

10 Thermal field theory

10.1 Overview

Introduction The Green functions we have considered so far were all defined as expectation value of products of fields in a pure state, the vacuum in the absence of real particles, $|0\rangle$. Out of these Green functions, we could built up our quantities of prime interest, decay or scattering amplitudes for $1 \rightarrow n$ and $2 \rightarrow n$ particles via the LSZ formalism. In this chapter, we discuss how Green functions should be calculated, if the vacuum is non-empty and described by a density matrix ρ . Examples for such systems are the early Universe or the dense, hot interior of a star. The simplest and at the same time most important cases of thermal systems are those in equilibrium.

In equilibrium statistical physics, the partition function is of central importance as all other thermodynamic quantities can be calculated from it. In the grand canonical ensemble (where both particles and energy may be exchanged between system and reservoir) it takes the form,

$$Z(V, T, \mu_1, \mu_2, \dots) = \sum_n \langle n | e^{-\beta H - \mu_i N_i} | n \rangle = e^{-\beta \Omega}, \quad (10.1)$$

where $\beta \equiv 1/T$ is the inverse temperature of the system and n denotes a complete set of quantum numbers. Finally, the Landau or grand canonical free energy

$$\Omega(U, S, N_i) = -T \ln Z = U - TS - \mu_i N_i = PV \quad (10.2)$$

connects the microscopic partition function to thermodynamics. As we already saw, the partition function is closely related to the corresponding field theory in Euclidean space, while we can derive from Ω all relevant thermodynamical quantities: For instance, we obtain the pressure P as $P = \partial \Omega / \partial V |_{T, \mu_i}$. In addition the expectation value of any observable, O , is given (setting in the following always $\mu_i = 0$) as

$$\langle O \rangle = Z^{-1} \text{Tr} \left[O e^{-\beta H} \right]. \quad (10.3)$$

Computational approaches Two main approaches to calculations are used in thermal field theory:

- In the real-time formalism one applies the formula (10.3) valid for any observable directly to Green functions, that is

$$G(x_1, \dots, x_n) = \langle T(\phi(x_1) \cdots \phi(x_n)) \rangle \quad (10.4)$$

$$= Z^{-1} \text{Tr} \left[e^{-\beta H} T(\phi(x_1) \cdots \phi(x_n)) \right]. \quad (10.5)$$

As main advantage, we note that this approach can be extended to the non-equilibrium case. As disadvantage, we note that the proper definition of the contours for the propagators becomes more complicated and one has to introduce therefore matrix-valued 2×2 propagators.

- In the imaginary time formalism, we rotate from Minkowski to Euclidean space, $t \rightarrow -it$, so that the transition amplitude from an initial state, $|q(t_i)\rangle$, to a final state, $|q(t_f)\rangle$, is given by

$$\langle q(t_f) | e^{-(t_f-t_i)H} | q(t_i) \rangle = \int_{q(t_i)}^{q(t_f)} \mathcal{D}q e^{-S}, \quad (10.6)$$

where S is now the Euclidean action. If we set the evolution time, $t_f - t_i$, equal to the inverse temperature β and integrate over all periodic paths $q(t_i) = q(t_f + \beta)$, we have

$$\sum_q \langle q | e^{-\beta H} | q \rangle = \int_{q(t)}^{q(t+\beta)} \mathcal{D}q e^{-S} = Z. \quad (10.7)$$

We now see that we have formally connected the path integral formulation of quantum field theory (in Euclidean space) to the partition function of statistical mechanics.

In contrast to $T = 0$ field theory, Green functions and the partition function in the Euclidean are not merely a mathematical tool but our main objects of interest. Since the Euclidean Green functions do depend on temperature instead of time, we are not able to describe time-dependent phenomena in this approach.

Thermal Green functions The trace in the partition function of statistical physics implies that we have to sum over configurations connecting the same physical state at t and $t + \beta$. In the path integral corresponding to statistical mechanics, the periodicity condition $q(t) = q(t + \beta)$ for the real coordinate q is clearly the only possible choice. In contrast, fields may only be observable through bilinear quantities, as e.g. $\bar{\psi}\Gamma\psi$ for a fermion field. This raises the question, if we should require periodic or anti-periodic boundary conditions for fermionic fields.

We consider first thermal Green functions G_β for a free scalar field. From the Heisenberg equation for the field operator,

$$\phi(t, \mathbf{x}) = e^{iHt} \phi(0, \mathbf{x}) e^{-iHt}, \quad (10.8)$$

we find¹

$$\begin{aligned} G_\beta^+(t', \mathbf{x}'; t, \mathbf{x}) &= \text{tr} [e^{-\beta H} \phi(t', \mathbf{x}') \phi(t, \mathbf{x})] / Z \\ &= \text{tr} [e^{-\beta H} \phi(t', \mathbf{x}') e^{\beta H} e^{-\beta H} \phi(t, \mathbf{x})] / Z \\ &= \text{tr} [\phi(t' + i\beta, \mathbf{x}') e^{-\beta H} \phi(t, \mathbf{x})] / Z \\ &= \text{tr} [e^{-\beta H} \phi(t, \mathbf{x}) \phi(t' + i\beta, \mathbf{x}')] / Z = G_\beta^-(t' + i\beta, \mathbf{x}'; t, \mathbf{x}). \end{aligned} \quad (10.9)$$

Hence for bosonic fields,

$$G_\beta^\pm(t', \mathbf{x}'; t, \mathbf{x}) = G_\beta^\mp(t' + i\beta, \mathbf{x}'; t, \mathbf{x}), \quad (10.10)$$

implying periodic boundary condition for the thermal propagators,

$$G_\beta(t', \mathbf{x}'; t, \mathbf{x}) = G_\beta(t' + i\beta, \mathbf{x}'; t, \mathbf{x}). \quad (10.11)$$

¹We set $G(t) = G^+\vartheta(t) + G^-\vartheta(-t)$ for bosonic and $S(t) = S^+\vartheta(t) - S^-\vartheta(-t)$ for fermionic propagators.

The derivation (10.9) goes through unchanged for fermionic fields,

$$S_{\beta}^{\pm}(t', \mathbf{x}'; t, \mathbf{x}) = S_{\beta}^{\mp}(t' + i\beta, \mathbf{x}'; t, \mathbf{x}). \quad (10.12)$$

But now the anti-commuting nature of fermionic fields—more specifically the minus sign in (6.40)—leads to anti-periodic boundary condition for their thermal propagators,

$$S_{\beta}(t', \mathbf{x}'; t, \mathbf{x}) = -S_{\beta}(t' + i\beta, \mathbf{x}'; t, \mathbf{x}). \quad (10.13)$$

Both periodicity conditions discretise the frequency spectrum of the wave-functions. Thus the Fourier transform of thermal fields contained in a box of size $V \times \beta$ is given by

$$\phi(t, \mathbf{x}) = \frac{1}{\sqrt{\beta V}} \sum_{n=-\infty}^{\infty} \sum_{\mathbf{p}} \phi_{n, \mathbf{p}} e^{-i(\omega_n t + \mathbf{p} \cdot \mathbf{x})} \quad (10.14)$$

with $\omega_n = 2n\pi T$ for bosonic and $\omega_n = (2n + 1)\pi T$ for fermionic fields, respectively, and $n \in \mathbb{Z}$. The frequencies ω_n are called Matsubara frequencies. Similarly, the Green function for a non-interacting scalar field in the limit $V \rightarrow \infty$ is given by

$$G_{\beta}(t, \mathbf{x}) = \frac{1}{\sqrt{\beta}} \sum_{n=-\infty}^{\infty} \int \frac{d^3 p}{(2\pi)^3} G_n(\omega_n, \mathbf{p}) \quad (10.15)$$

with

$$G_n(\omega_n, \mathbf{p}) = \frac{1}{\omega_n^2 + \mathbf{p}^2 + m^2}. \quad (10.16)$$

10.2 Interacting scalar gas

As main application of the formalism of thermal field theory we will calculate the first quantum correction to the equation of state of a gas of scalar particles interacting with a $\lambda\phi^4$ interaction. This $\mathcal{O}(\lambda)$ correction is given by a two-loop vacuum diagram and serves us also as demonstration how mixed UV divergences are cancelled by counter terms found at lower order. After that we discuss the IR behaviour of the massless $\lambda\phi^4$ theory. We will find that the plasma generates a thermal mass for the scalar particle, an effect we will try to imitate generating masses for the particles of the SM.

Equation of state of a scalar gas We aim now at the first-quantum correction to the equation of state of gas of massive scalar particles interacting with a $\lambda\phi^4$ interaction. The corresponding Hamiltonian is given by

$$H = \int d^3 x \left[\frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \right] \quad (10.17)$$

$$= H_0 + H_{\text{int}}, \quad (10.18)$$

where $H_0 = \sum_{\mathbf{k}} \omega_{\mathbf{k}} (1/2 + a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}})$. We expand also the Landau free energy in λ ,

$$\Omega = \Omega_0 + \lambda \Omega_1 + \lambda^2 \Omega_2 + \dots \quad (10.19)$$

The interaction term produces to first order in λ the following vacuum diagram,

$$\begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \circ \text{---} \end{array} = \frac{\lambda}{4!} \langle \phi^4 \rangle = \frac{3\lambda}{4!} \langle \phi^2 \rangle \langle \phi^2 \rangle = \frac{\lambda}{8} \left(\Delta(t=0, \mathbf{x}=0) \right)^2,$$

where the factor three has come from the possible ways of joining the lines to form the two loops seen in the diagram. (Alternatively, we can use that $\langle \phi^n \rangle$ is just the expectation value of ϕ^n times a Gaussian.) We could now insert into the formula the Green function derived in the imaginary time formalism. Instead, we apply a more instructive argument following the steps in Eqs. (2.44-2.45): At $T = 0$, the vacuum is empty and the $a_{\mathbf{k}}^\dagger a_{\mathbf{k}}$ term in H_0 gives zero contribution,

$$\Delta(t=0, x=0) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{k}}}. \quad (10.20)$$

For $T > 0$, the expectation value of the number operator $N_{\mathbf{k}} = a_{\mathbf{k}}^\dagger a_{\mathbf{k}}$ is just the number distribution $n_{\mathbf{k}}$ of ϕ particles,

$$\Delta(t=0, x=0) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_{\mathbf{k}}} \left(\frac{1}{2} + \langle a_{\mathbf{k}}^\dagger a_{\mathbf{k}} \rangle \right) \quad (10.21)$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1 + 2n_{\mathbf{k}}}{2\omega_{\mathbf{k}}}. \quad (10.22)$$

In thermal equilibrium the number density of a scalar field follows Bose-Einstein statistics, that is

$$n_{\mathbf{k}} = \frac{1}{e^{\beta\omega_{\mathbf{k}}} - 1}. \quad (10.23)$$

More formally, this result can be derived directly from the periodicity condition of the Green functions (problem 10.2). Note that we can view the propagator as the sum of a vacuum part (“1/2”) and a thermal part (“ $n_{\mathbf{k}}$ ”). In the latter, the high energy modes are exponentially suppressed and thus no UV divergences should appear in the temperature dependent parts of physical observables. Thus our standard renormalisation program at $T = 0$ should apply in the same way at $T > 0$.

Continuing the derivation of the first order correction to Ω we have

$$\frac{\lambda}{4!} \langle \phi^4 \rangle = \frac{\lambda}{8} \left[\sum_{\mathbf{k}} \frac{1 + 2n_{\mathbf{k}}}{2\omega_{\mathbf{k}}} \right]^2 \quad (10.24)$$

$$= \frac{\lambda}{8} \left[\underbrace{\left(\sum_{\mathbf{k}} \frac{1}{2\omega_{\mathbf{k}}} \right)^2}_{\text{vacuum term}} + \underbrace{\left(\sum_{\mathbf{k}} \frac{n_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^2}_{\text{finite T-dependent term}} + \underbrace{2 \left(\sum_{\mathbf{k}} \frac{1}{2\omega_{\mathbf{k}}} \right) \left(\sum_{\mathbf{k}} \frac{n_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right)}_{\text{infinite vacuum} \times \text{T-dependent term}} \right]. \quad (10.25)$$

The mixed term gives rise to a temperature dependent UV divergence, which from the previous argument should not appear in our calculation. Thus we expect that this term will be cancelled by an one-loop counter term.

We consider therefore the self-energy Σ , which is given by

$$\Sigma = \frac{\lambda}{2} \Delta(t=0, \mathbf{x}=0) = \frac{\lambda}{2} \sum_{\mathbf{k}} \frac{1 + 2n_{\mathbf{k}}}{\omega_{\mathbf{k}}}, \quad (10.26)$$

which contains the usual $T = 0$ divergence coming from the unsuppressed sum over frequencies. We use physical perturbation theory, adding an additional interaction term of the form $\delta m^2 \phi^2$ to the Lagrangian, where δm^2 is chosen such that m corresponds to the physical mass. Thus δm^2 is determined by

$$\delta m^2 + \frac{\lambda}{2} \sum_{\mathbf{k}} \frac{1}{2\omega_{\mathbf{k}}} = 0. \quad (10.27)$$

Because the product $\delta m^2 \Delta(0)$ is of $\mathcal{O}(\lambda)$, we have to add the following vacuum diagram to Ω_1 ,

$$\frac{1}{2} \delta m^2 \Delta(0,0) = \text{Diagram} = \frac{1}{2} \left(-\frac{\lambda}{2} \sum_{\mathbf{k}} \frac{1}{2\omega_{\mathbf{k}}} \right) \left(\sum_{\mathbf{k}} \frac{1+2n_{\mathbf{k}}}{2\omega_{\mathbf{k}}} \right). \quad (10.28)$$

Comparing this expression to the troublesome mixed term, we see that they agree but have the opposite sign. Thus the one-loop subdivergence cancels the temperature dependent UV divergence at two-loop in Ω_1 . As result, we obtain the consistent expression

$$\frac{\lambda}{4!} \langle \phi^4 \rangle = \frac{\lambda}{8} \left(\sum_{\mathbf{k}} \frac{n_{\mathbf{k}}}{\omega_{\mathbf{k}}} \right)^2 + \text{vac}, \quad (10.29)$$

which we can now calculate explicitly. For simplicity, we restrict ourselves to the high-temperature limit,

$$\sum_{\mathbf{k}} \frac{n_{\mathbf{k}}}{\omega_{\mathbf{k}}} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega} \frac{1}{e^{\beta\omega} - 1} = \frac{T^2}{2\pi^2} \underbrace{\int_0^\infty dx \frac{x}{e^x - 1}}_{\pi^2/6} = \frac{T^2}{12}, \quad (10.30)$$

where we have used that $m/T \rightarrow 0$ at high T . So our final result for Ω_1 is

$$\Omega_1 = \frac{\lambda}{8} \left(\frac{T^2}{12} \right)^2 = \frac{\lambda}{1152} T^4, \quad (10.31)$$

which we may compare to the non-interacting result, $\Omega_0 = \pi^2 T^4 / 90$: The ratio $\Omega_1 / \Omega_0 \approx 10^{-4} \lambda$ seems to indicate a fast convergence of the perturbative expansion of the pressure for any reasonable value of the coupling.

We simply quote the result to three loops or second order in λ from the literature,

$$P = \frac{\pi^2 T^4}{9} \left[\frac{1}{10} - \frac{1}{8} \frac{\lambda}{16\pi^2} + \frac{1}{8} \left(3 \ln \frac{\mu}{4\pi T} + \frac{31}{35} + C \right) \left(\frac{\lambda}{16\pi^2} \right)^2 \right]. \quad (10.32)$$

The parameter μ in Eq. (10.32) is not the chemical potential but as usually the renormalisation scale. As the pressure is a physical quantity, it should not depend upon such a parameter, but the truncation used in the expansion causes this μ dependence, which then leads to a μ dependence at $\mathcal{O}(\lambda^3)$. Examining this μ dependence of P we have

$$\mu \frac{dP}{d\mu} = \frac{\pi^2 T^4}{9} \left[-\frac{1}{8} \frac{1}{16\pi^2} \underbrace{\mu \frac{d\lambda}{d\mu}}_{\beta=3\lambda^2/16\pi^2} + \frac{3}{8} \left(\frac{\lambda}{16\pi^2} \right)^2 + \mathcal{O}(\lambda^3) \right] = \mathcal{O}(\lambda^3). \quad (10.33)$$

Thus the dependence on the renormalisation scale is of higher order in λ than the order of perturbation theory we are working. Still we can try to minimise the remaining dependence

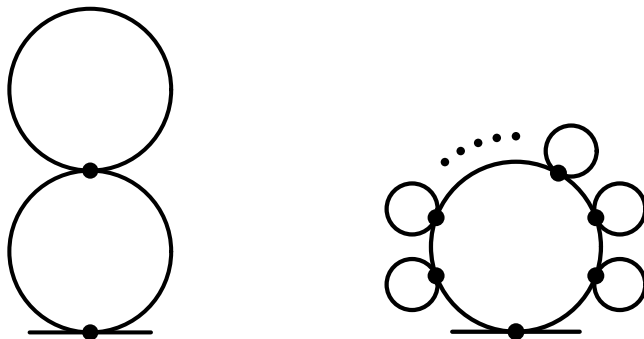


Figure 10.1: Left: A second order correction to the mass; Right: Ring diagram, each external bubble corresponds to an insertion of m_D^2 .

by a suitable choice of μ : Clearly, a sensible choice in this case is $\mu = 4\pi T$, for which the logarithm vanishes. Still, any other choice is mathematically as correct as this one. Instead of being worried about this dependence on the renormalisation scale, we may take advantage of it as follows: Varying the renormalisation scale μ in a “reasonable range”, say between $\mu = 2\pi T$ and $\mu = 8\pi T$, we obtain an error estimate for the missing higher-order corrections.

Finally, we note that $\mu \propto T$ implies that a QCD plasma becomes in the large temperature limit an asymptotically free gas of quarks and gluons, while at $T = \mathcal{O}(\Lambda_{\text{QCD}})$ a phase transition to colourless mesons and baryons happens.

IR behaviour We have found a fast convergence of the perturbative expansion from the results for the pressure of a scalar gas. For applications particularly interesting is the case of a massless particle and we consider now as a toy-model for QCD a $\lambda\phi^4$ theory with $m = 0$.

Looking back at our previous result for the self-energy, Eq. (10.26), setting $m \rightarrow 0$ and dropping the vacuum term, we have

$$\Sigma = \frac{\lambda}{2} \Delta(t=0, \mathbf{x}=0) \rightarrow \frac{\lambda}{2} \int \frac{d^3k}{(2\pi)^3} \frac{n_{\mathbf{k}}}{\omega_{\mathbf{k}}} = \frac{\lambda T^2}{2 \cdot 12}. \quad (10.34)$$

Thus the thermal part of the self-energy induces at first order in λ a thermal or Debye mass,

$$m_D^2 = \frac{\lambda T^2}{2 \cdot 12}. \quad (10.35)$$

Switching to the covariant form of the thermal propagator, Eq. (10.15), the contribution shown in the left panel of figure 10.2 at second order is

$$\Sigma_2 = -\frac{\lambda}{2} T \sum_n \int \frac{d^3k}{(2\pi)^3} \frac{m_D^2}{(\omega_n^2 + \mathbf{k}^2)^2}. \quad (10.36)$$

While in the terms with $n \neq 0$ the ω_n act as a IR cutoff, we see that for $n = 0$ and thus $\omega_0 = 0$ the integral is proportional to $\int dk/k^2$ and thus IR divergent. If go to higher terms in the expansion and add additional loops to the ‘primary loop’ as shown in the right panel of figure 10.2, then the IR divergence only increases in power.

The solution to this problem is to account for the thermal mass of the scalar particle properly. If we use an effective propagator which includes the Debye mass of the particle,

$$\frac{1}{\omega_n^2 + \mathbf{k}^2} \rightarrow \frac{1}{\omega_n^2 + \mathbf{k}^2 + m_D^2}, \quad (10.37)$$

then the $n = 0$ term of Eq. (10.36) with $\omega_0 = 0$ is given by

$$\Sigma_{2,n=0} = -\frac{\lambda}{2} T \int \frac{d^3k}{(2\pi)^3} \frac{m_D^2}{(\mathbf{k}^2 + m_D^2)^2} = -\frac{\lambda^2}{4} \left(\frac{T^2}{12}\right) \left(\frac{T}{8\pi m_D}\right) \quad (10.38)$$

and thus finite.

Since the Debye mass scales as $m_D \propto \lambda^{1/2}$, the contribution of $\Sigma_{2,n=0}$ in the perturbative expansion is not of order λ^2 but of order $\lambda^{3/2}$. Thus we obtained a term which is non-analytic in the coupling—which can not happen, if we sum a finite number of terms. As explanation, we have to look at the expansion of the effective propagator for small m_D (restricting ourselves again to the $n = 0$ term),

$$\frac{1}{\mathbf{k}^2 + m_D^2} = \frac{1}{\mathbf{k}^2} - \frac{m_D^2}{\mathbf{k}^4} + \frac{m_D^4}{\mathbf{k}^6} + \dots \quad (10.39)$$

Here we can view e.g. the m_D^4/\mathbf{k}^6 term as a ring diagram with three massless propagators \mathbf{k}^{-2} and three factors m_D^2 produced by self-energy insertions. Thus including the thermal mass corresponds to summing up the infinite sum of diagrams shown in figure 10.2.

We can formalise the inclusion of the Debye mass in the originally massless scalar theory as follows: We reorganise perturbation theory by adding a mass term to the free Lagrangian and subtracting it from the the interaction term,

$$\mathcal{L}_0 = \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 \quad (10.40)$$

$$\mathcal{L}_{\text{int}} = \frac{\lambda}{4!} \phi^4 - \frac{1}{2} m^2 \phi^2. \quad (10.41)$$

Her we may set $m^2 = m_D^2$ or keep it as a free parameter to be determined by, for example, that the free energy is independent of this parameter

$$\frac{d\Omega}{dm^2} = 0. \quad (10.42)$$

This reformulation of the perturbative expansion in thermal field theories is called screened or optimised perturbation theory.

Symmetry restoration at high temperature Let us reconsider the Hamiltonian (10.17). We have chosen the sign of λ such that H is bounded from below, and $m^2 > 0$ ensures that the minimal energy is obtained for $\langle \phi \rangle = 0$. If we add in this case the thermal mass,

$$V(\phi, T) = \frac{1}{2} \left(m^2 + \frac{\lambda}{24} T^2 \right) \phi^2 + \frac{\lambda}{4!} \phi^4, \quad (10.43)$$

we increase simply the effective mass of the ϕ particle. Something more drastic happens, if we switch the sign of the vacuum mass term and set $m^2 < 0$. Then for $T = 0$ the minimal

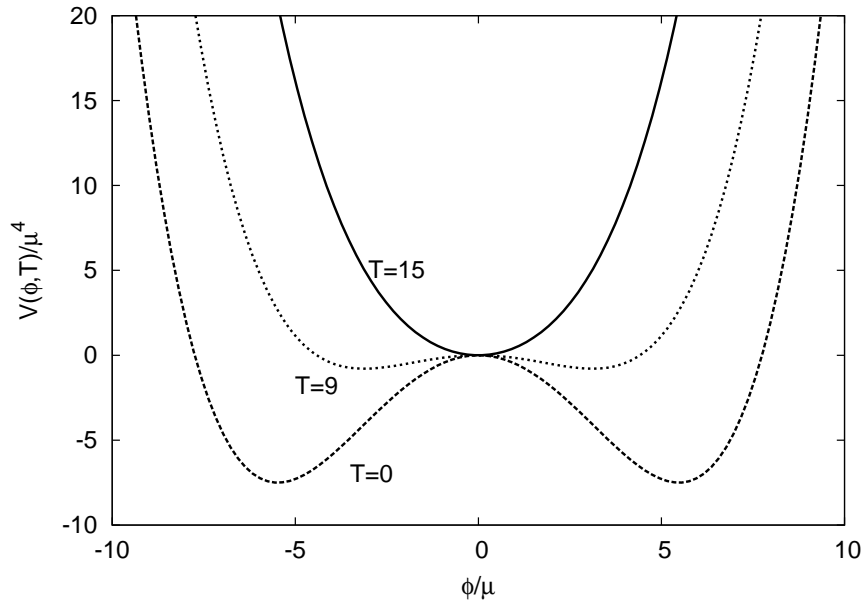


Figure 10.2: Potential with finite temperature with $\mu < 0$ and $T = 0$, $0 < T < T_c$ and $T = T_c$ showing restoration of symmetry.

energy is obtained for $\phi_0 = \pm\sqrt{-6m^2/\lambda}$. Thus we expect that the vacuum is non-empty and filled with non-zero, classical field ϕ , having either the value $\pm\phi_0$. Picking out one of the two breaks the $\phi \rightarrow -\phi$ symmetry of the Hamiltonian. Increasing the temperature, the positive thermal mass grows. Above some critical temperature, the effective mass becomes therefore again positive and the minimal energy configuration becomes $\phi_0 = 0$. This is shown graphically in figure 10.3.

This behaviour resembles the one we know from a ferromagnet: Below some critical temperature, spontaneous magnetisation breaks rotation invariance, which is above T_c restored.