11 Symmetries and symmetry breaking

We have seen in the last chapter that the discrete Z_2 symmetry of our standard $\lambda\phi^4$ Lagrangian could be hidden at low temperatures, if we choose a negative mass term in the zero temperature Lagrangian. Although such a choice seems at first sight unnatural, we will investigate this case in the following in more detail. Our main motivation is the expectation that hiding a symmetry by choosing a non-invariant ground-state retains the "good" properties of the symmetric Lagrangian. Coupling then such a scalar theory to a gauge theory, we hope to break gauge invariance in a "gentle" way which allows e.g. gauge boson masses without spoiling the renormalisability of the unbroken theory.

As additional motivation we remind that couplings and masses are not constants but depend on the scale considered. Thus it might be that the parameters determining the Lagrangian of the Standard Model at low energies originate from a more complete theory at high scales, where the mass parameter μ^2 is originally still positive. In such a scenario, $\mu^2(Q^2)$ may become negative only after running it down to the electroweak scale $Q = m_Z$.

11.1 Symmetry breaking and Goldstone's theorem

Let us start classifying the possible destinies of a symmetry:

- Symmetries may be exact. In the case of local gauge symmetries as U(1) or SU(3) for the electromagnetic and strong interactions, we expect that this holds even in theories beyond the SM. In contrast, there is no good reason why global symmetries of the SM as $B - L$ should be respected by higher-dimensional operators originating from a more complete theory valid at higher energy scales.

- A classical symmetry may be broken by quantum effects. As a result, the corresponding Noether currents are non-zero and the Ward identities of the theory are violated. If the anomalous symmetry is a local gauge symmetry, the theory becomes thereby non-renormalisable. Moreover, we would expect e.g. in case of QED that the universality of the electric charge does not hold exactly.

- The symmetry is explicitly broken by some "small" term in the Lagrangian. An example for such a case is isospin which is broken by the mass difference of the u and d quarks.

- The Lagrangian contains an exact symmetry but the ground-state is not symmetric under the symmetry. In field theory, the ground-state corresponds to the mass spectrum of particles. As a result, the symmetry of the Lagrangian is not visible in the spectrum of physical particles. If the ground-state breaks the original symmetry because one or several scalar fields acquire a non-zero vacuum expectation value, one calls this spontaneous symmetry breaking (SSB). As the symmetry is not really broken on the Lagrangian level, a perhaps more appropriate name would be "hidden symmetry."

In this and the following chapter, we discuss the case of spontaneous symmetry breaking, first in general and then applied to the electroweak sector of the SM. Since the breaking of an internal symmetry should leave Poincaré symmetry intact, we can give only scalar quantities a non-zero vacuum expectation value. This excludes non-zero vacuum expectation values for tensor fields, which would single out a specific direction. On the other hand, we can construct scalars as $\langle 0|\phi|0\rangle = \langle 0|\bar{\psi}\psi|0\rangle \neq 0$ out of the product of multiple fields with spin. In the following, we will always treat ϕ as an elementary field, but we should keep in mind the possibility that ϕ is a composite object, e.g. a condensate of fermion fields, $\langle\phi\rangle = \langle\bar{\psi}\psi\rangle$, similar to the case of superconductivity.

Spontaneous breaking of discrete symmetries We will first consider the simplest example of a theory with a broken symmetry: A single scalar field with a discrete reflection symmetry. Consider the familiar $\lambda\phi^4$ Lagrangian, but with a negative mass term which we include into the potential $V(\phi)$,

$$\mathscr{L} = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4}\phi^4 = \frac{1}{2}(\partial_\mu\phi)^2 - V(\phi)\,. \tag{11.1}$$

The Lagrangian is invariant under the symmetry operation $\phi \to -\phi$ for both signs of μ^2. The field configuration with the smallest energy is a constant field ϕ_0, chosen to minimise the potential

$$V(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4}\phi^4, \tag{11.2}$$

which has the two minima

$$\phi_0 \equiv v = \pm\sqrt{\mu^2/\lambda}\,. \tag{11.3}$$

In quantum mechanics, we learn that the wave-function of the ground state for the potential $V(x) = -\frac{1}{2}\mu^2x^2 + \lambda x^4$ will be a symmetric state, $\psi(x) = \psi(-x)$, since the particle can tunnel through the potential barrier. In field theory, such tunnelling can happen in principle too. However, the tunnelling probability is proportional to the volume L^3, and vanishes in the limit $L \to \infty$: In order to transform $\phi(x) = -v$ into $\phi(x) = +v$ we have to switch an infinite number of oscillators, which clearly costs an infinite amount of energy.

Thus in quantum field theory, the system has to choose between the two vacua $\pm v$ and the symmetry of the Lagrangian is broken in the ground state. Had we used our usual Lagrangian with $\mu^2 > 0$, the vacuum expectation value of the field would have been zero, and the ground state would respect the symmetry.

Quantising the theory with $\mu^2 < 0$ around the usual vacuum, $|0\rangle$ with $\langle 0|\phi|0\rangle = \phi_c = 0$, we find modes behaving as

$$\phi_k \propto \exp(-\mathrm{i}\omega t) = \exp(-\mathrm{i}\sqrt{-\mu^2 + |\boldsymbol{k}|^2}\,t)\,, \tag{11.4}$$

which can grow exponentially for $|\boldsymbol{k}|^2 < \mu^2$. More generally, exponentially growing modes exists, if the potential is concave at the position of ϕ_c, i.e. for

$$m^2_{\text{eff}}(\phi_c) = V''(\phi_c) = -\mu^2 + 3\lambda\phi_c^2 < 0 \tag{11.5}$$

or $|\phi_c| < 2\mu^2/(3\lambda)$.

Clearly, the problem is that we should, as always, expand the field around the ground-state v. This requires that we shift the field as

$$\phi(x) = v + \xi(x)\,, \tag{11.6}$$

splitting it into a classical part $\langle\phi\rangle = v$ and quantum fluctuations $\xi(x)$ on top of it. Then we express the Lagrangian as function of the field ξ,

$$\mathscr{L} = \frac{1}{2}(\partial_\mu\xi)^2 - \frac{1}{2}(2\mu^2)\xi^2 - \mu\sqrt{\lambda}\xi^3 - \frac{\lambda}{4}\xi^4 + \frac{1}{2}\frac{\mu^4}{\lambda}\,. \tag{11.7}$$

In the new variable ξ, the Lagrangian describes a scalar field with *positive* mass $m_\xi = \sqrt{2}\mu > 0$. The original symmetry is no longer apparent: Since we had to select one out of the two possible ground states, a term ξ^3 appeared and the $\phi \to -\phi$ symmetry is broken. The new cubic interaction term rises now the question, if our scalar $\lambda\phi^4$ theory becomes non-renormalisable after SSB: As we have no corresponding counter-term at our disposal, the renormalisation of μ and λ has to cure also the divergences of the $-\mu\sqrt{\lambda}\xi^3$ interaction.

Finally, we note that the contribution $\mu^4/(2\lambda)$ to the energy density of the vacuum is in contrast to the vacuum loop diagrams generated by $Z[0]$ classical and finite. Thus it unlikely that we can use the excuse that "quantum gravity" will solve this problem. Moreover, we see later that symmetries will be restored at high temperatures or at early times in the evolution of the Universe. Even if we take the freedom to shift the vacuum energy density, we have either before or after SSB an unacceptable large contribution to the cosmological constant (problem 11.1).

Spontaneous breaking of continuous symmetries As next step we look at a system with a global continuous symmetry. We discussed already in section 5.1 the case of N real scalar fields described by the Lagrangian

$$\mathscr{L} = \frac{1}{2}\left[(\partial_\mu\boldsymbol{\phi})^2 + \frac{1}{2}\mu^2\boldsymbol{\phi}^2\right] - \frac{\lambda}{4}(\boldsymbol{\phi}^2)^2\,. \tag{11.8}$$

Since $\boldsymbol{\phi} = \{\phi_1, \dots, \phi_N\}$ transforms as a vector under rotations in field space,

$$\phi^i \to R^{ij}\phi^j \tag{11.9}$$

with $R^{ij} \in O(n)$, the Lagrangian is clearly invariant under orthogonal transformations.

Before we consider the general case of arbitrary N, we look at the case $N = 2$ for which the potential is shown in Fig. 11.1. Without loss of generality, we choose the vacuum pointing in the direction of ϕ_1: Thus $v = \langle\phi_1\rangle = \sqrt{\mu^2/\lambda}$ and $\langle\phi_2\rangle = 0$. Shifting the field as in the discrete case gives

$$\mathscr{L} = \frac{1}{2}\frac{\mu^4}{\lambda} + \frac{1}{2}(\partial_\mu\boldsymbol{\xi})^2 - \frac{1}{2}(2\mu^2)\xi_1^2 + \mathscr{L}_{\rm int}\,, \tag{11.10}$$

i.e. the two degrees of freedom of the field $\boldsymbol{\phi}$ split after SSB into one massive and one massless mode.

Since the mass matrix consists of the coefficients of the terms quadratic in the fields, the general procedure for the determination of physical masses is the following: Determine first the minimum of the potential $V(\boldsymbol{\phi})$. Expand then the potential up to quadratic terms,

$$V(\boldsymbol{\phi}) = V(\boldsymbol{\phi}_0) + \frac{1}{2}(\phi - \phi_0)_i(\phi - \phi_0)_j \underbrace{\frac{\partial^2 V}{\partial\phi_i\partial\phi_j}}_{M_{ij}} + \dots \tag{11.11}$$

The term of second derivatives is a symmetric matrix with elements $M_{ij} \geq 0$, because we evaluate it by assumption at the minimum of V. Diagonalising M_{ij} gives as eigenvalues the squared masses of the fields. The corresponding eigenvectors are called the mass eigenstates or physical states. Propagators and Green functions describe the evolution of fields with definite masses and should be therefore build up on these states. If the potential has $n > 0$ flat directions, the vacuum is degenerated and n massless modes appear.

Looking at Fig. 11.1 suggests to use polar instead of Cartesian coordinates in field space. In this way, the rotation symmetry of the potential and the periodicity of the flat direction is reflected in the variables describing the scalar fields. Introducing first the complex field $\phi = (\phi_1 + \mathrm{i}\phi_2)/\sqrt{2}$, the Lagrangian becomes

$$\mathscr{L} = \partial_\mu \phi^\dagger \partial^\mu \phi + \mu^2 \phi^\dagger \phi - \lambda(\phi^\dagger \phi) . \tag{11.12}$$

Next we set

$$\phi(x) = \rho(x) \mathrm{e}^{\mathrm{i}\vartheta(x)} \tag{11.13}$$

and use $\partial_\mu \phi = [\partial_\mu \rho + \mathrm{i}\rho \partial_\mu \vartheta] \mathrm{e}^{\mathrm{i}\vartheta}$ to express the Lagrangian in the new variables,

$$\mathscr{L} = (\partial_\mu \rho)^2 + \rho^2 (\partial_\mu \vartheta)^2 + \mu^2 \rho^2 - \lambda \rho^4 . \tag{11.14}$$

Shifting finally again the fields as $\rho = v + \xi$, we find

$$\begin{aligned} \mathscr{L} = \frac{1}{2}\frac{\mu^4}{\lambda} + \frac{\mu^2}{2\lambda}(\partial\vartheta)^2 + (\partial_\mu \xi)^2 - 2\mu^2\xi^2 - \mu\sqrt{2\lambda}\xi^3 - \lambda\xi^4 \\ + \left[\sqrt{\frac{2\mu^2}{\lambda}}\xi + \xi^2\right](\partial_\mu \vartheta)^2 . \end{aligned} \tag{11.15}$$

The phase ϑ which parametrises the flat direction of the potential $V(\vartheta, \xi)$ remained massless. This mode is called Goldstone (or Nambu-Goldstone) boson and has derivative couplings to the massive field ξ, given by the last term in Eq. (11.15). This is a general result, implying that static Goldstone bosons do not interact. Another general property of Goldstone boson is that they carry the quantum number of the corresponding symmetry generator. Thus they are scalar or pseudo-scalar particles, except in the case where we consider the SSB of a supersymmetry which has fermionic generators.

Let us now discuss briefly the case of general N for the Lagrangian (11.8). The lowest energy configuration is again a constant field. The potential is minimised for any set of fields $\boldsymbol{\phi}_0$ that satisfies

$$\boldsymbol{\phi}_0^2 = \frac{\mu^2}{\lambda}. \tag{11.16}$$

This equation only determines the length of the vector, but not its direction. It is convenient to choose a vacuum such that $\boldsymbol{\phi}_0$ points along one of the components of the field vector. Aligning $\boldsymbol{\phi}_0$ with its Nth component,

$$\boldsymbol{\phi}_0 = \left(0, \ldots, 0, \sqrt{\tfrac{\mu^2}{\lambda}}\right) , \tag{11.17}$$

we now follow the same procedure as in the previous example. First we define a new set of fields, with the Nth field expanded around the vacuum

$$\boldsymbol{\phi}(x) = (\phi^k(x), v + \xi(x)) , \tag{11.18}$$

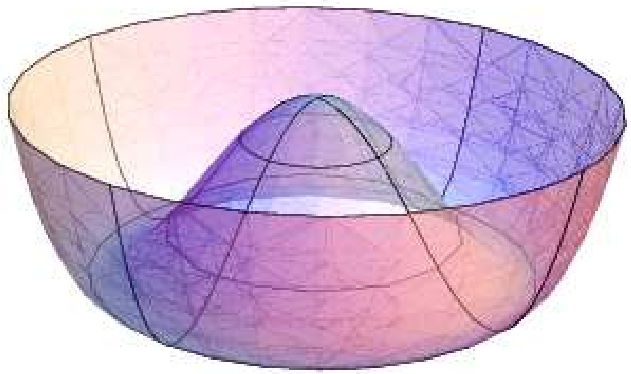

Figure 11.1: Scalar potential symmetric under $O(2)$

where k now runs from 1 to $N-1$. Then we insert this, and the value $v = \sqrt{\mu^2/\lambda}$ for the vacuum expectation value into the Lagrangian, and obtain

$$\begin{aligned}\mathscr{L} &= \frac{1}{2}(\partial_\mu \phi^k)^2 + \frac{1}{2}(\partial_\mu \xi)^2 - \frac{1}{2}(2\mu^2)\xi^2 + \frac{1}{4}\frac{\mu}{\lambda} \\ &- \sqrt{\lambda}\mu\xi^3 - \sqrt{\lambda}\mu\xi(\phi^k)^2 - \frac{\lambda}{2}\xi^2(\phi^k)^2 - \frac{\lambda}{4}\left[(\phi^k)^2\right]^2 - \frac{\lambda}{4}\xi^4 \,.\end{aligned}$$

This Lagrangian describes $N-1$ massless fields and a single massive field ξ, with cubic and quartic interactions. The $O(N)$ symmetry is no longer apparent, leaving as symmetry group the subgroup $O(N-1)$, which rotates the ϕ^k fields among themselves. This rotation describes movements along directions where the potential has a vanishing second derivative, while the massive field corresponds to oscillations in the radial direction of V. This can be visualised for $N = 2$, where we get the "Mexican hat" potential shown in figure 11.1.

Goldstone's theorem The observation that massless particles appear in theories with spontaneously broken *continuous* symmetries is a general result, known as Goldstone's theorem. The first example for such particles was suggested by Nambu in 1960: He showed that a massless quasi-particle appears in a magnetised solid, because the magnetic field breaks rotation invariance. Goldstone applied soon after that this idea to relativistic QFTs and showed that massless scalar elementary particles appear in theories with SSB. Since no massless scalar particles are known to exist, this theorem appeared to be a dead end for the application of SSB to particle physics. So our task is two-fold: First we should derive Goldstone's theorem and then we should find out how we can bypass the theorem applying it our case of interest, gauge theories.

The theorem is obvious at the classical level: Consider a Lagrangian with a symmetry G and a vacuum state invariant under a subgroup H of G. For instance, choosing a Lagrangian invariant under $G = O(3)$ and picking out a vacuum along ϕ_3, the subgroup $H = O(2)$ of rotation around ϕ_3 keeps the vacuum invariant. Let us denote with $U(g)$ a representation of G acting on the fields $\boldsymbol{\phi}$ and with $U(h)$ a representation of H, respectively. Since we consider constant fields, derivative terms in the fields vanish and the potential V alone has to be symmetric under G, i.e.

$$V(U(g)\boldsymbol{\phi}) = V(\boldsymbol{\phi})\,. \tag{11.19}$$

Moreover, we know that the vacuum is kept invariant for all h, $\boldsymbol{\phi}_0' = U(h)\boldsymbol{\phi}_0$, but changes for some g, $\boldsymbol{\phi}_0' \neq U(g)\boldsymbol{\phi}_0$. Using the invariance of the potential and expanding $V(U(g)\boldsymbol{\phi}_0)$ for an infinitesimal group transformation gives

$$V(\boldsymbol{\phi}_0) = V(U(g)\boldsymbol{\phi}_0) = V(\boldsymbol{\phi}_0) + \frac{1}{2}\left.\frac{\partial^2 V}{\partial\phi_i\partial\phi_j}\right|_0 \delta\phi_i\delta\phi_j + \dots\,, \tag{11.20}$$

where $\delta\phi_i$ denotes the resulting variation of the field. Equation (11.20) implies that

$$M_{ij}\delta\phi_i\delta\phi_j = 0\,. \tag{11.21}$$

The variation $\delta\phi_i$ depends on whether the transformation belong to $U(h)$ or not: In the former case, the vacuum $\boldsymbol{\phi}_0$ is unchanged, $\delta\phi_i = 0$ and (11.21) is automatically satisfied. If on the other hand g does not belong to H, i.e. is a member of the left coset G/H, then $\delta\phi_i \neq 0$, implying that the mass matrix M_{ij} has a zero eigenvalue. It is now clear that the number of massless particles is simply determined by the dimensions of the two groups G and H: The number of Goldstone bosons is equal to the dimension of the left coset G/H, or the number of symmetries spontaneously broken.

Quantum case The previous discussion was based on the classical potential. Thus we should address the question if this picture survives quantum corrections.

Noether's theorem tells us that every continuous symmetry has associated to its generators g_i conserved charges Q_i. On the quantum level this means the operators Q_i commute with the Hamiltonian, $[H, Q_i] = 0$. Subtracting the cosmological constant, we have $H\,|0\rangle = 0$. If the vacuum is invariant under the symmetry Q, then $\exp(\mathrm{i}\vartheta Q)\,|0\rangle = |0\rangle$. For the infinitesimal form of the symmetry transformation, $\exp(\mathrm{i}\vartheta Q) \approx 1 + \mathrm{i}\vartheta Q$, and we conclude that the charge annihilates the vacuum,

$$Q\,|0\rangle = 0\,. \tag{11.22}$$

Or, in simpler words, the vacuum has the charge 0.

Now we came to the case we are interested in, namely that the symmetry is spontaneously broken and thus $Q\,|0\rangle \neq 0$. We first determine the energy of the state $Q\,|0\rangle$. From

$$HQ\,|0\rangle = (HQ - \underbrace{QH)\,|0\rangle}_{H|0\rangle=0} = [H, Q]\,|0\rangle \;\; = 0\,, \tag{11.23}$$

we see that at least another state $Q\,|0\rangle$ exists which has as the vacuum $|0\rangle$ zero energy.

We represent the charge operator as the volume integral of the time-like component of the corresponding current operator,

$$Q = \int \mathrm{d}^3x J^0(t, \boldsymbol{x})\,. \tag{11.24}$$

The state

$$|s\rangle = \int \mathrm{d}^3x\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} J^0(t, \boldsymbol{x})\,|0\rangle \to Q\,|0\rangle \qquad \text{for} \quad k \to 0 \tag{11.25}$$

becomes in the zero-momentum limit equal to the state $Q\,|0\rangle$ we are searching for. Moreover, applying $\boldsymbol{P}$ on $|s\rangle$ gives (problem 11.5)

$$\boldsymbol{P}\,|s\rangle = \boldsymbol{p}\,|s\rangle\ . \tag{11.26}$$

Thus the SSB of the vacuum, $Q\,|0\rangle \neq 0$, implies excitations of the system with a frequency that vanishes in the limit of long wavelengths. In the relativistic case, Goldstone's theorem predicts massless states, while in the non-relativistic case relevant for solid states the theorem predicts collective excitations with zero energy gap.

11.2 Renormalisation of theories with SSB

When we went through the SSB of the scalar field, we saw that new ϕ^3 interactions were introduced. The question then arises, are new renormalisation constants needed when a symmetry is spontaneously broken? This would make these theories non-renormalisable.

We can address this questions in two ways. One possibility is to repeat our analysis of the renormalisability of the scalar theory in section 9.3.2, but now for the broken case with $\mu^2 < 0$. Then we would find that the ϕ^3 term becomes finite, renormalising fields, mass and coupling as in the unbroken case. This is not unexpected, because shifting the field $\phi \to \tilde{\phi} = \phi - v$, which is an integration variable in the generating functional, should not affect physics. On the other hand, such a shift reshuffles the splitting $\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_{\rm int}$ in our standard perturbative expansion in the coupling constant. To avoid this problem, we analyse SSB in the following in a different way: We develop as a new tool the loop expansion which is based on the effective action formalism. Additionally of being not affected by a shift of the fields, this formalism allows us to calculate the potential including all quantum corrections in the limit of constant fields.

First we recall the definition of the classical field,

$$\phi_c(x) = \frac{\delta W}{\delta J(x)} = \frac{1}{Z}\int \mathcal{D}\phi\phi(x)\exp\{\mathrm{i}(S + \int \mathrm{d}^4x' J(x')\phi(x')\} \tag{11.27}$$

and of the effective action[1]

$$\Gamma[\phi_c] = W[J] - \int \mathrm{d}^4x' J(x')\phi_c(x')\} \equiv W[J] - \langle J\phi\rangle\,, \tag{11.28}$$

which lead to the converse relation for J,

$$J(x) = -\frac{\delta\Gamma[\phi_c]}{\delta\phi(x)}\,. \tag{11.29}$$

Effective potential In general we will not be able to solve the effective action. Studying SSB, we can however make use of a considerable simplification: The fields we are interested in are constant, and it should be therefore useful to perform a gradient expansion of the effective action $\Gamma[\phi]$,

$$\Gamma[\phi] = \int \mathrm{d}^4x \left[-V_{\rm eff}(\phi) + \frac{1}{2}Z(\phi)(\partial_\mu\phi)^2 + \ldots\right]. \tag{11.30}$$

[1] We will suppress the subscript c on the classical field from now on and use brackets $\langle J\phi\rangle$ to indicate integration.

Here, we introduced also the effective potential $V_{\text{eff}}(\phi)$ as the zeroth order term of the expansion in $(\partial_\mu\phi)^2$, i.e. the only term surviving for constant fields.

If we now choose the source $J(x)$ to be constant, the field $\phi(x)$ has to be uniform too, $\phi(x) = \phi$, by translation invariance. Together this implies that

$$\Gamma = -\Omega V_{\text{eff}}\,, \tag{11.31a}$$
$$-J = \frac{\delta\Gamma[\phi]}{\delta\phi} = -\Omega\,\frac{\partial V_{\text{eff}}(\phi)}{\partial\phi}\,, \tag{11.31b}$$

where Ω is the space-time volume. Hence, as announced, we only have to calculate the effective potential, not the full effective action. In the absence of external sources, $J = 0$, Eq. (11.31b) simplifies to $V'_{\text{eff}}(\phi) = 0$. Thus this is the quantum version of our old approach where we minimised the classical potential $V(\phi)$ in order to find the vacuum expectation value of ϕ. Note that Eq. (11.31b) contains all quantum corrections to the classical potential, the only approximation made so far neglecting gradients of the classical field.

In order to proceed, we use that we know the classical potential and we assume that quantum fluctuations are small. Then we can perform a saddle-point expansion around the classical solution ϕ_0, given by the solution to

$$\left.\frac{\delta\{S[\phi] + \langle J\phi\rangle\}}{\delta\phi}\right|_{\phi_0} = 0 \tag{11.32}$$

or,

$$\Box\phi_0 + V'(\phi_0) = J(x)\,. \tag{11.33}$$

We write the field as $\phi = \phi_0 + \tilde{\phi}$, i.e. as a classical solution with quantum fluctuations on top. Then we can approximate the path integral by

$$Z = \exp\{\frac{\mathrm{i}}{\hbar}W\} \approx \exp\{\frac{\mathrm{i}}{\hbar}\big[S[\phi_0] + \langle J\phi_0\rangle\big]\}\int\mathcal{D}\tilde{\phi}\exp\left\{\frac{\mathrm{i}}{\hbar}\int\mathrm{d}^4x\frac{1}{2}\left[(\partial_\mu\tilde{\phi})^2 - V''(\phi_0)\tilde{\phi}^2\right]\right\}, \tag{11.34}$$

where V'' is the second derivative of the potential term of the theory. Planck's constant $\hbar$ has been restored to indicate that what we are doing here is an expansion in $\hbar$, or a loop expansion, which we will show later. The functional integral over $\tilde{\phi}$ is quadratic and can be formally solved directly, it is equal to $(\det(\Box + V'')^{-\frac{1}{2}}$. Using the identity $\ln\det A = \operatorname{tr}\ln A$, we find

$$W = S[\phi_0] + \langle J\phi_0\rangle + \frac{\mathrm{i}\hbar}{2}\operatorname{tr}\ln[\Box + V''(\phi_0)] + \mathcal{O}(\hbar^2)\,. \tag{11.35}$$

To evaluate the operator trace, we write out the definition and insert two complete set of plane waves,

$$\begin{aligned}\operatorname{tr}\ln[\Box + V''] &= \int\mathrm{d}^4x\langle x|\ln[\Box + V'']|x\rangle \\ &= \int\mathrm{d}^4x\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{d}^4q}{(2\pi)^4}\langle x|k\rangle\langle k|\ln[\Box + V'']|q\rangle\langle q|x\rangle \\ &= \int\mathrm{d}^4x\frac{\mathrm{d}^4k}{(2\pi)^4}\ln[-k^2 + V'']\,.\end{aligned}$$

Performing the Legendre transform and putting everything together, we obtain for the effective potential including the first quantum corrections

$$V_{\text{eff}}(\phi) = V(\phi) - \frac{\mathrm{i}\hbar}{2} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \ln\left[k^2 - V''(\phi)\right] + \mathcal{O}(\hbar^2)\,. \tag{11.36}$$

As an example we can use the $\lambda\phi^4$ theory, with

$$V''(\phi) = \mu^2 + \frac{1}{2}\lambda\phi^2\,. \tag{11.37}$$

The second term can be interpreted as an effective mass contribution due to the constant background field ϕ.

Not surprisingly, the effective potential is divergent and we have to introduce counter-terms that can eliminate the divergent parts. Our effective potential is then

$$V_{\text{eff}}(\phi) = V(\phi) + \frac{\hbar}{2} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \ln\left(\frac{k_E^2 + V''(\phi)}{k_E^2}\right) + B\phi^2 + C\phi^4 + \mathcal{O}(\hbar^2)\,. \tag{11.38}$$

Here, we Wick rotated the integral to Euclidean space and subtracted an infinite constant in order to make the logarithm dimensionless. (Equivalently we could have added an additional constant counter-term A renormalising the vacuum energy density.) The integral can be solved in different regularisation schemes, here we will expand the logarithm,

$$\ln\left(1 + \frac{V''}{k_E^2}\right) = \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{V''}{k_E^2}\right)^n\,, \tag{11.39}$$

and cutoff the integral at some large momenta Λ. The first two terms of the sum will depend on the cutoff, being proportional to Λ^2 and $\ln(\Lambda^2/V'')$, respectively. Performing the integral and throwing away terms that vanishes for large Λ, we obtain

$$V_{\text{eff}}(\phi) = V(\phi) + \frac{\Lambda^2}{32\pi^2} V''(\phi) + \frac{V''(\phi)^2}{64\phi^2} \ln\left(\frac{V''(\phi)}{\Lambda^2}\right)\,. \tag{11.40}$$

Now we see that if we start out with a massless $\lambda\phi^4$, our cutoff-dependent terms are

$$V'' = \frac{1}{2}\lambda\phi^2, \quad \text{and} \quad (V'')^2 = \frac{\lambda^2}{4}\phi^4, \tag{11.41}$$

which both can be absorbed into the counter-terms B and C by imposing appropriate renormalisation conditions.

Let us stress the important point in this result: The renormalisation of the $\lambda\phi^4$ theory using the effective potential approach is not affected at all by a shift of the field: We are free to use both signs of μ^2 and any value of the classical field ϕ in Eq. (11.40). Independently of the sign of μ^2, we need only symmetric counter-terms, as a cubic term does not appear at all.

We can rephrase this point as follows: If we renormalise before we shift the fields, we know that we obtain finite renormalised Green functions. But shifting the fields does not change the total Lagrangian. Thus the effective action and the effective potential are unchanged too. Consequently the theory has to stay renormalisable after SSB.

Let us now discuss what happens with a non-renormalisable theory in the effective potential approach. Including e.g. a ϕ^6 term leads to $(V'')^2 \propto \phi^8$ which requires an additional counter-term $D\phi^8$, generating in turn even higher order terms and so forth. Thus in this case an

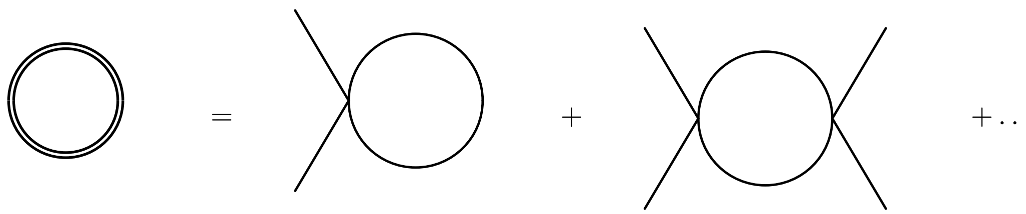

Figure 11.2: Perturbative expansion of the one-loop effective potential $V_{\text{eff}}^{(1)}$ for the $\lambda\phi^4$ theory; all external legs have zero momentum.

infinite number of counter-terms is needed for the calculation of $V_{\text{eff}}^{(1)}$. How does this finding goes together with our statement that non-renormalisable theories are predictive below a certain cutoff scale Λ? The reason for this apparent contradiction becomes clear, if we look again at the series expansion of the logarithm in the one loop contribution $V_{\text{eff}}^{(1)}$,

$$V_{\text{eff}}^{(1)} = \mathrm{i}\sum_{n=1}^{\infty} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{2n} \left[\frac{V''(\phi)}{k^2}\right]^n . \tag{11.42}$$

This contribution is an infinite sum of single loops with progressively more pairs of external legs with zero-momentum attached, see Fig. 11.2 for the case of $V''(\phi) = \frac{1}{2}\lambda\phi^2$. (The factor i appeared, because we are back in Minkowski space; the symmetry factor $2n$ appearing automatically in this approach accounts for the symmetry of a graph with n vertices under rotations and reflection.) As we saw, the superficial degree of divergence increases with the number of external particles for a $\lambda\phi^n$ theory and $n > 4$. Hence every single diagram in the infinite sum contained in $V_{\text{eff}}^{(1)}$ diverges and requires a counter-term of higher order in ϕ^n. Therefore the effective potential approach is not useful for non-renormalisable theories.

Expansion in $\hbar$ as a loop expansion To show that the expansion in $\hbar$ is really a loop expansion, we introduce artificially a parameter a into our Lagrangian so that

$$\mathscr{L}(\phi, \partial_\mu\phi, a) = a^{-1}\mathscr{L}(\phi, \partial_\mu\phi) . \tag{11.43}$$

Now we want to determine the power P of a in an arbitrary Feynman diagram, a^P: A propagator is the *inverse* of the quadratic form in $\mathscr{L}$ and contributes thus a positive power a, while each vertex $\propto \mathscr{L}_{\text{int}}$ adds a factor a^{-1}. The number of loops in an 1PI diagram is given by $L = I - V + 1$, cf. Eq. (9.45), where I is the number of internal lines and V is the number of vertices. Putting this together we see that

$$P = I - V = L - 1 \tag{11.44}$$

and it is clear that the power of a gives us the number of loops.

We should stress that using a loop expansion does not imply a semi-classical limit, $S \gg \hbar$: Our fictitious parameter a is not small; in fact, it is one. The loop expansion is not affected by a shift in the fields, since a multiplies the whole Lagrangian. Thus this procedure is particularly useful discussing the renormalisation of SSB.

Effective action as generating functional for 1PI Green functions We have now all the necessary tools in order to show that the tree-level graphs generated by the effective action $\Gamma[\phi]$ correspond to the complete scattering amplitudes of the corresponding action $S[\phi]$. We compare our familiar generating functional

$$Z[J] = \int \mathcal{D}\phi \, \exp\{\mathrm{i}S + \langle J\phi\rangle\} = \mathrm{e}^{\mathrm{i}W[J]}\,, \tag{11.45}$$

with the functional $V_a[J]$ of a fictitious field theory whose action S is the effective action $\Gamma[\phi]$ of the theory (11.45) we are interested in,

$$V_a[J] = \int \mathcal{D}\phi \, \exp\left\{\frac{\mathrm{i}}{a}\{\Gamma[\phi] + \langle J\phi\rangle\}\right\} = \mathrm{e}^{\mathrm{i}U_a[J]}\,. \tag{11.46}$$

Additionally, we introduced the parameter a with the same purpose as in (11.43): In the limit $a \to 0$, we can perform a saddle-point expansion and the path integral is dominated by the classical path. From (11.34), we find thus

$$\lim_{a\to 0} aU_a[J] = \Gamma[\phi] + \langle J\phi\rangle = W[J]\,, \tag{11.47}$$

where we used the definition of the effective action, Eq. (11.28), in the last step. The RHS is the sum of all connected Green functions of our original theory. The LHS is the classical limit of the fictitious theory $V_a[J]$, i.e. it is the sum of all connected tree graphs of this theory. Equation 9.146 shows the vertices of this theory are given by $\Gamma^{(n)}(x_1, ..., x_n)$, i.e. the 1PI Green functions of our original theory. Thus we can represent the connected graphs of W as tree graphs whose effective vertices are the sum of all 1PI graphs with the appropriate number of external lines.

Another proof of the Goldstone theorem With the help of the effective potential we can give another simple proof of the Goldstone theorem. We know that the zero of the inverse propagator determines the mass of a particle. From Eq. (9.147), the exact inverse propagator in momentum space for a set of scalar fields is given by

$$\boldsymbol{\Delta}_{ij}^{-1}(p^2) = \int \mathrm{d}^4x \, \mathrm{e}^{\mathrm{i}p(x-x')} \frac{\delta^2\Gamma}{\delta\phi_i(x)\delta\phi_j(x')}\,. \tag{11.48}$$

Massless particles correspond to zero eigenvalues of this matrix equation for $p^2 = m^2$. If we set $\boldsymbol{p} = 0$, the fields are constant. But differentiating the effective action w.r.t. to constant fields is equivalent to differentiating simply the effective potential,

$$\frac{\partial^2 V_{\text{eff}}}{\partial\phi_i(x)\partial\phi_j(x')} = 0\,. \tag{11.49}$$

The effective potential has the same symmetry properties as the classical potential, but accounts for all quantum effects. Thus our previous analysis of Goldstone's theorem using the classical potential is not modified by quantum corrections.

Coleman-Weinberg Problem Sidney Coleman and Erick Weinberg [CW73] used this formalism to investigate if quantum fluctuations could trigger SSB in an initially massless theory. Rewriting the effective potential a bit we have

$$V_{\text{eff}}(\phi) = \left[\frac{\Lambda^2}{64\pi^2}\lambda + B\right]\phi^2 + \left[\frac{\lambda}{4!} + \frac{\lambda^2}{(16\pi)^2}\ln\frac{\phi^2}{\Lambda^2} + C\right]\phi^4 \,. \tag{11.50}$$

Now we impose the renormalising conditions, first

$$\left.\frac{\mathrm{d}^2 V_{\text{eff}}}{\mathrm{d}\phi^2}\right|_{\phi=0} = 0\,, \tag{11.51}$$

which implies that

$$B = -\frac{\lambda\Lambda^2}{64\pi^2}\,. \tag{11.52}$$

When renormalising the coupling constant, we have to pick a different point than $\phi = 0$, due to the logarithm being ill-defined there. This means that we have to introduce a scale μ. Taking the fourth derivative and ignoring terms that are independent of ϕ, we find

$$\left.\frac{\mathrm{d}^4 V_{\text{eff}}}{\mathrm{d}\phi^4}\right|_{\phi=\mu} = \lambda = 24\frac{\lambda^2}{(16\pi)^2}\ln\frac{\mu^2}{\Lambda^2}\,. \tag{11.53}$$

We can convince ourselves that this expression gives the correct beta function,

$$\beta(\mu) = \mu\frac{\partial\lambda}{\partial\mu} = \frac{3}{16\pi^2}\lambda^2 + \mathcal{O}(\lambda^3)\,. \tag{11.54}$$

Using the complete expression for Eq. (11.53), we can determine C and obtain for the renormalised effective potential (problem 11.6)

$$V_{\text{eff}}(\phi) = \frac{\lambda(\mu)}{4!}\phi^4 + \frac{\lambda^2(\mu)}{(16\pi)^2}\phi^4\left[\ln\frac{\phi^2}{\mu^2} - \frac{25}{6}\right] + \mathcal{O}(\lambda^3)\,. \tag{11.55}$$

This potential has two minima outside of the origin, so it seems that SSB does indeed happen. These minima lie however outside the expected range of validity of the one loop approximation: Rewriting the potential as $V_{\text{eff}}(\phi) = \lambda\phi^4/4!(1 + a\lambda\ln(\phi^2/\mu^2) + \ldots$ suggest that we can trust the one-loop approximation only as long as $(3/32\pi^2)\lambda\ln(\phi^2/\mu^2) \ll 1$.

11.3 Abelian Higgs model

After we have shown that the renormalisability is not affected by SSB, we now try to apply this idea to a the case of a gauge symmetry. First of all, because we aim to explain the masses of the W and Z bosons as consequence of SSB. Secondly, we saw that SSB of global symmetries leads to massless scalars which are however not observed. As SSB cannot change the number of physical degrees of freedom, we hope that each of the two diseases is the cure of the other: The Goldstone bosons which would remain massless in a global symmetry disappear becoming the required additional longitudinal degrees of freedom of massive gauge bosons in case of the SSB of a gauged symmetry.

The Abelian Higgs model, which is the simplest example for this mechanism, is obtained by gauging a complex scalar field theory. Introducing in the Lagrangian (11.12) the covariant derivative

$$\partial_\mu \to D_\mu = \partial_\mu - \mathrm{i}eA_\mu \tag{11.56}$$

and adding the free Lagrangian of an U(1) gauge field gives

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^\dagger(D^\mu\phi) + \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2\,. \tag{11.57}$$

The symmetry breaking and Higgs mechanism is best discussed changing to polar coordinates in field-space, $\phi = \rho\exp\{\mathrm{i}\vartheta\}$. Then we insert

$$D_\mu\rho = \left[\partial_\mu\rho + \mathrm{i}\rho(\partial_\mu\vartheta - eA_\mu)\right]e^{\mathrm{i}\vartheta} \tag{11.58}$$

into the Lagrangian, obtaining

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \rho^2(\partial_\mu\vartheta - eA_\mu)^2 + (\partial_\mu\rho)^2 + \mu^2\rho - \lambda\rho^4\,. \tag{11.59}$$

The only difference to the ungauged model is the appearance of the gauge field in the prospective mass term $\rho^2(\partial_\mu\vartheta - eA_\mu)^2$. This allows us to eliminate the angular mode ϑ which shows up nowhere else by performing a gauge transformation on the field A_μ: The action of a $U(1)$ gauge transformation on the original field ϕ is just a phase shift, hence ρ is unchanged and ϑ is shifted by a constant. This means that if we do the transformation

$$A_\mu \to B_\mu = A_\mu - \frac{1}{e}\partial_\mu\vartheta\,,$$

we eliminate ϑ completely, as $F_{\mu\nu}$ is gauge invariant,

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + e^2\rho^2(B_\mu)^2 + (\partial_\mu\rho)^2 + \mu^2\rho - \lambda\rho^4\,. \tag{11.60}$$

It is now evident that the Goldstone mode ϑ has disappeared, while the new gauge field B_μ obtained a mass and interaction term $e\rho$. Eliminating the field ρ in favour of fluctuations ξ around the vacuum $v = \sqrt{\mu^2/\lambda}$, i.e. shifting as usually the field as

$$\rho = \frac{1}{\sqrt{2}}(v + \chi)\,, \tag{11.61}$$

we obtain after some algebra as new Lagrangian

$$\begin{aligned}\mathscr{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2(B_\mu)^2 + e^2v\chi(B_\mu)^2 + \frac{1}{2}e^2\chi^2(B_\mu)^2 \\ &+ \frac{\mu^2}{4\lambda^2} + \frac{1}{2}(\partial_\mu\chi)^2 - \frac{1}{2}(\sqrt{2}\mu)^2\chi^2 - \sqrt{\lambda}\mu\chi^3 - \frac{\lambda}{4}\chi^4\,.\end{aligned} \tag{11.62}$$

As in the ungauged model we obtain a χ^3 self-interaction and a contribution to the vacuum energy density. But the gauge field B_μ acquired the mass $M = ev$, therefore having now three spin degrees of freedom. The additional longitudinal one has been delivered by the Goldstone boson which in turn disappeared: The gauge field has eaten the Goldstone boson, so to speak. We also see that the number of degrees of freedom before SSB $(2+2)$ matches the number afterwards (3+1). The phenomenon that breaking spontaneously a gauge symmetry does not lead to massless Goldstone bosons because they become the longitudinal degree of freedom of massive gauge bosons is called the Higgs effect.

The gauge transformation we used to eliminate the ϑ field corresponds to the Higgs model in the unitary gauge, where only physical particles appear in the Lagrangian. The massive gauge boson is decribed by the Procca Lagrangian and we know that the resulting propagator becomes constant for large momenta. Hence, this gauge is convenient for illustrating the concept of the Higgs mechanism, but not suited for loop calculations.

A different way to consider the model is to keep the Cartesian fields $\phi = \frac{1}{\sqrt{2}}(\phi_1 + \mathrm{i}\phi_2)$. Then the Lagrangian is

$$\begin{aligned}\mathscr{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left[(\partial_\mu\phi_1 + eA_\mu\phi_2)^2 + (\partial_\mu\phi_2 - eA_\mu\phi_1)^2\right] \\ &+ \mu^2(\phi_1^2 + \phi_2^2) - \lambda(\phi_1^2 + \phi_2^2)^2 .\end{aligned} \tag{11.63}$$

Performing the shift due to the SSB, $\phi_1 = v + \tilde{\phi}_1$ and $\phi_2 = \tilde{\phi}_2$, the Lagrangian becomes

$$\begin{aligned}\mathscr{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2(A_\mu)^2 - evA^\mu\partial_\mu\tilde{\phi}_2 \\ &+ \frac{1}{2}\left[(\partial_\mu\tilde{\phi}_1)^2 - 2\mu^2\tilde{\phi}_1^2\right] + \frac{1}{2}(\partial_\mu\tilde{\phi}_2)^2 + \dots ,\end{aligned} \tag{11.64}$$

where we have omitted interaction and vacuum terms not relevant to the discussion. As we see, the Goldstone boson $\tilde{\phi}_2$ does not disappear and it couples to the gauge field A_μ. On the other hand, the mass spectrum of the physical particles is the same as in the unitary gauge. The degrees of freedom before and after breaking the symmetry do not match, hence there is an unphysical degree of freedom in the theory, namely that corresponding to $\tilde{\phi}_2$.

Gauge fixing and gauge boson propagator In order to make the generating functional $Z[J^\mu, J, J^*]$ of the abelian Higgs model well-defined, we have to remove the gauge freedom of the classical Lagrangian. Using the Faddeev-Popov trick to achive this implies to add a gauge-fixing and a Faddeev-Popov ghost term to the classical Lagrangian,

$$\mathscr{L}_{\text{eff}} = \mathscr{L}_{\text{cl}} + \mathscr{L}_{\text{gf}} + \mathscr{L}_{\text{FP}} = \mathscr{L}_{\text{cl}} - \frac{1}{2\xi}G^2 - \bar{c}\frac{\partial G}{\partial\vartheta}c \,. \tag{11.65}$$

Here $G = 0$ is a suitable gauge condition, ϑ are the generators of the gauge symmetry and $c, \bar{c}$ are grassmannian ghost fields.

In the unbroken abelian case we used as gauge condition $G = \partial_\mu A^\mu$. With the gauge transformation $A^\mu \to A^\mu - \partial^\mu\vartheta$ the ghost term becomes simply $\mathscr{L}_{\text{FP}} = \bar{c}(-\Box)c$. Thus the ghost fields completely decouple from any physical particles, and the ghost term can be absorbed in the normalisation.

In the present case of a theory with SSB, we want to use the Fadeev-Popov term to cancel the mixed $A^\mu\partial_\mu\phi_2$ term. Therefore we include the Goldstone boson ϕ_2 in the gauge condition,

$$G = \partial_\mu A^\mu + \xi ev\phi_2 = 0 \,. \tag{11.66}$$

From

$$\phi_2 = -\frac{\partial_\mu A^\mu}{\xi ev}, \tag{11.67}$$

we see that the unitary gauge corresponds to $\xi \to \infty$. We calculate first G^2,

$$G^2 = (\partial_\mu A^\mu)(\partial_\nu A^\nu) + 2\xi ev\phi_2\partial_\mu A^\mu + \xi^2 e^2 v^2 \phi_2^2\,, \tag{11.68}$$

integrate partially the cross term and insert the result into $\mathscr{L}_{\text{gf}}$,

$$\mathscr{L}_{\text{gf}} - \frac{1}{2\xi}G^2 = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2 + evA_\mu\partial^\mu\phi_2 - \frac{1}{2}\xi(ev)^2\phi_2^2\,. \tag{11.69}$$

Now we see that the second term cancels the unwanted mixed term in $\mathscr{L}_{\text{cl}}$, while a ξ dependent mass term ξM^2 for ϕ_2 appeared.

If we write out the terms in $\mathscr{L}_{\text{eff}}$ quadratic in A_μ and ϕ_2,

$$\mathscr{L}_{\text{eff},2} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2A_\mu^2 - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \frac{1}{2}(\partial_\mu\phi_2^\mu)^2 - \frac{1}{2}\xi M^2\phi_2^2\,, \tag{11.70}$$

we can find the boson propagator. Using the antisymmetry of $F_{\mu\nu}$ and a partial integration, we transform F^2 into standard form,

$$\begin{aligned}
-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} &= -\frac{1}{2}\big(\partial_\mu A^\nu\partial^\mu A_\nu - \partial^\nu A_\mu\partial^\mu A_\nu\big)\\
&= \frac{1}{2}\big(A^\nu\partial_\mu\partial^\mu A_\nu - A_\mu\partial^\mu\partial^\nu A_\mu\big)\\
&= \frac{1}{2}A_\mu(g^{\mu\nu}\Box - \partial^\mu\partial^\nu)A_\nu\,.
\end{aligned}$$

The part of the Lagrangian quadratic in A_μ then reads

$$\begin{aligned}
\mathscr{L}_A &= \frac{1}{2}A_\mu\big[g^{\mu\nu}\Box - \partial^\mu\partial^\nu\big]A_\nu + \frac{1}{2}A_\mu g^{\mu\nu}M^2A_\nu + \frac{1}{2\xi}A_\mu\partial^\mu\partial^\nu A_\nu\\
&= \frac{1}{2}A_\mu\big[g^{\mu\nu}(\Box + M^2) - (1-\xi^{-1})\partial^\mu\partial^\nu\big]A_\nu\,.
\end{aligned}$$

To find the propagator we want to invert the term in the bracket, denote this term by $P^{\mu\nu}(k)$. If we go to momentum space, then

$$P^{\mu\nu} = -(k^2 - M^2)g^{\mu\nu} + (1-\xi^{-1})k^\mu k^\nu\,. \tag{11.71}$$

This can be split into a transverse and a longitudinal part by factoring out terms proportional to $P_T^{\mu\nu} = g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}$,

$$\begin{aligned}
P^{\mu\nu} &= -(k^2 - M^2)\big(P_T^{\mu\nu} + \frac{k^\mu k^\nu}{k^2}\big) + (1-\xi^{-1})k^\mu k^\nu\\
&= -(k^2 - M^2)P_T^{\mu\nu} - \left(\frac{k^2 - M^2}{k^2} - 1 + \xi^{-1}\right)k^\mu k^\nu\\
&= -(k^2 - M^2)P_T^{\mu\nu} - \xi^{-1}(k^2 - \xi M^2)P_L^{\mu\nu}\,,
\end{aligned} \tag{11.72}$$

where the longitudinal part is $P_L^{\mu\nu} = \frac{k^\mu k^\nu}{k^2}$. Since $P_T^{\mu\nu}$ and $P_L^{\mu\nu}$ as projection operators are orthogonal to each other, we can invert the two parts separately and obtain

$$\begin{aligned}
\mathrm{i}D_F^{\mu\nu}(k^2) &= \frac{-\mathrm{i}P_T^{\mu\nu}}{k^2 - M^2 + \mathrm{i}\varepsilon} + \frac{-\mathrm{i}\xi P_L^{\mu\nu}}{k^2 - \xi M^2 + \mathrm{i}\varepsilon}\\
&= \frac{-\mathrm{i}}{k^2 - M^2 + \mathrm{i}\varepsilon}\left[g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2 - \xi M^2 + \mathrm{i}\varepsilon}\right].
\end{aligned} \tag{11.73}$$

As we see, the transverse part propagates with mass M^2, while the longitudinal part propagates with mass ξM^2. $\xi \to \infty$ corresponds again to the unitary gauge and $\xi = 1$ corresponds to the easier Feynman-'t Hooft gauge. For finite ξ we see that the propagator is proportional to k^{-2} and no problems arise in loop calculations, as it did in the unitary gauge.

The Goldstone boson ϕ_2 has the usual propagator of a scalar particle, however with gauge-dependent mass ξM^2.

Ghosts Using the Faddeev-Popov ansatz for the gauge introduces ghosts field through the term

$$\mathscr{L}_{\text{FP}} = -\bar{c}\,\frac{\delta G}{\delta\vartheta}\,c \tag{11.74}$$

into the Lagrangian. To calculate $\delta G/\delta\vartheta$, we have to find out how the gauge fixing condition G changes under an infinitesimal gauge transformation. Looking first at the change of the complex field,

$$\phi \to \tilde{\phi} = \phi - \mathrm{i}e\vartheta\phi = \phi - \mathrm{i}e\vartheta\frac{1}{\sqrt{2}}(v + \phi_1 + \mathrm{i}\phi_2)\,, \tag{11.75}$$

we see that the fields ϕ_1 and ϕ_2 are mixed under the gauge transformation.

$$A_\mu \to \tilde{A}_\mu = A_\mu + \partial_\mu\vartheta \tag{11.76}$$

$$\phi_1 \to \tilde{\phi}_1 = \phi_1 + e\vartheta\phi_2 \tag{11.77}$$

$$\phi_2 \to \tilde{\phi}_2 = \phi_2 - e\vartheta(v + \phi_1)\,. \tag{11.78}$$

Inserting this into the gauge fixing condition (11.66) and differentiating with respect to the generator, we obtain

$$\frac{\delta G}{\delta\vartheta} = \frac{\delta}{\delta\vartheta}\left(\partial_\mu\tilde{A}^\mu - \xi ev\tilde{\phi}_2\right) = \Box + \xi e^2 v(v + \phi_1)\,. \tag{11.79}$$

That is, with spontaneous symmetry breaking the ghost particles will get a ξ-dependent mass and they interact with the Higgs field ϕ_1. To see this explicitly we insert $\delta G/\delta\vartheta$ into the ghost Lagrangian,

$$\begin{aligned}\mathscr{L}_{\text{FP}} &= -\bar{c}\left[\Box + \xi e^2 v(v + \phi_1)\right] c\\ &= (\partial^\mu\bar{c})(\partial_\mu c) - \xi M^2\bar{c}c - \xi e^2 v\phi_1\bar{c}c\,.\end{aligned}$$

The second term corresponds to the mass $\xi e^2 v^2 = \xi M$ for the ghost field, the third one describes ghost-ghost-Higgs interaction.

To sum this up, we have the following propagators in the R_ξ gauge, where we follow common practise and denote with h the physical Higgs boson and with ϕ the Goldstone boson:

Gauge boson A_μ with mass $M_A = ev$

μ ν $$= \frac{\mathrm{i}}{k^2 - M^2 + \mathrm{i}\epsilon}\left[-g_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2 - \xi M^2 + \mathrm{i}\epsilon}\right]$$

Higgs boson h with mass squared $M_h^2 = 2\mu^2$

$$= \frac{\mathrm{i}}{k^2 - M_h^2 + \mathrm{i}\epsilon}$$

Goldstone boson ϕ with mass squared ξM_A^2

$$= \frac{\mathrm{i}}{k^2 - \xi M_A^2 + \mathrm{i}\epsilon}$$

Ghost c with mass squared ξM_A^2

$$= \frac{\mathrm{i}}{k^2 - \xi M_A^2 + \mathrm{i}\epsilon}$$

Before we finish this chapter, we should answer why the Goldstone theorem does not apply to the case of the Higgs model. The characteristic property of gauge theories that no manifestly covariant gauge exists which eliminates all gauge freedom is also responsible for the failure of the Goldstone theorem: In the first version of our proof, we may either choose a gauge as the Coulomb gauge. Then only physical degrees of freedom of the photon propagate, but the potential $A^0(x)$ drops only as $1/|\boldsymbol{x}|$ and the charge Q defined in (11.24) becomes ill-defined. Alternatively, we can use a covariant gauge as the Lorentz gauge. Then the charge is well-defined, but unphysical scalar and longitudinal photons exist. The Goldstone does apply, but the massless Goldstone bosons do not couple to physical modes.

In the second version of our proof, the effective potential for the scalar and for the gauge sector do not decouple and mix by the same reason after SSB. This invalidates our analysis including only scalar fields.

Summary of chapter Examining spontaneous symmetry breaking of internal symmetries, we found three different behaviours: If the broken symmetry is discrete, no problem arises. For a broken global continuous symmetry, Goldstone's theorem predicts the existence of massless scalars. In the case of broken approximate symmetries, this can explain the existence of light scalar particles—an example are pions. The case of broken global continuous symmetry which are exact seems to be not realised in nature, since no massless scalar particles are observed. If we gauge the broken symmetry, the would-be massless Goldstone bosons become the longitudinal degrees of freedom required for massive spin-1 bosons.

We developed the effective potential as tool to study the renormalisability of spontaneously broken theories: This approach allows the calculation of all quantum corrections to the classical potential in the limit of constant[1] fields and is invariant under a shift of fields. Thereby we could establish that renormalisability is not affected by SSB.

Further reading

Our discussion of the effective potential is based on the Erice lectures of S. Coleman [Col88].

Exercises

11.1 Contribution to the vacuum energy density from SSB.

Calculate the difference in the vacuum energy density before and after SSB in the SM using $v = 256\,\text{GeV}$ and $m_h^2 = 2\mu^2 = (125)^2\,\text{GeV}^2$. Compare this to the observed value of the cosmological constant.

11.2 Scalar Lagrangian after SSB.

Derive Eq. (11.10) and write down the explicite form of $\mathscr{L}_{\text{int}}$.

11.3 Quantum corrections to $\langle\phi\rangle$.

We implicitely assumed that quantum corrections are small enough to that the field stays at the chosen classical minimum. Calculate $\langle\phi(0)^2\rangle$ for d space-time dimensions and show that this assumption is violated for $d \leq 2$.

11.4 Instability of $\langle\phi\rangle$.

Calculate the imaginary part of the self-energy for a scalar field with the Lagrangian (11.1), i.e. with a negative squared mass $\mu^2 < 0$. Discuss the physical interpretation.

11.5 Goldstone mode as zero mode.

Show that the state $|s\rangle$ defined in Eq. (11.25) has zero energy for $\boldsymbol{k} \to 0$.

11.6 Coleman-Weinberg problem.

Derive Eq. (11.55), find the minima of the potential and discuss the validity of the one-loop approximation.