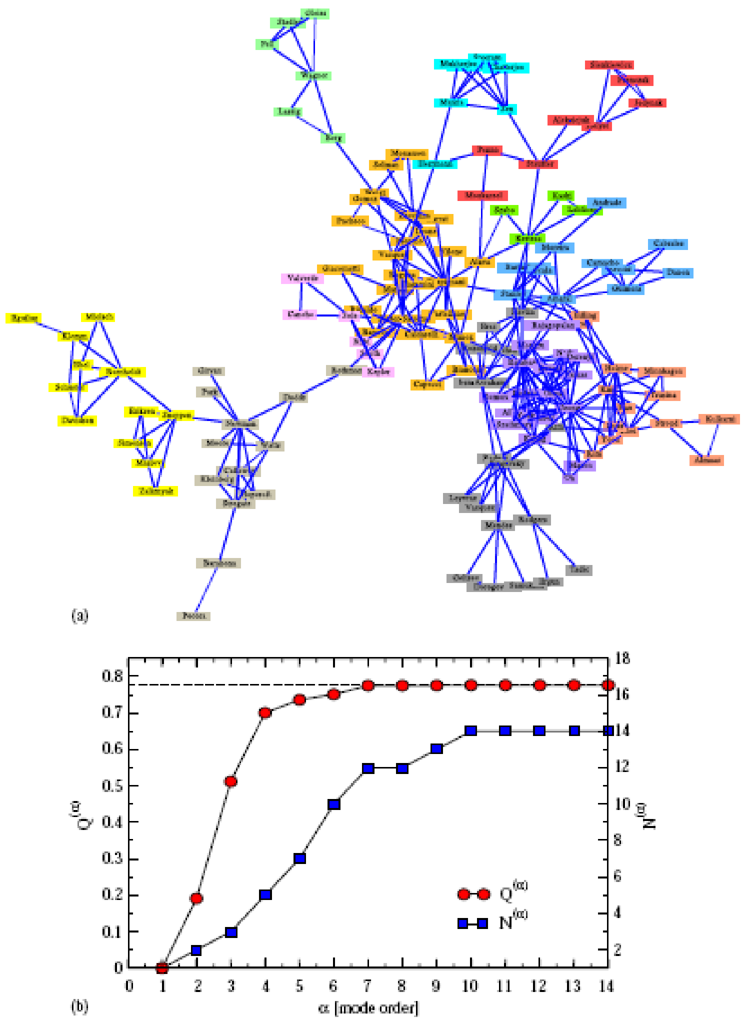
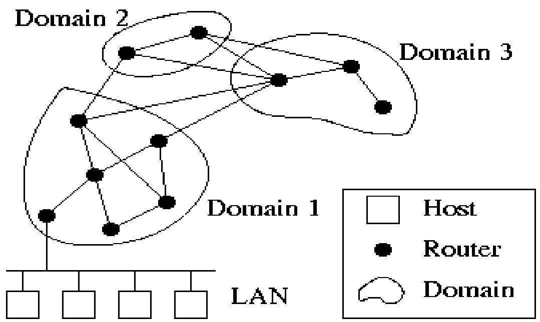


Fig. 4. A "network" scientists collaboration network (see Refs. [9,19] for details). (a) The largest component, consists of $N = 145$ vertices, of this network. Figure after Ref. [9], where colors correspond to communities found there. (b) Same as Fig. 3, but for the network of Fig. 4(a). The optimal partitioning is found for 14 communities characterized by the modularities $Q = 0.78$ and $Q_A = 0.70$.

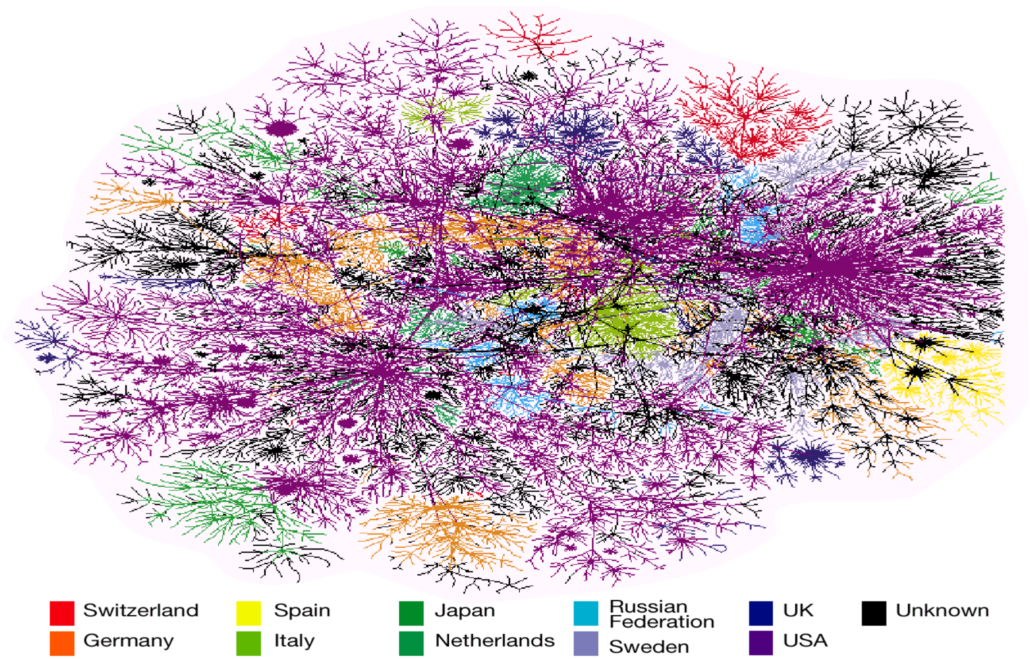
Autonomous System Network



We now consider a larger network!

$$N = 6474$$
$$L = 12572$$

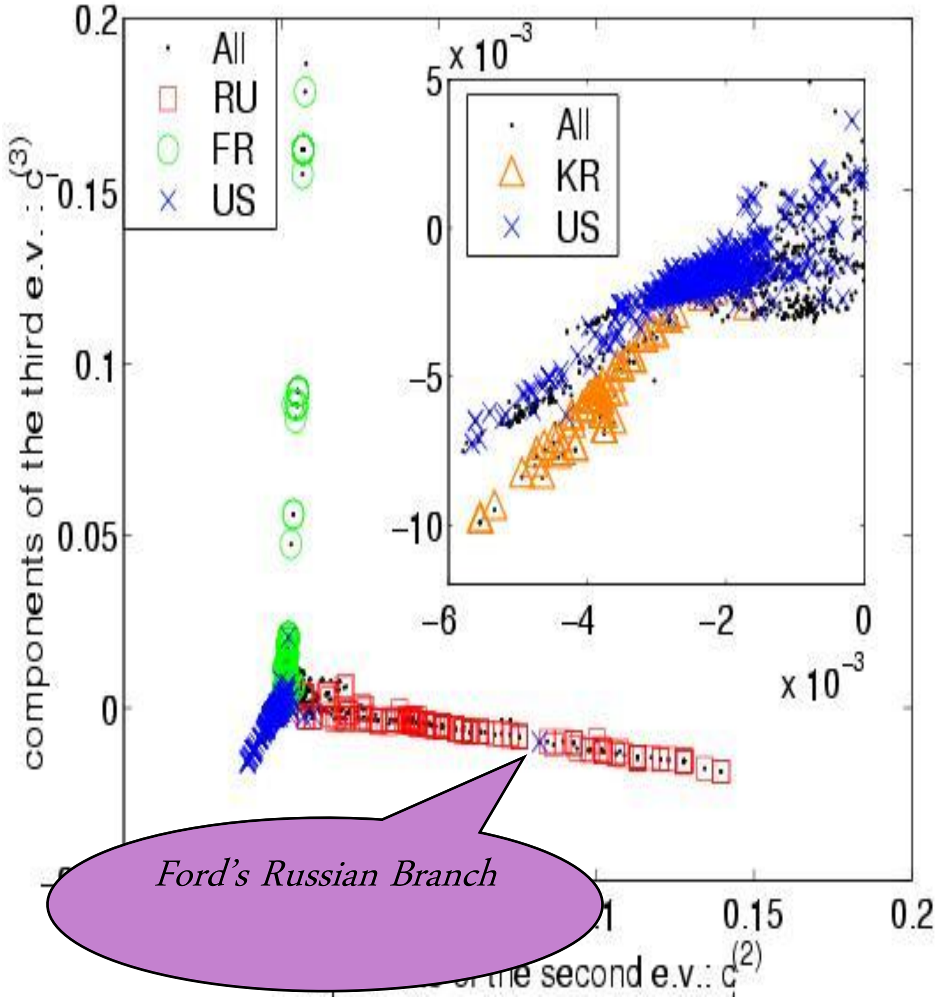
Definition: An Autonomous System (AS) is a connected segment of a network consisting of a collection of subnetworks interconnected by a set of routers.



Nodes : AS numbers

Links : Information sharing

Autonomous System Network (*cont.*)



The Internet is *indeed* structured

The structure follows roughly the national/political structure

The *extreme edges* of the Internet are represented by *Russian and US military* sites !

These country modules *could not* have been detected using spectral analysis of A

Ref : PRE **64**, 026704 (2001)

Summary : Network Topology

- The current projection technique probes large scale structures of the network

- References
  - K. Eriksen et al., Phys. Rev. Lett. **90**, 148701 (2003)
  - I. Simonsen, Physica A **357**, 317 (2005)

Cascading Failure of Networks

Application II

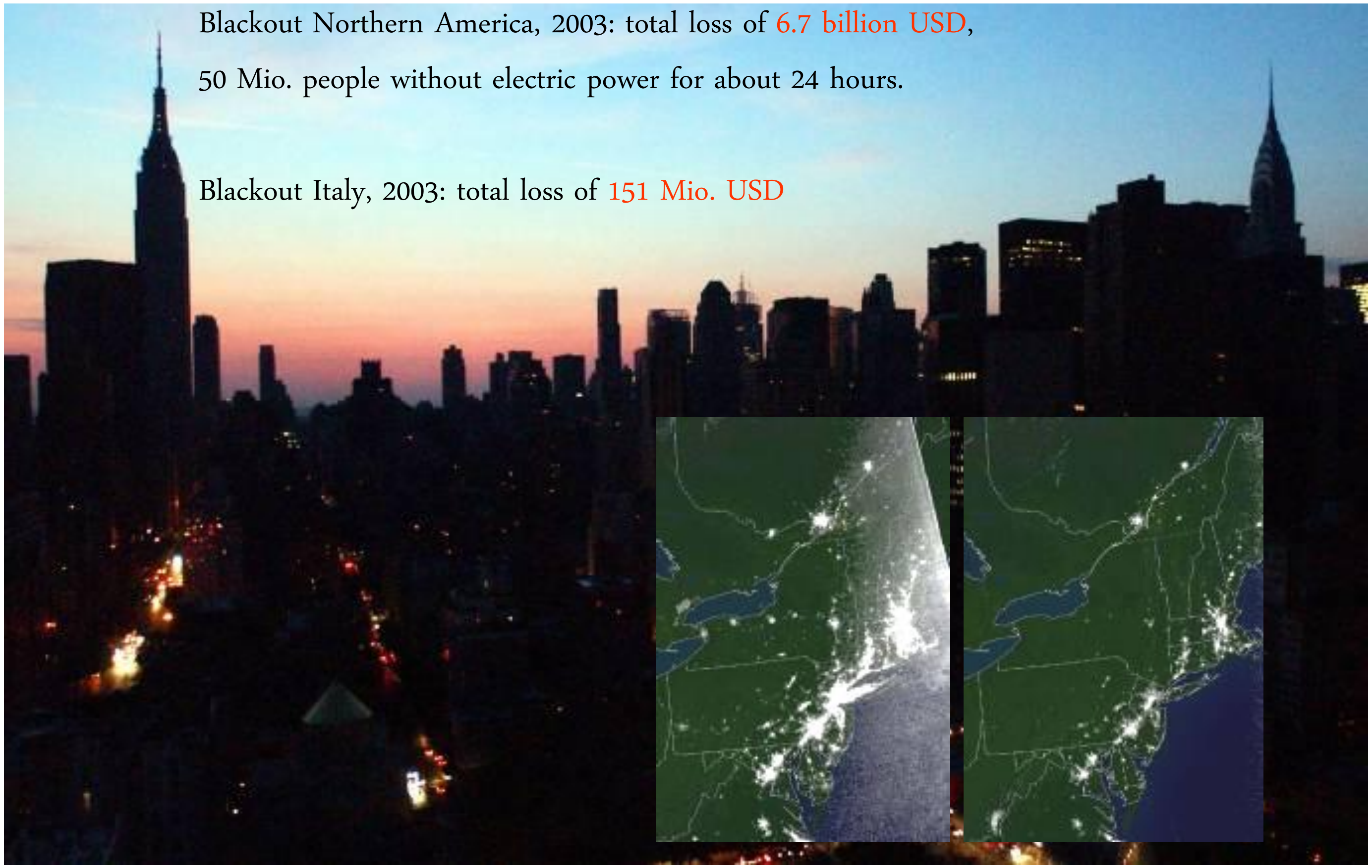
I. Simonsen, et al., Phys. Rev. Lett. **100**, 218701 (2008).

Motivation

New York, August 14, 2003

Rome, September 28, 2003

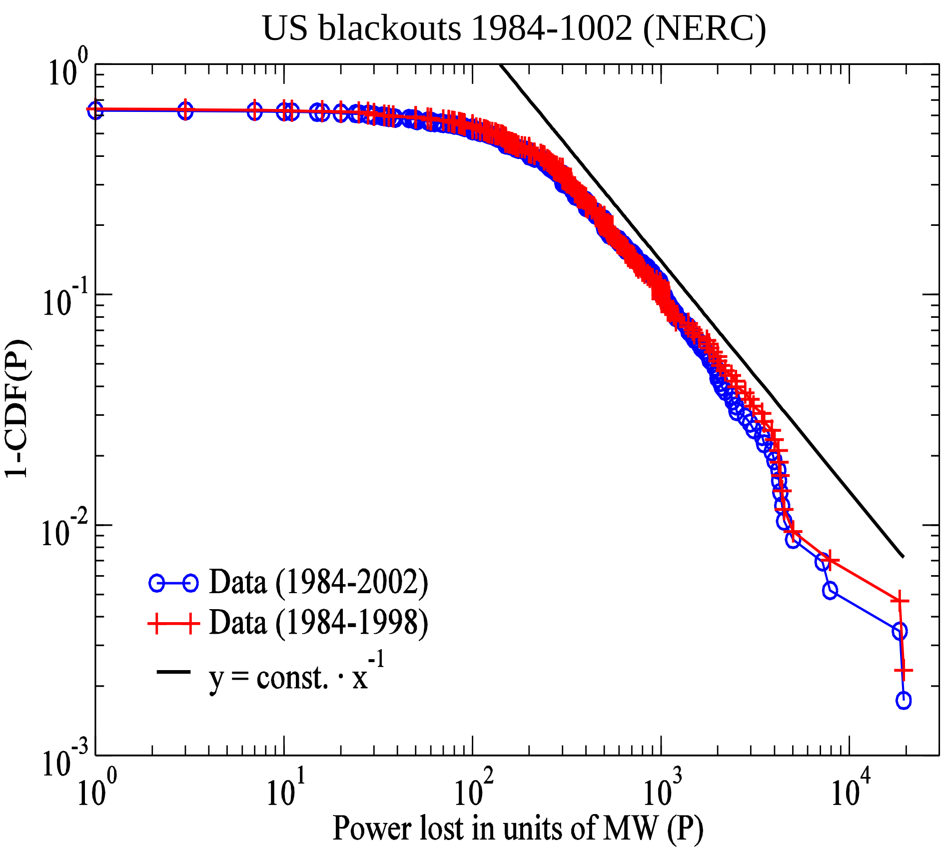
Motivation



Blackout in parts of the USA and Canada (2003), an impressive example of the long-reaching accompaniments of supply network failures.

Risk of Power Blackouts

- There are rather few *large* blackouts
  - So why should we care at all?
- Risk = Probability . Cost
- Large Power Blackouts are the most RISKY!



Source : Weron and Simonsen (2005)

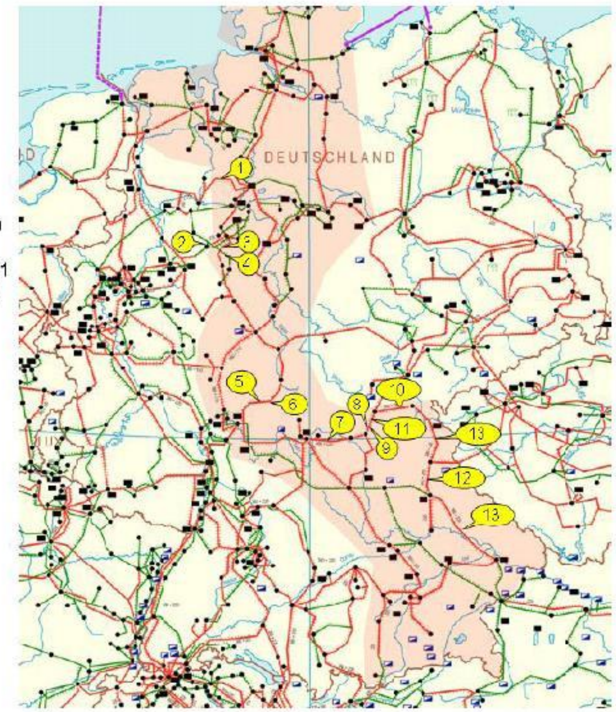
Power Blackouts: Real-Life examples

Europe Nov. 2006: What happened…?

State of the power grid shortly before the incident

Sequence of events on November 4, 2006

Nr.	Zeit	kV	Leitung
1	22:10:13	380	Wehrendorf-Landesbergen
2	22:10:15	220	Bielefeld/Ost-Spexard
3	22:10:19	380	Bechterdissen-Elsen
4	22:10:22	220	Paderborn/Süd-Bechterdissen/Gütersloh
5	22:10:22	380	Dipperz-Großkrotzenburg 1
6	22:10:25	380	Großkrotzenburg-Dipperz 2
7	22:10:27	380	Oberhaid-Grafenrheinfeld
8	22:10:27	380	Redwitz-Raitersaich
9	22:10:27	380	Redwitz-Oberhaid
10	22:10:27	380	Redwitz-Etzenricht
11	22:10:27	220	Würgau-Redwitz
12	22:10:27	380	Etzenricht-Schwandorf
13	22:10:27	220	Mechlenreuth-Schwandorf
14	22:10:27	380	Schwandorf-Pleinting



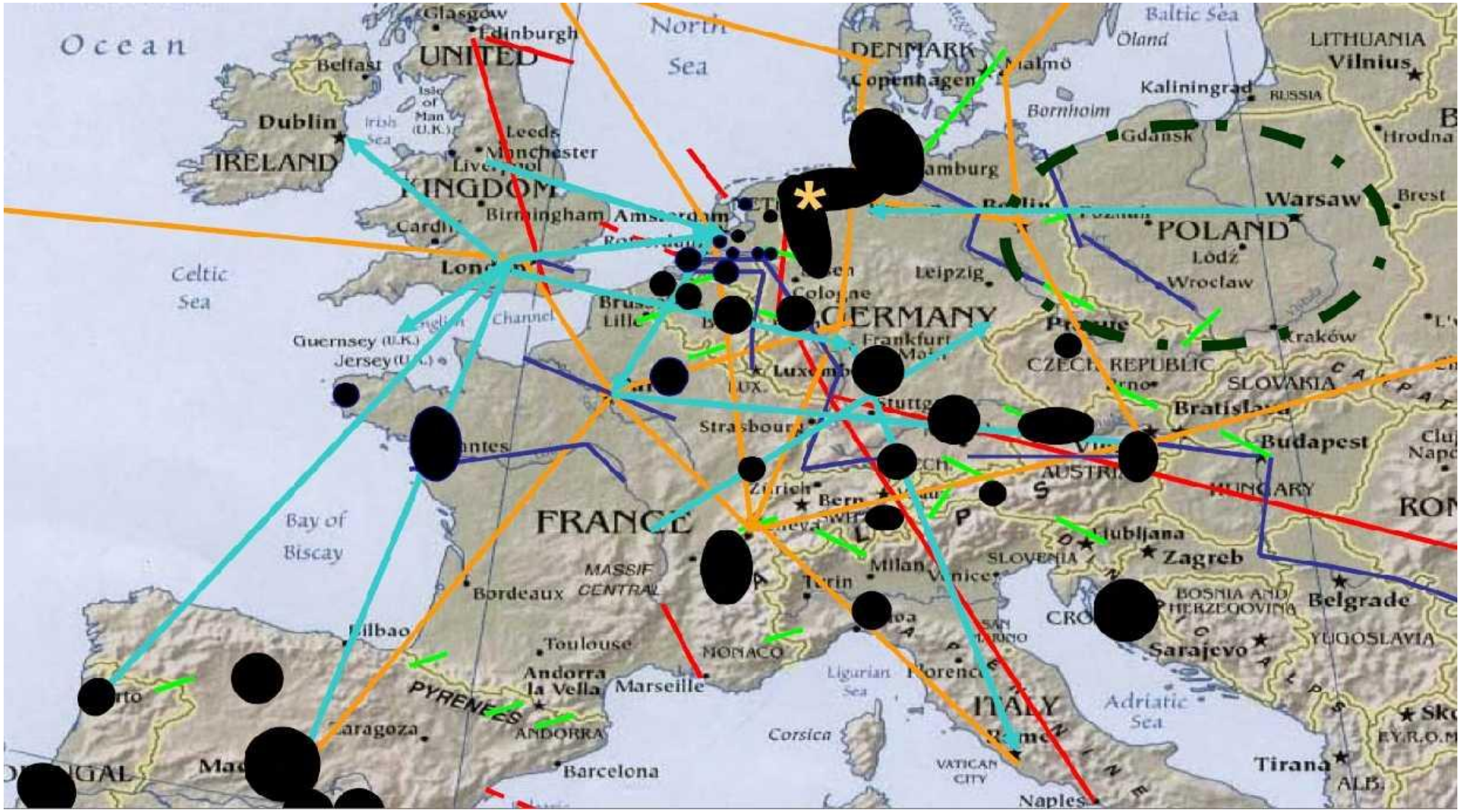
1,3,4,5 – lines switched off for construction work

2 – line switched off for the transfer of a ship by *Meyer-Werft*

Source : Report on the system incident of November 4, 2006, E.ON Netz GmbH

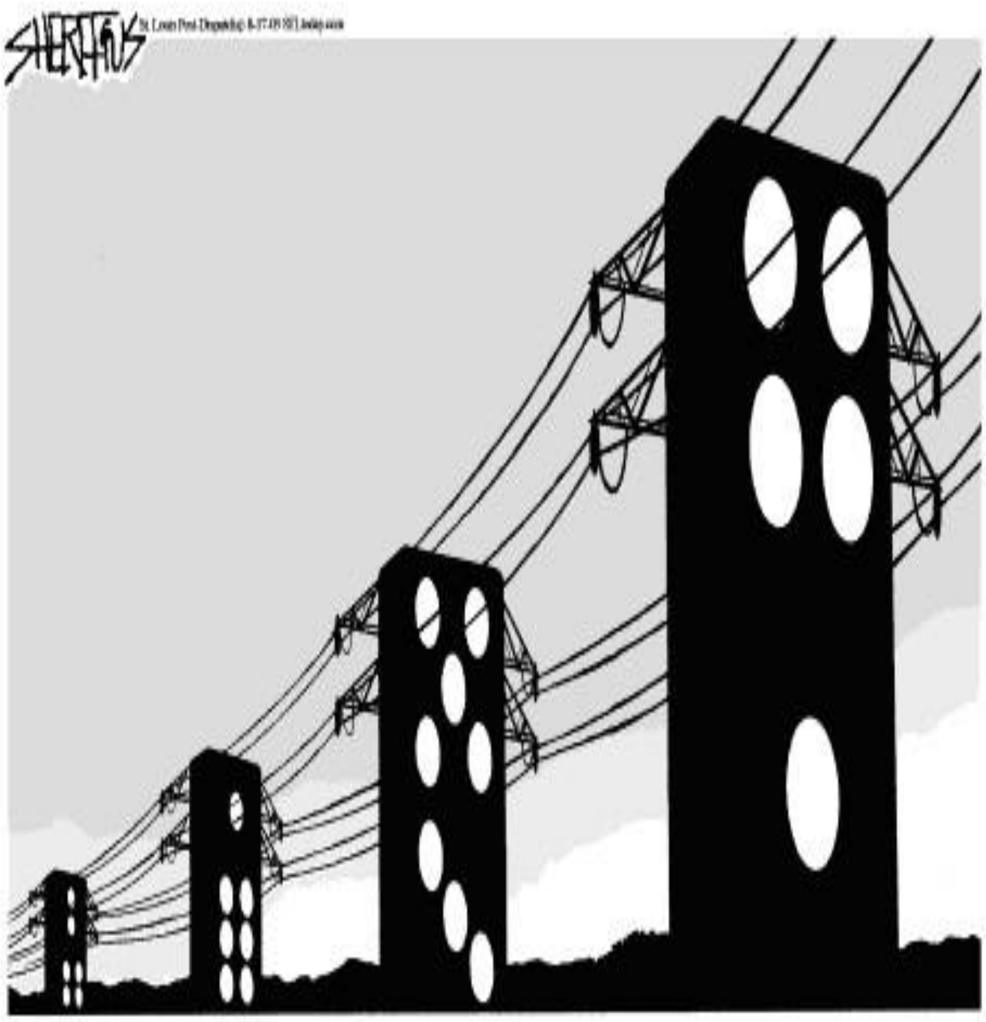
Power Blackouts: Real-Life examples

Failure in the continental European electricity grid on November 4, 2006



EU project IRRIIS: E. Liuf (2007) Critical Infrastructure protection, R&D view

Power Blackouts:
The Domino Effect (Cascading failure)



"Under certain conditions, a network component shutting down can cause current fluctuations in neighboring segments of the network, though this is unlikely, leading to a cascading failure of a larger section of the network. <u>This may range from a building, to a block, to an entire city, to the entire electrical grid."</u>

Source :Wikipedia

Cascading Failures Exist in Real Systems

- Examples
  - The power grid
  - Telecommunication networks
  - Transportation systems
  - Computer networks/ the Internet
  - Pipe line systems (water/gas/oil)
- They can be very *costly*
- They typically affect many people

Question : *How* can one protect (supply) network systems against cascading failures?

A few words on System Design

- The systems are designed with a *given load* in mind
- To ensure stability, the engineering approach, is to introduce some *overcapacity* into the system (security margins)
- …but overcapacity is *costly*!
- System robustness is often ONLY evaluated locally

- Cascading failure: When an initial perturbation occurs, loads have to redistributes. If the resulting loads exceed the capacities of link/nodes, new failures can result…. “the Domino effect”

Why do we have blackouts…..?

- System load (throughput)
  - optimized to get the maximum out of the system
  - high load means small operating margins
  - has impact on interactions and component failures
- Tradeoff between load and risk of failure
  - at system level
  - for system components
- What is the role of the deregulation?

Some Terminology

- Node Capacity:

$$(1+\alpha)L_i^{(0)}$$

  - Load in the stationary state : $L_i^{(0)}$
  - Overcapacity (tolerance) : α

- Overload when :

$$L_i(t) > (1+\alpha)L_i^{(0)}$$

- Fraction of nodes remaining (in the Giant Component)

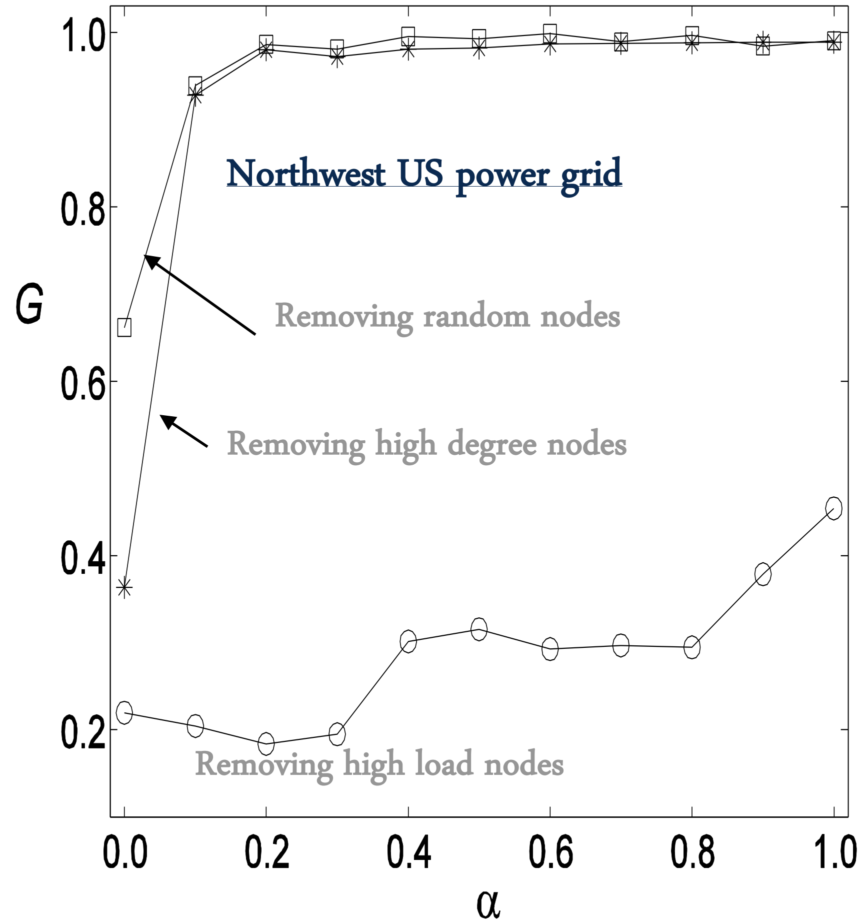
$$G(\alpha) = \frac{N_{GC}^{(\alpha)}(t \to \infty)}{N}$$

Previous physics works : Cascading Failures Motter and Lai: PRE 66, 065102R (2002)

Load calculated as betweeness centrality

- Overload checked only for the <u>stationary state</u>

- No sinks/sources



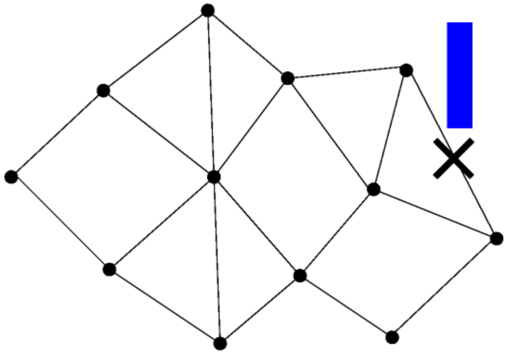
Previous works : Summary/Open Questions

- Previous works of cascading failures exclusively considered the stationary state
- But…why should the system *not* experience additional failures due to overloading during the transient period?

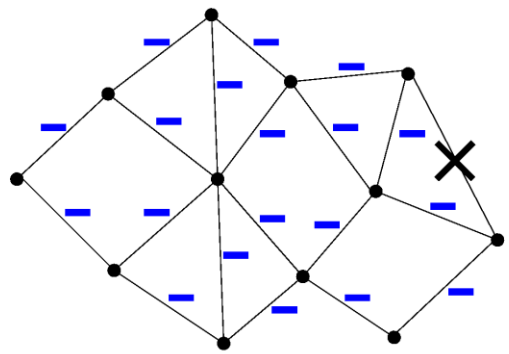
- Question to address:
  - What is the role played by the dynamics in cascading failures on complex networks
- A dynamical model is needed for such a study
  - .... But which one to choose?

Expected difference between a static and a dynamic model for flow redistribution

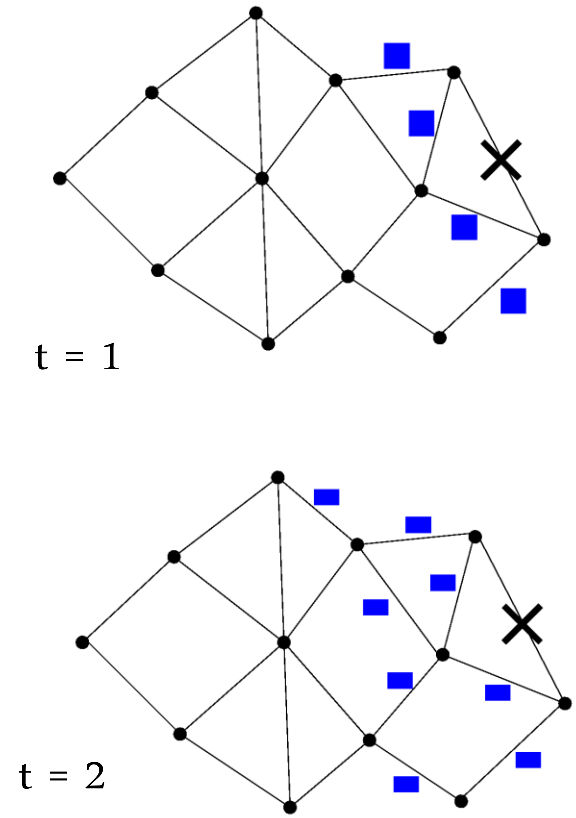
Initial failure



Stationary model



Dynamic model



Model : Requirements

- It should be:
  - Generic : no particular physical process is addressed
  - As simple as possible, but not simpler…

- Important ingredient (in our opinion)
  - The flowing quantity should be *CONSERVED*

Our solution : *A Random Walk (or Diffusion type)* model !

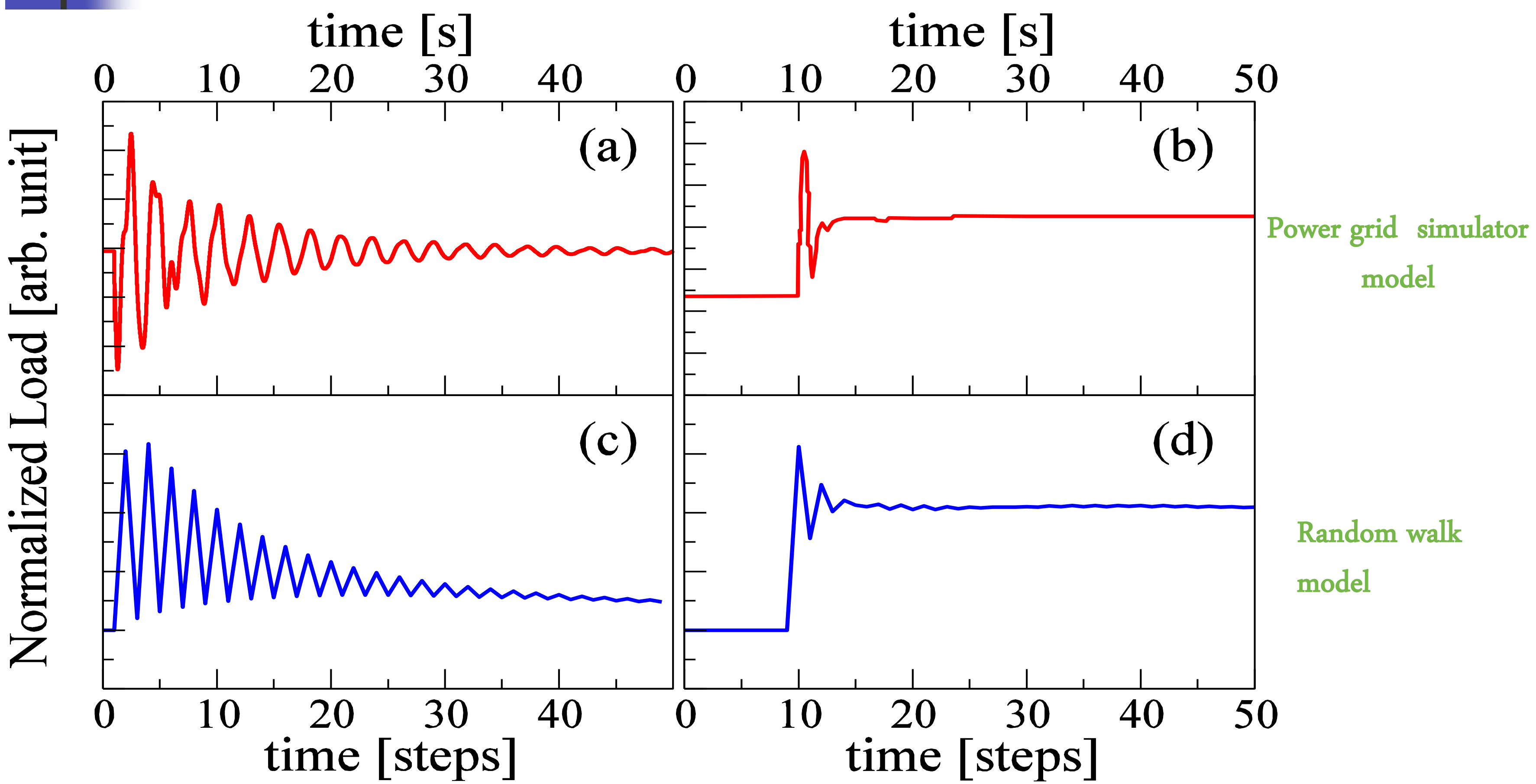
The Master Equation

- Our simple dynamical model incorporates:
  - Flow conservation
  - Network topology
  - Load redistribution

$$\vec{c}(t+1) = T^T \vec{c}(t) + \vec{j}^{\pm}(t); \qquad T_{ij} = \frac{W_{ij}}{w_j}$$

$c_i(t)$: The outgoing current from node *i* per link weight unit

Model Dynamics: Is it realistic?



Source : R. Sadikovic, Power flow control with UPFC, (internal report)

EUROSTAG power simulator : wwwaurostag.epfl.ch

Currents and Loads

- Link current on the link from node *j* to *i*

$$C_{ij}(t) = W_{ij} c_j(t)$$

- Loads (on the same link)

$$L_{ij}(t) = C_{ij}(t) + C_{ji}(t) = W_{ij} c_j(t) + W_{ij} c_i(t)$$

Stationary Solution

- Equation

$$(1-T^T)\vec{c}(\infty)=\vec{j}^{\pm};$$

- Solution

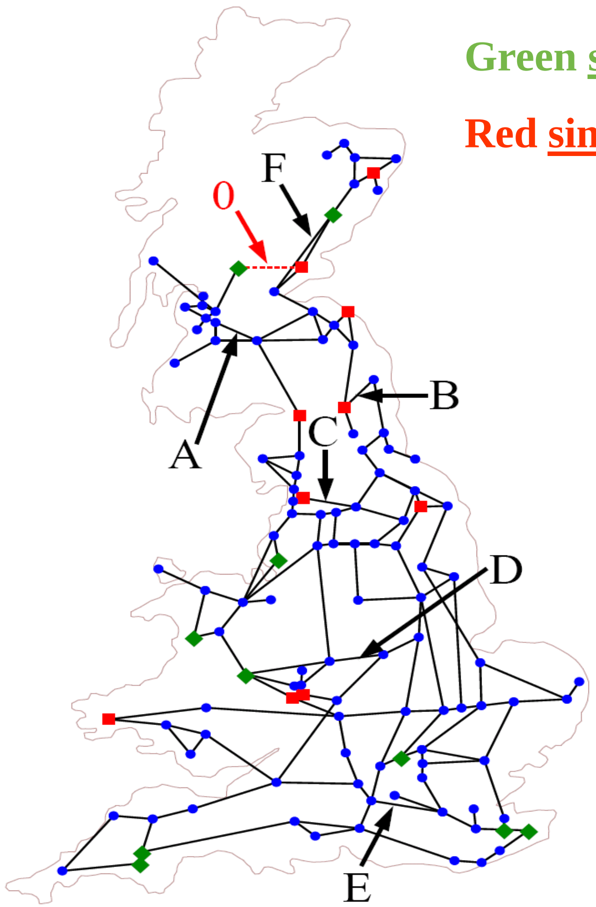
$$\vec{c}(\infty)=\vec{c}^{(0)}(\infty)+(1-T^T)^{+}\vec{j}^{\pm}$$

where

- Homogeneous solution $c_i^{(0)}(\infty)=1/N$
- $(1-T^T)^{+}$ is the generalized inverse of $1-T^T$

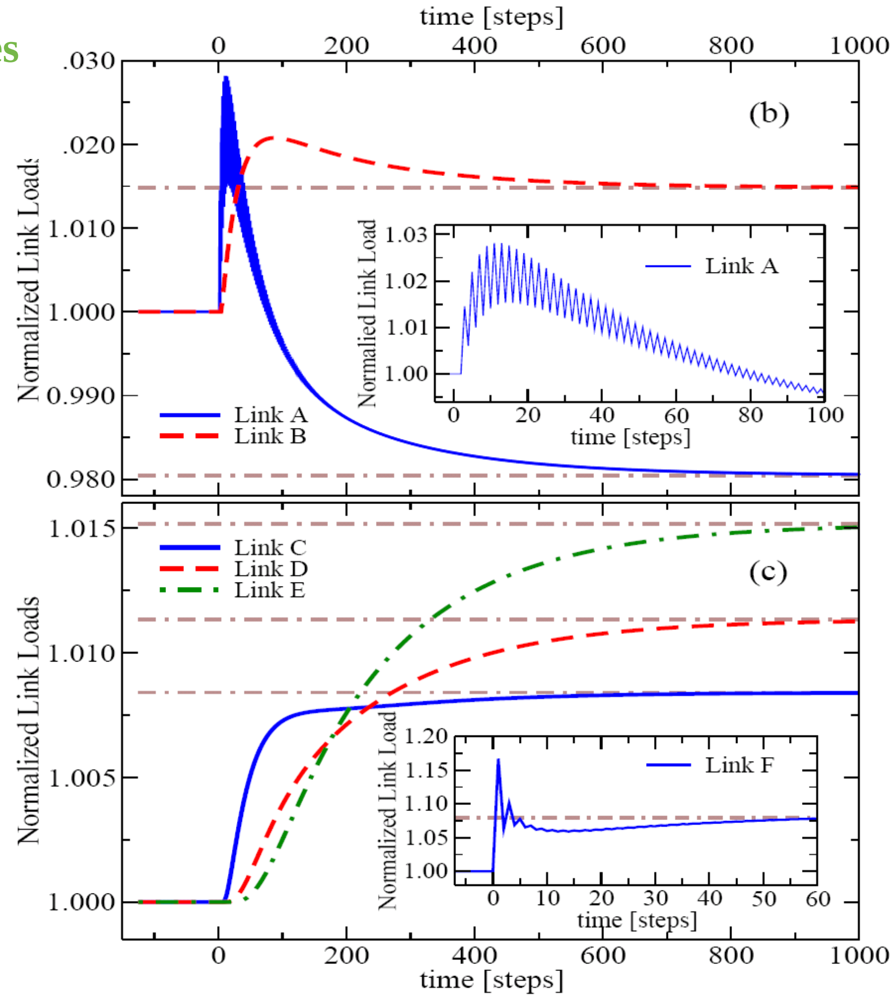
- Link capacities $(1+\alpha)L_{ij}^{(0)}$

Model Dynamics:
UK high voltage power grid (300-400kV)



Green source nodes

Red sink nodes



At t=0, link 0 is broken!

When does a link/node fail?

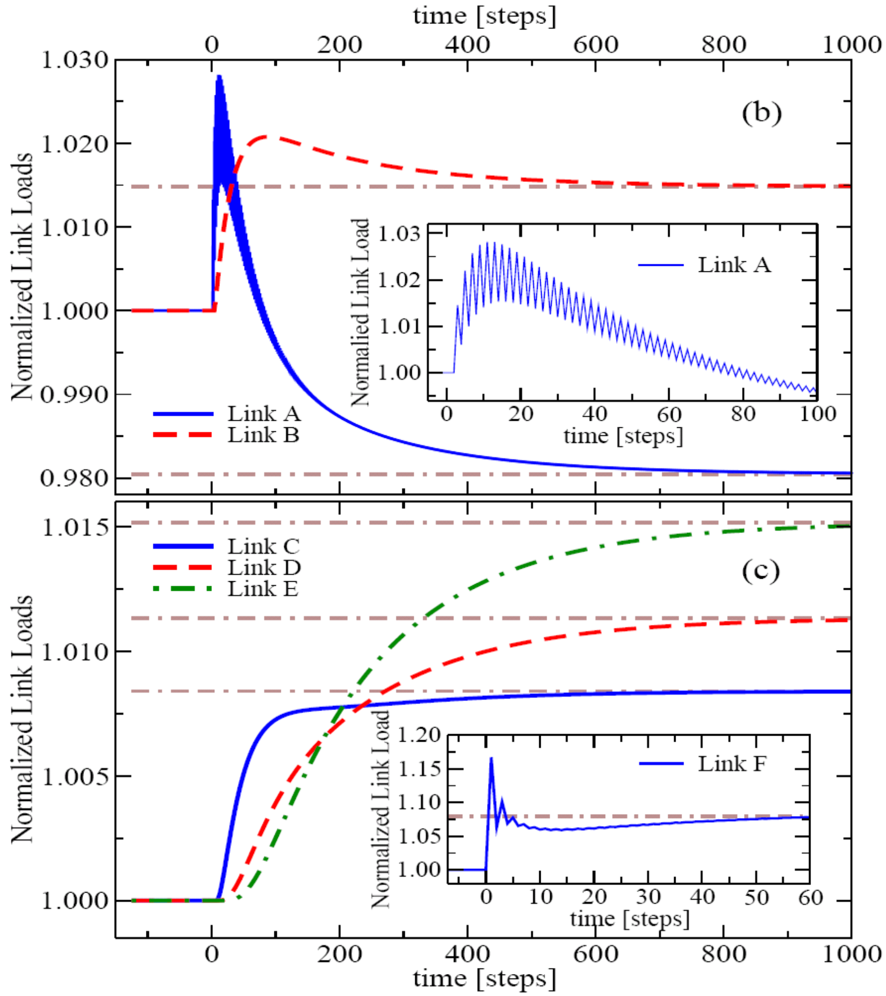
- Link/node capacities relative to the <u>undisturbed</u> state (L_{ij}) via a *tolerance parameter α*

$$(1+\alpha)L_{ij}^{(0)}$$

- A link/node fails whenever its *current* load, $L_{ij}(t)$ exceeds the capacity of *that* link/node

Failure if :

$$L_{ij}(t) > (1+\alpha)L_{ij}^{(0)}$$



Main steps of the simulations

The simulations consist of the following steps:

1. A *triggering event* (t=0) [remove a random link]
2. Calculate the link loads $L_{ij}(t)$
3. Check if any links are *overloaded* via $L_{ij}(t) > (1+\alpha)L_{ij}^{(0)}$
   1. If so remove such overloaded links
4. Repeat step 2 and 3 till no more links are overloaded
5. Average the results over the triggering event of pnt. 1 (and source and slinks locations)

Stationary Model vs. Dynamic Model : The northwestern US power transmission grid

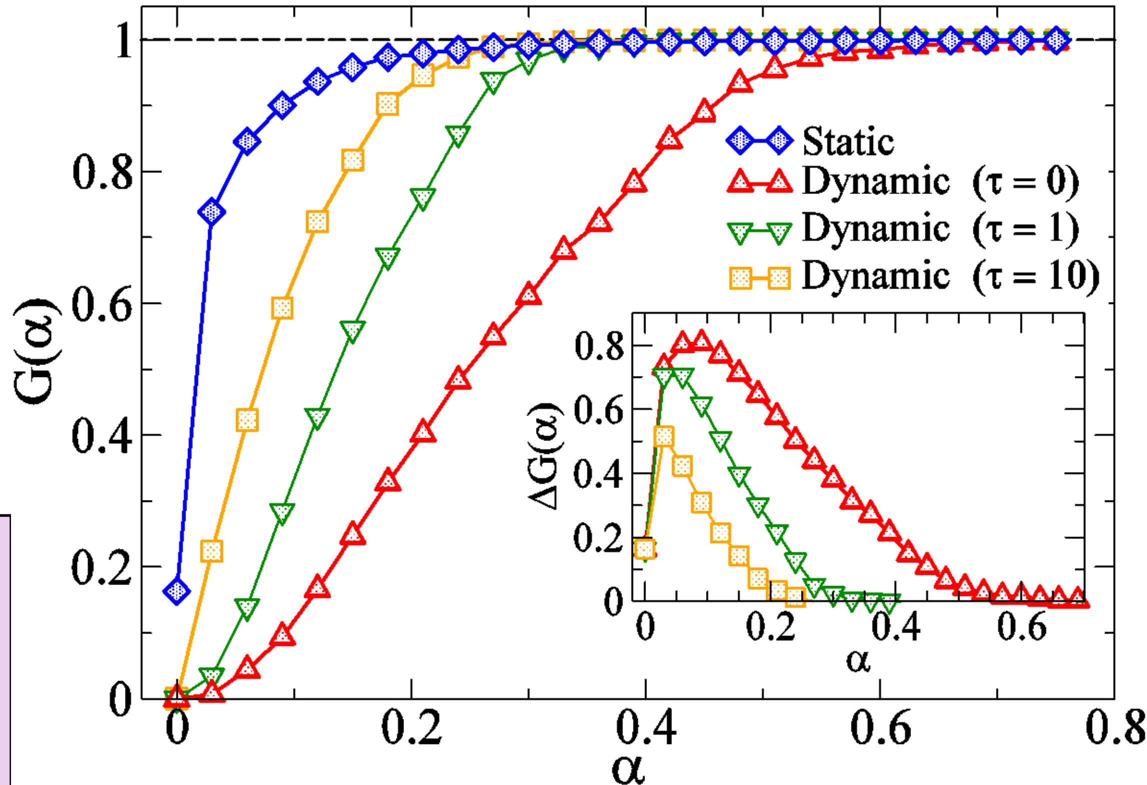
Phys. Rev. Lett. **100**, 218701 (2008)

$|\mathcal{N}|$ number of nodes (5000)

$|\mathcal{L}|$ number of links

$|\mathcal{N}_R|$ number of remaining nodes

$|\mathcal{L}_R|$ number of remaining links



$$G_{\mathcal{L}}(\alpha) = \frac{|\mathcal{L}_R|}{|\mathcal{L}|} \approx G_{\mathcal{N}}(\alpha) = \frac{|\mathcal{N}_R|}{|\mathcal{N}|} = G(\alpha)$$

The stationary model can *overestimates robustness* by more than 80% (in this case)

Overload exposure times may be relevant and will increase the robustness……

Stationary Model vs. Dynamic Model : The role of the two time-scales

- There are two characteristic time-scales in the problem:
  - Overload exposure time (protection system response time): τ
  - Typical transient time for the dynamics: τ_0
- Control parameter :

$$\chi = \frac{\tau}{\tau_0}$$

  - Static cascading failure model: $\chi >> 1$
  - Dynamical ($\tau=0$) cascading failure model : $\chi=0$

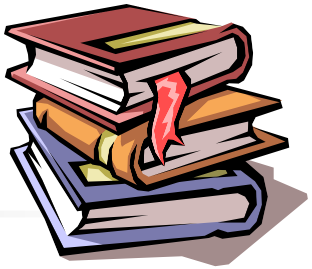
- The real situation is probably somewhere in between….

Summary: Cascading Failure

- *The dynamical* process on the network may be important to consider when evaluating network robustness (cascading)
  - Using a stationary model may dramatically overestimate (by 80-95%) the robustness of the underlying network
  - The actual overestimation does depend on the actual overload exposure time

- In a dynamical model:
  - links may fail that otherwise would not have done so (overshooting)
  - The proximity to a disturbance is more important in a dynamical model

References



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- See also : Phys. Rev. Lett. **90**, 14870 (2003).

  Physica A **357**, 317 (2005).
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  - Bakke *et al*. Europhys. Lett. ***76,*** *717 (2006).*

Epidemology on Networks

Application III

Colizza et al., Nature Physics **3**, 276 (2007).

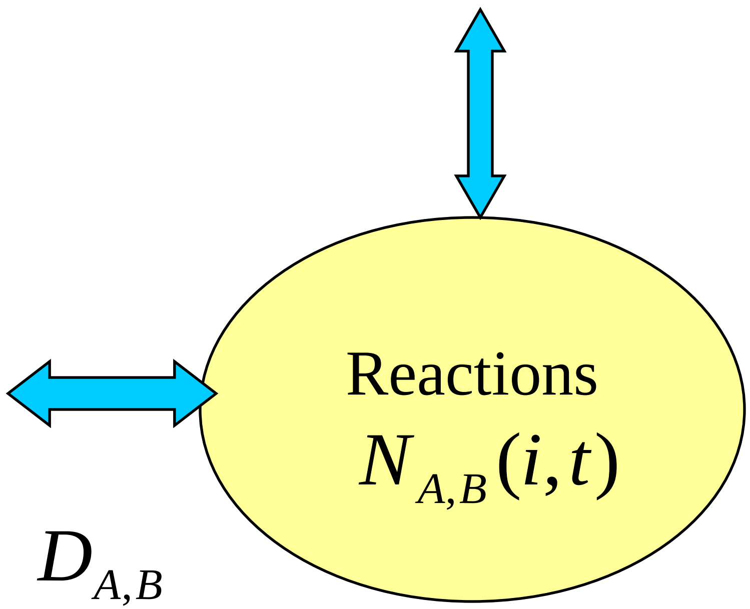
Lund et al. Work in progress (2010)

Epidemiology: The Questions

- What is the ratio of infected people in a population?
  - Does the ratio remain small or is it close to unity
  - What is the steady state behavior

- Other important questions:
  - How fast does an infection spread?
  - Can we stop it and how?

Reaction-Diffusion on Networks



(Diffusion constants)

- More then one type of particle A, B, …
- Two processes
  - Reactions on nodes
  - Diffusion between nodes

The SIS-Model

- The SIS Model
  - Individuals can become infected with the decease and recover from it with no immunity
  - Two types of individuals: Susceptible (A) and Infected (B)

- Reactions

$$A + B \xrightarrow{\beta} 2B$$

$$B \xrightarrow{\mu} A$$

Parameters:

Contact/Infection rate : β

Recovery rate : μ

SIS-Reactions Only
No diffusion

$$A + B \xrightarrow{\beta} 2B$$

$$B \xrightarrow{\mu} A$$

- No. of A and B-particles at node *i* at time *t*

$$\partial_t N_A(i,t) = \mu N_B(i,t) - \beta \Gamma_\nu(i,t) = -\partial_t N_B(i,t)$$

- Reaction Kernels

$$\Gamma_\nu(i,t) = \begin{cases} N_A(i,t) N_B(i,t); & \nu = 1 \\ \dfrac{N_A(i,t) N_B(i,t)}{N(i)}; & \nu = 2 \\ \dfrac{N_A(i,t) N_B(i,t)}{N_\times + N(i,t)}; & \nu = 3 \end{cases}$$

Type I “the village”

Type II “the city”

Type M (for Mixed)

Reactions-Diffusion on Networks

- Each time step consists of two processes
  - SIS-reactions on a node
  - Diffusion to neighboring nodes

$$\vec{N}_A(t+1) = \overbrace{(1-D_A)\vec{N}_A(t) + D_A T \vec{N}_A(t)}^{\text{Diffusion with diffusion const. } D_A} \underbrace{+\mu \vec{N}_B(t) - \beta \vec{\Gamma}_v(t)}_{\text{Reaction}}$$

$$\vec{N}_B(t+1) = (1-D_B)\vec{N}_B(t) + D_B T \vec{N}_B(t) - \mu \vec{N}_B(t) + \beta \vec{\Gamma}_v(t)$$

The Stationary State for Infected "particles"

- Quantities:

$$\rho = \frac{N}{V}; \qquad \rho_B(t) = \frac{N_B(t)}{V}; \qquad \frac{\rho_B(t)}{\rho} = \frac{N_B(t)}{N}$$

- Population ratio of infected in the stationary state

$$\frac{\bar{\rho}_B}{\rho} = \lim_{t \to \infty} \frac{N_B(t)}{N}$$

Special Cases

- Cases
  - $D_A=0$; $D_B=0$; (not so interesting)
  - $D_A=1$; $D_B=0$; (not so interesting)
  - $D_A=0$; $D_B=1$
  - $D_A=1$; $D_B=1$

- Assumptions
  - Mean field approximation
  - Uncorrelated network

Phase Transitions
Analytic Mean Field Results (Colizza et al. 2007)

- Type I : "the village"

$$\bar{\rho}_B = \begin{cases} \rho - \rho_c & \rho \geq \rho_c \\ 0 & \rho < \rho_c \end{cases} \qquad \rho_c = \begin{cases} \dfrac{\mu}{\beta} & D_A = 0 \\ \dfrac{\mu}{\beta} \dfrac{\langle k \rangle^2}{\langle k^2 \rangle} & D_A = 1 \end{cases}$$

- Type II: "the city"

$$\bar{\rho}_B = \begin{cases} \rho\left(1 - \dfrac{\mu}{\beta}\right) & \mu < \beta \\ 0 & \mu \geq \beta \end{cases} \qquad D_A \in \{0,1\}$$

Analytic Mean Field Results (Lund et al. 2010)

- Phase Transition

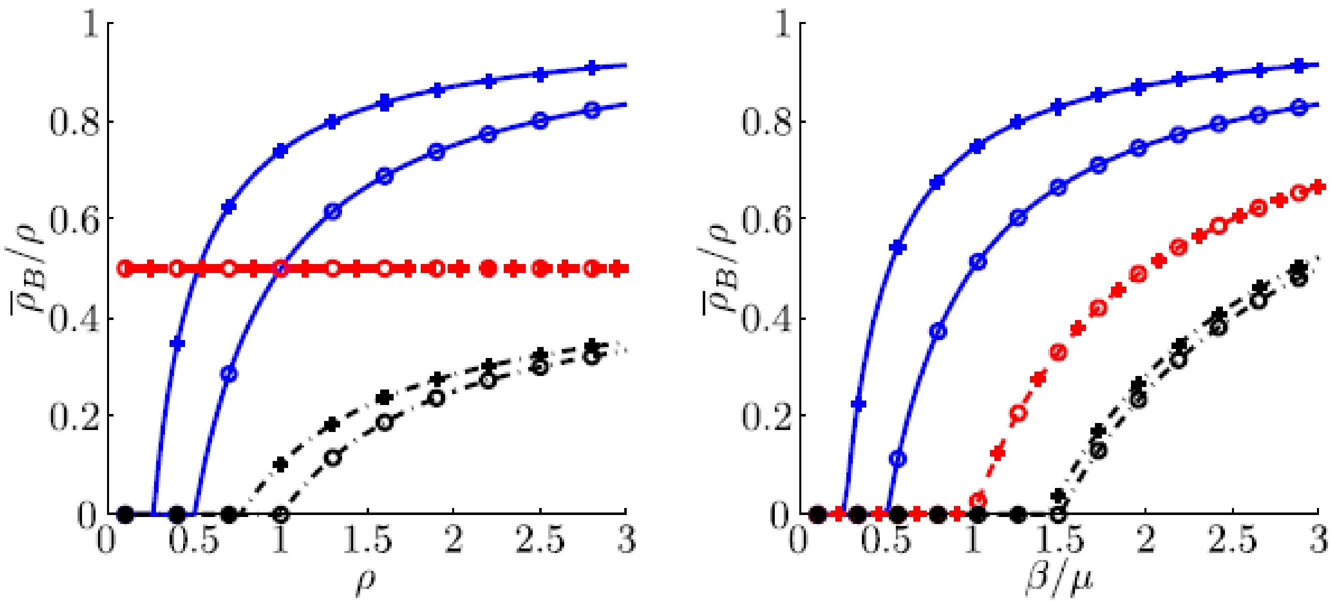
$$\bar{\rho}_B = \begin{cases} \rho - \rho_c & \rho \geq \rho_c \\ 0 & \rho < \rho_c \end{cases}$$

- where the critical nodal density is (DB=1)

$$\rho_c = \frac{\mu}{\beta - \mu}\rho_\times \qquad for \quad D_A = 0$$

$$\rho_c = \frac{\frac{\mu}{\beta}\langle k \rangle}{\sum_k p(k)\frac{k^2}{\langle k \rangle \rho_\times + k\rho}} \qquad for \quad D_A = 1$$

Analytic Results



(a) $\beta/\mu = 2$ (b) $\rho = 2$

Type I

Type II

Type M

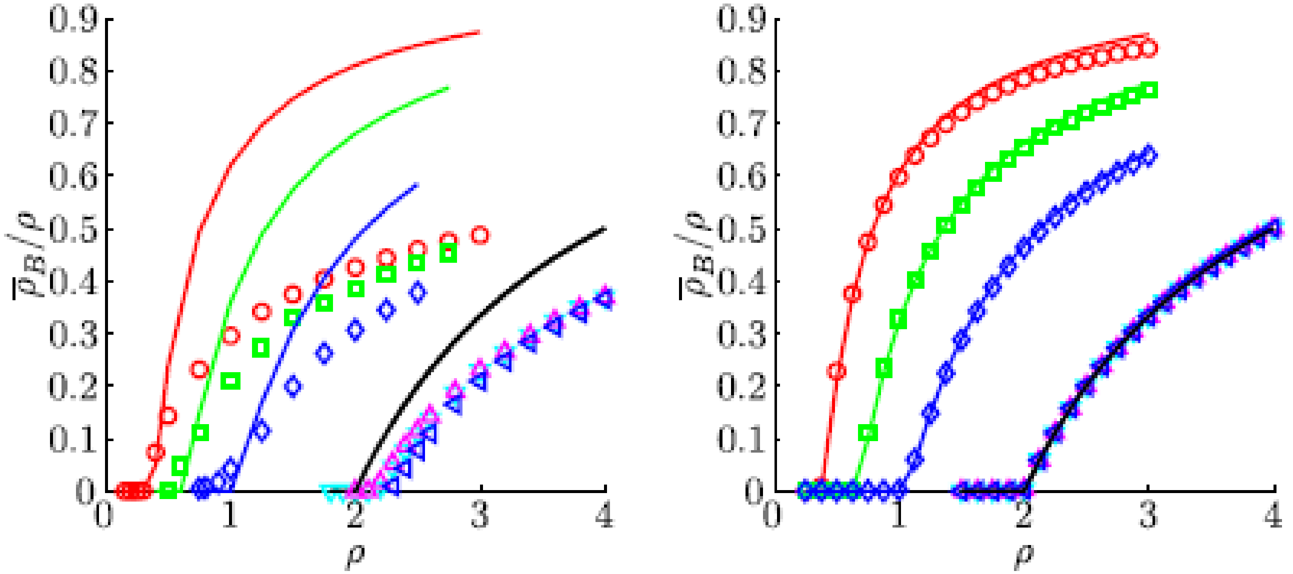
o D_A=0

+ D_A=1

Figure 4.1: Analytical expressions for stationary states of type I, II and M ($\rho_\times = 1$) with diffusing B particles ($D_B = 1$), from Eqs. (4.14), (4.16), (4.18), (4.20), (4.21) and (4.22). Solid lines are type I, dashed lines are type II and dash-dot lines are type M. Circles indicate $D_A = 0$, cross (+) indicates $D_A = 1$. The network is scale-free with $V = 1000$ nodes and $\gamma = 2.5$.

$$\frac{\beta}{\mu} = 2$$

Numerical Results : Type I "the village"

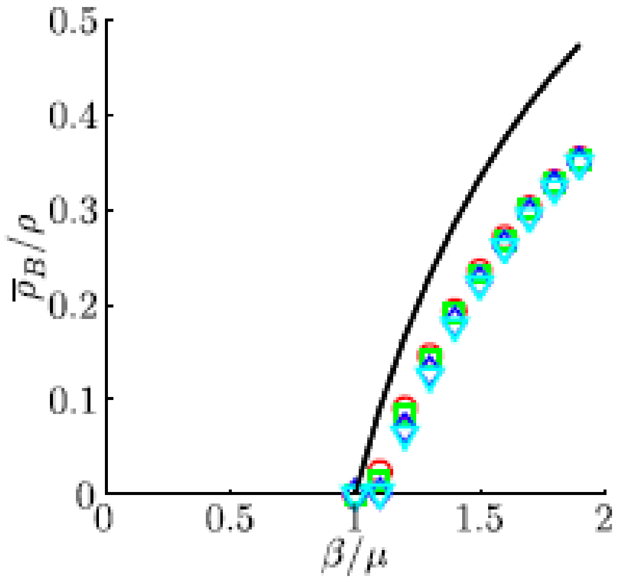


(a) Colizza (stochastic, μ and β unkown). Simulation results read directly from figure in Ref. [8], analytical results superimposed by us.

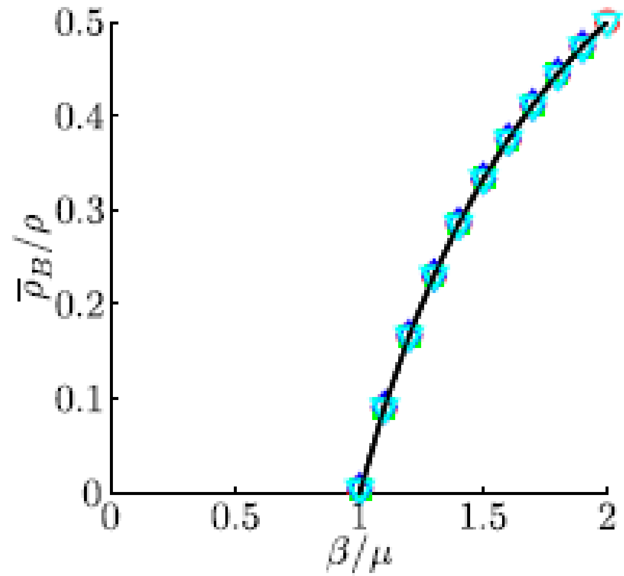
(b) Lund [2] (deterministic, $\mu = 0.02$, $\beta = 0.01$)

Figure 4.2: Comparison of (a) Colizza's [8] and (b) Lund's [2] simulation results for type I reactions, with $\mu/\beta = 2$ and $D_B = 1$. Lines indicate analytical results (Eqs. (4.14) and (4.20)), symbols are simulation results. Each symbol with properties (γ, V, D_A): Circles ○ (2.5, 10^5, 1), squares □ (2.5, 10^4, 1), diamonds ◇ (2.5, 10^3, 1), downward triangle ▽ (2.5, 10^4, 0), upward triangle △ (2.5, 10^5, 0), leftward triangle ◁ (3.0, 10^4, 0).

Numerical Results : Type II "the city"



(a) Colizza (stochastic, μ and β unkown). Simulation results read directly from figure in Ref. [8], analytical results superimposed by us.



(b) Lund [2] (deterministic, $\mu = 0.1$)

Figure 4.3: Comparison of (a) Colizza's and (b) Lund's simulation results for type II reactions, with $\rho = 20$ and $D_B = 1$. Lines indicate analytical result (Eq. (4.21)), symbols are simulation results. Each symbol with properties (γ, V, D_A): Circles ○ $(2.5, 10^4, 1)$, squares □ $(3.0, 10^4, 1)$, diamonds ◇ $(2.5, 10^4, 0)$ and triangles ▽ $(3.0, 10^4, 0)$.

Type M : The Mixed Reaction

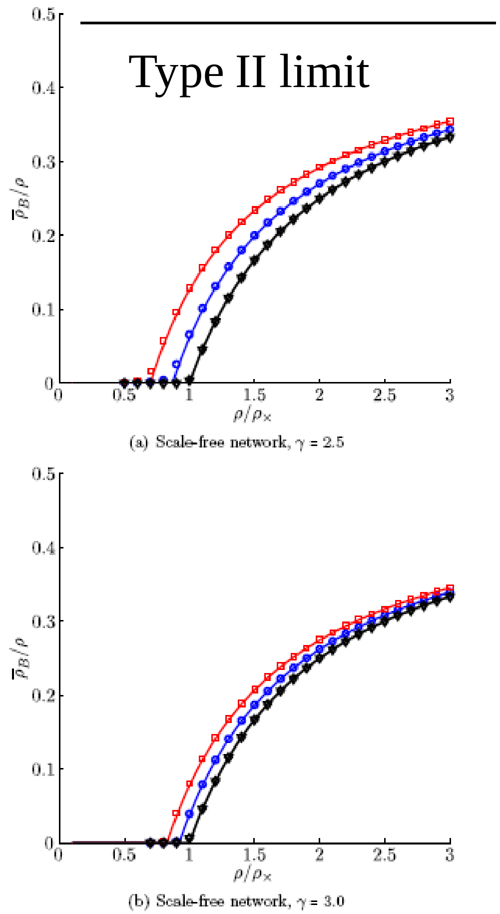


Figure 4.6: Simulation results for type M reactions, with $D_B = 1$ and $\beta/\mu = 2$. Lines indicate analytical result (Eqs. (4.18) and (4.22)), symbols are simulation results. Each symbol with properties (V, D_A): Circles ○ $(10^4, 1)$, squares □ $(10^2, 1)$, diamonds ◇ $(10^4, 0)$ and triangles ▽ $(10^2, 0)$.

$$\frac{\beta}{\mu} = 2$$

$$Type\ M = \begin{cases} Type\ I & \rho/\rho_\times << 1 \\ Type\ II & \rho/\rho_\times >> 1 \end{cases}$$

Summary: Epidemology on Networks

- Reaction-Diffusion can be used to study epidemic spreading on complex networks

- Phase transitions will occur depending on
  - Diffusion constants
  - Reaction rates
  - Network properties
  - Nodal densities

Conclusions

Conclusions

- Random walk (RW) models are simple and powerful
- Combined with complex networks (CN) they shine

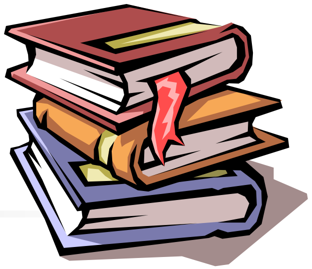
- The combination RW+CN shown useful for
  - Networks topology studies
  - Cascading failures
  - Epidemiology studies

Take home message

A model is almost never too simple to be useful!

Thank you for your attention!

References



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The End!