# 1 Main sequence stars and their evolution

## 1.1 Equations of stellar structure

We look for spherical symmetric, static solutions of the equations of stellar structure. This requires that rotation, convection, magnetic fields $B$, and other effects that break rotational symmetry have only a minor influence on the star.

### 1.1.1 Mass continuity and hydrostatic equilibrium

We denote by $M(r)$ the mass enclosed inside a sphere with radius $r$ and density $\rho(r)$,

$$M(r) = 4\pi \int_0^r \mathrm{d}r' r'^2 \rho(r') \tag{1.1}$$

or in differential form

$$\boxed{\frac{\mathrm{d}M(r)}{\mathrm{d}r} = 4\pi r^2 \rho(r)\,.} \tag{1.2}$$

Although trivial, this constitutes the first ("continuity equation") of the five equations needed to describe the structure and evolution of stars. An important application of the continuity equation is to express physical quantities not as function of the radius $r$ but of the enclosed mass $M(r)$. This facilitates the computation of the stellar properties as function of time, because the mass of a star remains nearly constant during its evolution, while the stellar radius can change considerably.

A radial-symmetric mass distribution $M(r)$ produces according Gauß' law the same gravitational acceleration, as if it would be concentrated at the center $r = 0$. Therefore the gravitational acceleration $g(r)$ produced by $M(r)$ is

$$g(r) = -\frac{GM(r)}{r^2}\,. \tag{1.3}$$

If the star is in equilibrium, this acceleration has to be balanced by a pressure gradient from the center of the star to its surface. Since pressure is defined as force per area, $P = F/A$, a pressure change along the distance $\mathrm{d}r$ corresponds to an increment

$$\mathrm{d}F = \underbrace{A\,\mathrm{d}P}_{\text{force}} = \underbrace{\rho(r)A\mathrm{d}r}_{\text{mass}} \quad \underbrace{a(r)}_{\text{acceleration}} \tag{1.4}$$

of the force $F$ produced by the pressure gradient $\mathrm{d}P$. Hydrostatic equilibrium, $g(r) = -a(r)$, requires then

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho(r)g(r) = -\frac{GM(r)\rho(r)}{r^2}\,. \tag{1.5}$$

If the pressure gradient and gravity do not balance each other, the layer at position $r$ is accelerated,

$$a_{\mathrm{tot}}(r) = \frac{GM(r)}{r^2} + \frac{1}{\rho(r)}\frac{\mathrm{d}P}{\mathrm{d}r}\,. \tag{1.6}$$

In general, we need an equation of state, $P = P(\rho, T, Y_i)$, that connects the pressure $P$ with the density $\rho$, the (not yet) known temperature $T$ and the chemical composition $Y_i$ of the star. For an estimate of the central pressure $P_c = P(0)$ of a star in hydrostatic equilibrium, we integrate (1.5) and obtain with $P(R) \approx 0$,

$$P_c = \int_0^R \frac{\mathrm{d}P}{\mathrm{d}r}\,\mathrm{d}r = G\int_0^M \mathrm{d}M\,\frac{M}{4\pi r^4}\,, \tag{1.7}$$

where we used the continuity equation to substitute $\mathrm{d}r = \mathrm{d}M/(4\pi r^2\rho)$ by $\mathrm{d}M$. If we replace furthermore $r$ by the stellar radius $R \geq r$, we obtain a lower limit for the central pressure,

$$P_c = G\int_0^M \mathrm{d}M\,\frac{M}{4\pi r^4} > G\int_0^M \mathrm{d}M\,\frac{M}{4\pi R^4} = \frac{M^2}{8\pi R^4}\,. \tag{1.8}$$

Inserting values for the Sun, it follows

$$P_c > \frac{M^2}{8\pi R^4} = 4\times 10^8\mathrm{bar}\left(\frac{M}{M_\odot}\right)^2\left(\frac{R_\odot}{R}\right)^4\,. \tag{1.9}$$

### 1.1.2 Gas and radiation pressure

A (relativistic or non-relativistic) particle in a box of volume $L^3$ collides per time interval $\Delta t = 2L/v_x$ once with the $yz$-side of the box, if the $x$ component of its velocity is $v_x$. Thereby it exerts the force $F_x = \Delta p_x/\Delta t = p_x v_x/L$. The pressure produced by $N$ particles is then $P = F/A = Np_x v_x/(LA) = np_x v_x$ or for an isotropic velocity distribution with $\langle v^2\rangle = \langle v_x^2\rangle + \langle v_y^2\rangle + \langle v_z^2\rangle = 3\langle v_x^2\rangle$

$$P = \frac{1}{3}nvp\,. \tag{1.10}$$

If the particles have a distribution $n_p$ of momenta with

$$N = V\int_0^\infty dp\,n_p = V\int_0^\infty \mathrm{d}v\,n_v\,, \tag{1.11}$$

then we obtain instead of Eq. (1.10) the so-called pressure integral

$$P = \frac{1}{3}\int_0^\infty \mathrm{d}p\,n_p vp\,. \tag{1.12}$$

Although the derivation assumed classical trajectories of the particles, the result holds for any kind of non-interacting particles, in particular also if quantum effects are important (cf. Ex. ).

The two most important cases in astrophysics are a classical, non-relativistic gas of atoms and a gas of photons. In the first case, we can derive the momentum distribution noting that the states describing a free particle are labelled by the continuous three-momentum $\mathbf{p}$. Thus the sum over discrete quantum numbers in the Boltzmann factor is replaced by an integration over the momenta $\mathrm{d}^3p$ and the volume $\mathrm{d}^3x$ occupied by the system,

$$\sum_i \exp(-E/kT) \to V\int \frac{\mathrm{d}^3p}{(2\pi)^3}\exp\left(-\frac{mv^2}{2kT}\right) = \frac{V}{(2\pi)^3}\int_0^\infty \exp\left(-\frac{mv^2}{2kT}\right)4\pi m^3v^2\mathrm{d}v\,. \tag{1.13}$$

If we compare the RHS with Eq. (1.11) we see that we need only to normalize correctly $n_v$. The integral can be evaluated by substituting $\alpha = m/(2kT)$ and noting that

$$-\frac{\partial}{\partial\alpha}\left\{\int_{-\infty}^{\infty} dx \exp(-\alpha x^2)\right\} = \int_{-\infty}^{\infty} dx x^2 \exp(-\alpha x^2) = -\frac{\partial}{\partial\alpha}\sqrt{\frac{\pi}{\alpha}} = \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}}\,. \tag{1.14}$$

Multiplying the integrand with $4(\alpha/\pi)^{3/2}$, we obtain the Maxwell-Boltzmann distribution of velocities for a classical gas,

$$n_v \mathrm{d}v = n\left(\frac{m}{2\pi kT}\right)^{2/3} \exp\left(-\frac{mv^2}{2kT}\right) 4\pi v^2 \mathrm{d}v\,. \tag{1.15}$$

Because of $n_p \mathrm{d}p = n_v \mathrm{d}v$, we can insert now $n_v$ into the pressure integral (1.12),

$$P = \frac{1}{3}\int_0^{\infty} \mathrm{d}v\, n_v v p = n\left(\frac{\alpha}{\pi}\right)^{2/3} \int_0^{\infty} \mathrm{d}x\, x^4 \exp\left(-\alpha x^2\right) = nkT\,. \tag{1.16}$$

The integral $\int dx\, x^4 \exp(-\alpha x^2)$ has been calculated with the same method, but now differentiating twice the Gaussian integral with respect to $\alpha$. Since we use generally the mass density $\rho$ instead of the particle number density $n$, it is more convenient to introduce the gas constant $R = k/m_H$ and the mean atomic weigth $\mu$ defined by $n = \rho/(\mu m_H)$. Then the ideal gas law becomes

$$\boxed{P = nkT = R\rho T/\mu\,.} \tag{1.17}$$

A fully ionized plasma consisting mainly of hydrogen has $\mu \approx 1/2$ because of $m_p \gg m_e$.

The second important example is the pressure $P_{\mathrm{rad}}$ of radiation, i.e. the pressure of a photon gas. With $p = h\nu/c$ and $n_\nu \mathrm{d}\nu = n_p \mathrm{d}p$ it follows

$$P_{\mathrm{rad}} = \frac{1}{3}\int_0^{\infty} \mathrm{d}\nu\, n_\nu h\nu\,. \tag{1.18}$$

Noting that the spectral energy density $u_\nu$ and the intensity $B_\nu$ of a thermal photon gas are connected by $u_\nu \mathrm{d}\nu = 4\pi/c B_\nu \mathrm{d}\nu$, it follows

$$\boxed{P_{\mathrm{rad}} = aT^4/3\,.} \tag{1.19}$$

Here we introduced also the radiation constant $a = 4\sigma/c$.

### 1.1.3 Virial theorem

The virial theorem is an important link between the (gravitational) potential energy and the internal (kinetic) energy of any system in equilibrium. In order to derive it for the special case of a star, we multiply both sides of the hydrostatic equilibrium Eq. (1.5) with $4\pi r^3$ and integrate over $r$,

$$\int_0^R \mathrm{d}r\, 4\pi r^3\, P' = -\int_0^R \mathrm{d}r\, 4\pi r^3\, \frac{GM(r)\rho}{r^2}\,. \tag{1.20}$$

Next we insert $\mathrm{d}M(r) = 4\pi r^2 \rho \mathrm{d}r$ on the RHS and integrate partially the LHS,

$$\left[4\pi r^3\, P\right]_0^R - 3\int_0^R \mathrm{d}r\, 4\pi r^2 P = -\int_0^M \mathrm{d}M\, \frac{GM(r)}{r}\,. \tag{1.21}$$

The RHS is the gravitational potential energy $U_{\rm pot}$ of the star. Using the boundary conditions $V(0)=0$ and $P(R)=0$, we see that the first term of the LHS vanishes. We can rewrite the remaining second term of the LHS as

$$-3\int_0^M \mathrm{d}M\,\frac{P}{\rho} = U_{\rm pot}\,. \tag{1.22}$$

For the special case of an ideal gas, $P = nkT = \frac{2}{3}U_{\rm kin}/V$, and we obtain the virial theorem,

$$\boxed{-2U_{\rm kin} = U_{\rm pot}\,.} \tag{1.23}$$

Hence the average energy of a single atom or molecule of the gas is $\langle E_{\rm kin}\rangle = -\frac{1}{2}\langle E_{\rm pot}\rangle$. This is the same result as for a free hydrogen atom, indicating that only the shape not the strength of the potential $V(r)\propto r^{-\alpha}$ determines the ratio of kinetic and potential energy.

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**Ex.:** Estimate the central temperature of the Sun with the virial theorem.
We estimate the gravitational potential energy of a proton at the center of the Sun as

$$\langle E_{\rm grav}\rangle \sim -\frac{GM_\odot m_p}{R_\odot} \approx -3.2\mathrm{keV}/c^2$$

For a thermal velocity distribution of a Maxwell-Boltzmann gas we obtain

$$\langle E_{\rm kin}\rangle = \frac{3}{2}kT = -\frac{1}{2}\langle E_{\rm grav}\rangle \approx 1.6\,\mathrm{keV}/c^2\,.$$

Hence our estimate for the central temperature of the Sun is $T_{\rm c}\approx 1.1\,\mathrm{keV}/c^2 \approx 1.2\times 10^7\,\mathrm{K}$ – compared to $T_{\rm c}\sim 1.3\,\mathrm{keV}/c^2$ in the so-called Solar Standard Model.

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### 1.1.4 $***$ Stability of stars $***$

We want to generalize the virial theorem to a gas with an polytropic equation of state, $P = K\rho^\gamma$. To do so, we have to express the energy density as function of the pressure and the polytropic index $\gamma$. Combining $\mathrm{d}P/P = -\gamma\mathrm{d}\rho/\rho$ and $\mathrm{d}\rho/\rho = -\mathrm{d}V/V$, we obtain

$$V\mathrm{d}P = -\gamma P\mathrm{d}V\,. \tag{1.24}$$

Next we add $p\mathrm{d}V$ to both sides,

$$\mathrm{d}(VP) = V\mathrm{d}P + P\mathrm{d}V = -(\gamma-1)P\mathrm{d}V\,, \tag{1.25}$$

or

$$\mathrm{d}\left(\frac{VP}{\gamma-1}\right) = -P\mathrm{d}V = \mathrm{d}U\,. \tag{1.26}$$

Hence the pressure and the (kinetic) energy density are connected by $P=(\gamma-1)U/V$. For an ideal gas, $U/V = 3/2kT$ and $P = nkT$, the adiabatic index is $\gamma = 5/3$, while for radiation, $U/V = aT^4$ and $P = aT^4/3$, and thus $\gamma = 4/3$.

The relation $P = (\gamma-1)U/V$ allows us to re-express the LHS of Eq. (1.21) as

$$-3(\gamma-1)U_{\rm kin} = U_{\rm pot}\,. \tag{1.27}$$

A star can be only stable, if its total energy $U_{\rm tot} = U_{\rm kin} + U_{\rm pot}$ is smaller than zero,

$$U_{\rm tot} = (4-3\gamma)U_{\rm kin} = \frac{3\gamma-4}{3\gamma-3}\,U_{\rm pot} < 0\,. \tag{1.28}$$

For $\gamma = 5/3$, we obtain back our old result for an ideal gas. A star with $\gamma = 4/3$ has zero energy and marks the border of matter that can become gravitationally bound. Adding an arbitrary small amount of energy would disrupt such a star, while subtraction would lead to its collapse. Important examples for matter with $\gamma = 4/3$ are all relativistic particles, i.e. not only photons but also relativistic electrons and nucleons.

Stars with $\gamma > 4/3$, or more generally all gravitationally bound systems, have surprising thermodynamical properties: Consider e.g. the heat capacity using the EoS of an ideal gas, $C_V = \partial U_{\rm tot}/\partial T = -\partial U_{\rm kin}/\partial T = -\frac{3}{2}Nk$: Losing energy, the star becomes hotter.

### 1.1.5 Energy transport

There exist three different mechanism for the transport of energy: i) radiative energy transfer, i.e. energy transport by photons, ii) conduction, i.e. scattering of electrons or atoms, and iii) macroscopic matter flows. Conduction plays a prominent role as energy transport only for dense systems, and is therefore only relevant in the dense, final stages of stellar evolution.

#### Radiative energy transport

For the energy flux $\mathcal{F}$ emitted per area and time by an layer of the star at radius $r$ and temperature $T$, a transport equation similar to Eq. (**??**) for the intensity $I$ holds,

$$\mathrm{d}\mathcal{F} = -\sigma n \mathcal{F}\mathrm{d}r = -\kappa\rho\mathcal{F}\mathrm{d}r\,. \tag{1.29}$$

Here we introduced also the opacity $\kappa$ being the cross section per mass of a certain material. Absorption of radiation means also a transfer of momentum to the medium. Since the momentum of photons is $p = E/c$ and $\mathcal{F} = E/(At)$, a slab of matter absorbs the momentum $\mathcal{F}/c$ per area and time. According to Newton's second law, $F = \mathrm{d}p/\mathrm{d}t$, this has to be equal to the net force applied to the layer. This force is simply the difference in radiation pressure $dP_{\rm rad}$ times the area. Thus

$$\frac{1}{c}\,\mathrm{d}\mathcal{F} = \frac{\mathrm{d}p}{\mathrm{d}A\,\mathrm{d}t} = \mathrm{d}P_{\rm rad} \tag{1.30}$$

and

$$\mathcal{F} = -\frac{1}{\kappa\rho}\,\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}r} = -\frac{c}{\kappa\rho}\,\frac{\mathrm{d}P_{\rm rad}}{\mathrm{d}r}\,. \tag{1.31}$$

The pressure of radiation is $P_{\rm rad} = aT^4/3$ and hence

$$\mathcal{F} = -\frac{c}{3\kappa\rho}\,\frac{\mathrm{d}(aT^4)}{\mathrm{d}r} = -\frac{4acT^3}{3\kappa\rho}\,\frac{\mathrm{d}T}{\mathrm{d}r}\,. \tag{1.32}$$

The luminosity of a shell at radius $r$ and temperature $T$ is thus connected with a temperature gradient $\mathrm{d}T/\mathrm{d}r$ as

$$L(r) = 4\pi r^2\mathcal{F}(r) = -\frac{16\pi r^2\,acT^3}{3\kappa\rho}\,\frac{\mathrm{d}T}{\mathrm{d}r}\,. \tag{1.33}$$

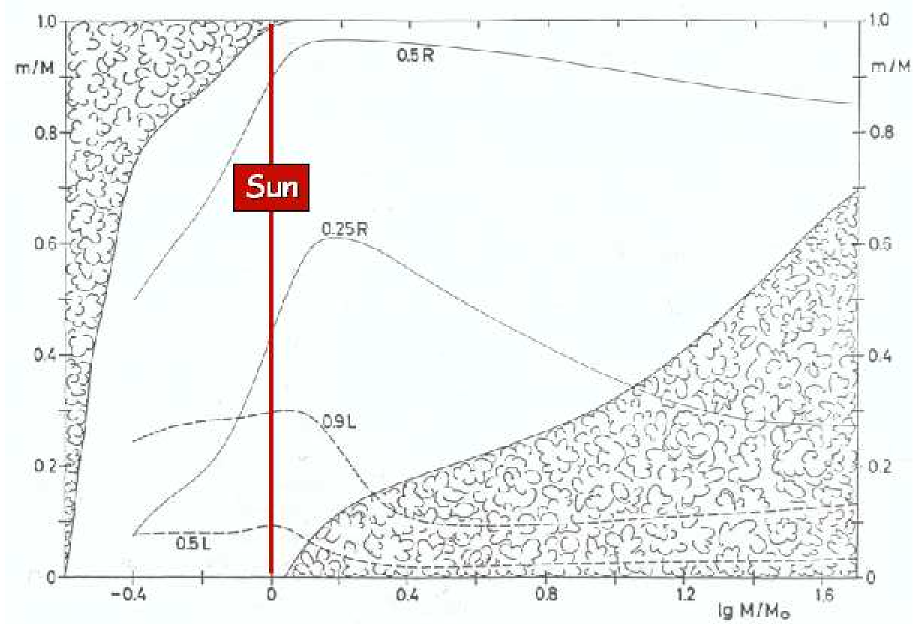


Figure 1.1: In the shadowed regions convection is important for Main Sequence stars, the $x$ axis labels the total mass of the stars, $x = \log(M/M_\odot)$, while the $y$ axis labels the position in the star, $y = M(r)/M$.

### Convection

Convection is a cyclic mass motion carrying energy outwards, if the temperature gradient becomes too large (or $L > L_{\rm Edd}$ as discussed later) – a phenomen familiar to everybody from water close to the boiling point. In the shadowed regions of Fig. 1.1 convection is important for Main Sequence stars of various masses. In the case of the Sun, convection takes places in its outer layer, $M(r) > 0.98 M_\odot$.

## 1.1.6 Thermal equilibrium and energy conservation

Thermal equilibrium and energy conservation require that the energy density $\epsilon$ produced per time and mass by all possible processes corresponds to an increase of the luminosity $L$,

$$\frac{\mathrm{d}L}{\mathrm{d}r} = 4\pi r^2 \epsilon\rho\,. \tag{1.34}$$

The energy production rate per time and mass unit, $\epsilon = \mathrm{d}E/(\mathrm{d}t)$, consists of three terms,

$$\epsilon = \epsilon_{\rm grav} + \epsilon_{\rm nuc} - \epsilon_\nu\,, \tag{1.35}$$

where $\epsilon_{\rm nuc}$ accounts for the energy production by nuclear processes and $\epsilon_\nu$ for the energy carried away by neutrinos. Both effects will be discussed in the next chapter in more detail. The term $\epsilon_{\rm grav}$ is the only one that can be both positive (contraction) or negative (expansion of the star) and is therefore crucial for the stability of a star.

and insert it into Eq. (1.36),

$$\frac{dP_{\rm rad}}{dP} = \frac{\kappa\eta}{4\pi cG}\,\frac{L}{M}\,. \tag{1.43}$$

At the surface, $\eta = 1$ by definition. In general, $\kappa$ increases for small $r$, while $\eta$ decreases ($L(r) \approx$ const. and $M(r) \to 0$ for $r \to 0$). To proceed, Eddington made the assumption that the product $\kappa\eta$ is approximately independent from the radius $r$, $\kappa\eta \equiv \kappa_s =$ const. Then we can integrate Eq. (1.43) immediately,

$$P_{\rm rad} = \frac{\kappa_s L}{4\pi cGM}\,P\,. \tag{1.44}$$

Defining $\beta$ as the fraction the gas contributes to the total pressure, $P_{\rm rad} = (1-\beta)P$ and $P_{\rm gas} = \beta P$, we have

$$\frac{P_{\rm rad}}{1-\beta} = P = \frac{P_{\rm gas}}{\beta}\,. \tag{1.45}$$

Assuming an ideal gas law, $P_{\rm gas} = R\rho T/\mu$, and inserting $P_{\rm rad} = aT^4/3$, we have

$$\frac{aT^4}{3(1-\beta)} = \frac{R}{\beta\mu}\,\rho T\,. \tag{1.46}$$

Now we can express the temperature as function of the density,

$$T = \left(\frac{3R(1-\beta)}{a\beta\mu}\right)^{1/3} \rho^{1/3}\,. \tag{1.47}$$

The total equation of state, $P = P_{\rm gas}/\beta = (R\rho/\beta\mu)T$, is therefore

$$P = \underbrace{\left(\frac{3R^4(1-\beta)}{a\beta^4\mu^4}\right)^{1/3}}_{K} \rho^{4/3}\,, \tag{1.48}$$

where the factor $K$ is a constant. An equation of state with $P = K\rho^\gamma$ is called polytropic with index $\gamma = 1 + 1/n$. Values of $n$ that are of special interest are $n = 5/3$ (nr. degenerate) 4/3 (rel. degenerate) and 3 (fully convective star).

**Lane-Emden equation** For a polytropic equation of state, the continuity and the hydrostatic equation decouple from the other equations of stellar structure. We want to combine now these two first order differential equation into one of second order. Thus we multiply the hydrostatic equation by $r^2/\rho$ and differentiate it then with respect to $r$,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = -G\,\frac{\mathrm{d}M(r)}{\mathrm{d}r} = -4\pi G\rho(r)\,, \tag{1.49}$$

where we inserted the continuity equation in the last step. Using also the equation of state with $P = K\rho^\gamma$, we obtain

$$\frac{(n+1)}{n}\,\frac{K}{4\pi G\,r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho^{\frac{n-1}{n}}}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) = -\rho(r)\,, \tag{1.50}$$

In order to solve this equation, we need two boundary conditions: One of them, $\rho(R) = 0$, is obvious, another one, $\mathrm{d}\rho/\mathrm{d}r(r = 0) = 0$, follows from $\mathrm{d}P/\mathrm{d}r(0) = 0$. It is convenient to go over to a new dimensionless variable $\vartheta \in [0 : 1]$ defining $\rho = \rho_c\vartheta^n$ with $\rho_c$ as central density. Then

$$\underbrace{\frac{(n+1)K}{4\pi G\rho_c^{\frac{n-1}{n}}}}_{\alpha^2} \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}\vartheta}{\mathrm{d}r}\right) = -\vartheta^n . \tag{1.51}$$

Since $\vartheta$ is dimensionless, the parameter $\alpha$ has the dimension of a length. Hence we can use $\alpha$ to make the variable $r$ dimensionless, $r = \alpha\xi$, obtaining finally

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2 \frac{\mathrm{d}\vartheta}{\mathrm{d}\xi}\right) = -\vartheta^n . \tag{1.52}$$

This second order differential equation was studied already a century ago by Lane and Emden. Apart from the cases $n = 0, 1, \infty$, the Lane-Emden equation has to be solved numerically. For $n < 5$, the solutions $\vartheta(\xi)$ decrease monotonically and become zero at a finite value $\vartheta(\xi_0) = 0$, corresponding to the stellar radius $R_n = \alpha\xi_0$.

**Mass-radius relation** The radius $R$ of a star is given by $\xi_0$, i.e. the value when $\vartheta$ becomes zero, via $R_n = \alpha\xi_0$. The mass of the star is then

$$M = 4\pi \int_0^R \mathrm{d}r r^2 \rho = 4\pi\alpha^3\rho_c \int_0^{\xi_0} \mathrm{d}\xi\xi^2\vartheta^n . \tag{1.53}$$

We can replace $\vartheta^n$ by the Lane-Emden equation,

$$M = -4\pi\alpha^3\rho_c \int_0^{\xi_0} \mathrm{d}\xi \, \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2 \frac{\mathrm{d}\vartheta}{\mathrm{d}\xi}\right) = -4\pi\alpha^3\rho_c\xi_1^2 \left.\frac{\mathrm{d}\vartheta}{\mathrm{d}\xi}\right|_{\xi_0} \equiv 4\pi\alpha^3\rho_c M_n . \tag{1.54}$$

Inserting $\alpha = R_n = /\xi_0$ and solving for $\rho_c$, we find

$$\rho_c = \frac{M}{\frac{4\pi}{3}R^3}\frac{\xi_0^3}{M_n} \equiv \bar{\rho}D_n . \tag{1.55}$$

In the last step, we inserted the average density $\bar{\rho}$ and defined

$$D_n = M_n/\xi_0^3 = -\left[\frac{3}{\xi_0} \left.\frac{\mathrm{d}\vartheta}{\mathrm{d}\xi}\right|_{\xi_0}\right]^{-1} . \tag{1.56}$$

Using first the definition of $\alpha$ to eliminate $\rho_c$ in Eq. (1.54) and then $\alpha = R/\xi_0 = R/R_n$ to eliminate $\alpha$, we find the total mass $M$ as function of the radius $R$ as

$$\left(\frac{GM}{M_n}\right)^{n-1}\left(\frac{R}{R_n}\right)^{3-n} = \frac{[(n+1)K]^n}{4\pi G} . \tag{1.57}$$

For the case of interest, $n = 3$, the mass is independent of the radius and is determined only by $K$,

$$M = 4\pi M_3 \left(\frac{K}{\pi G}\right)^{3/2} , \tag{1.58}$$

Table 1.1: Numerical constants from the integration of the Lane-Emden equation.

| $n$ | $R_n = \xi_0$ | $M_n = \xi_0^2 \, (\mathrm{d}\vartheta/\mathrm{d}\xi)_{\xi_0}$ | $D_n$ | $B_n$ |
|---|---|---|---|---|
| 1 | 3.14 | 3.14 | 3.29 | |
| 2 | 4.35 | 2.41 | 4.35 | |
| 3 | 6.90 | 2.02 | 54.2 | |

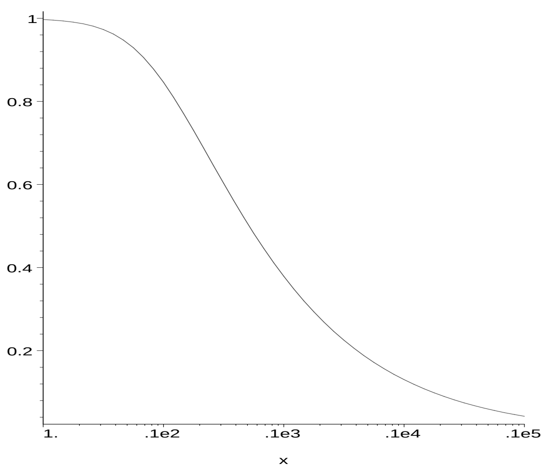


Figure 1.2: The solution $\beta$ of Eddington's quartic equation as function of $x = \mu^2 M/M_\odot$.

where $M_3 \approx 2.02$. Inserting $K$ gives

$$M \propto \frac{(1-\beta)^{1/2}}{\mu^2\beta^2} \,. \tag{1.59}$$

Squaring and inserting the numerical values of the constants, we obtain "Eddington's quartic equation",

$$1 - \beta = 0.003 \left(\frac{M}{M_\odot}\right)^2 \mu^4\beta^4 \,. \tag{1.60}$$

Its solution is shown in Fig. 1.2.

What can one learn about the structure and evolution of stars from this result?

- Remember the meaning of $\beta = P_{\text{gas}}/P$. Thus $\beta \to 0$ corresponds to a free gas of photons, $\beta \to 1$ to a "primordial", cold cloud of gas. Only in the small range of $\mu^2 M$ where $\beta$ has an intermediate value stars can exist.
- Inserting

  $$L = \frac{4\pi c G M}{\kappa_s}(1-\beta) \tag{1.61}$$

  into Eq. (1.60), we obtain

  $$\frac{L}{L_\odot} = \frac{4\pi c G M_\odot}{\kappa_s L_\odot}\mu^4\beta^4\left(\frac{M}{M_\odot}\right)^3 \,, \tag{1.62}$$

  close to a power-law and observations of main-sequence stars!

| | n | p | 2p+2n | $^4$He |
|---|---|---|---|---|
| $m/\mathrm{u}$ | 1.0090 | 1.0081 | 4.0342 | 4.0039 |

Table 1.2: The masses of nucleons and $^4$He in atomic mass units u.

- The variation in $\mu$ can explain scatter in $L$-$M$ plot.
- For stars of given composition (fixed $\mu$), $\beta$ increases as $M$ increases. Thus, radiation pressure is more important in massive stars.
- Nuclear reaction cause a gradual increase of $\mu$ and therefore an decrease of $\beta$. Thus, radiation pressure becomes more important when stars become older. In late stages, stars may eject part of their envelope (stellar winds).

## 1.4 Nuclear processes in stars

The origin of the radiation energy emitted by the Sun was questioned already 1846, soon after the establishment of the law of energy conservation, by J.R. Meyer. It remained mysterious for ninety years. Since the temperature on the Earth was approximately constant during the last $\tau \approx 4 \times 10^9$ years, the solar luminosity should be also roughly constant. Thus we can estimate the energy output of the Sun as $\tau L_\odot \approx 6 \times 10^{50}$ erg.

### 1.4.1 Nuclear fusion

The total mass $m(Z,N)$ of a nuclei with $Z$ protons and $N$ neutrons is because of its binding energy $E_b$ somewhat reduced compered to its constituent mass,

$$E_b/c^2 = Zm_H + Nm_n - m(Z,N)\,. \tag{1.63}$$

Measured values of the binding energy per nucleon $E_b/A$ as function of nucleon number $A$ are shown in Fig. 1.3. In Tab. 1.2, the masses of $^4$He and its constituent nucleons are compared.

- The binding energy per nucleon $E_b/A$ has its maximum at $A \sim 56$, i.e. iron $^{56}$Fe is the most stable element.
- Energy can be released by fusing two light nuclei or 'breaking' a heavy one.
- While $E_b/A$ is a rather smooth function of $A$ for $A > 20$, there several peaks visible for small $A$: $^4$He, $^{12}$C, $^{14}$N, $^{20}$Ne, and $^{16}$O are energetically much more favourable than their neighbouring elements. The bound states of nucleons in nuclei have a similar shell structure as electrons in atoms. Nuclei with filled shells are especially stable, as noble gases are especially stable atoms.
- Thus the fusion of four protons to $^4$He releases 26.2 MeV energy, a factor $10^7$ more than in our estimate for chemical reactions. Converting a solar mass into helium releases $M_\odot/(4m_p) \times 26.2\,\mathrm{MeV} \approx 1.25 \times 10^{52}$ erg and thus around 5% of the Sun have been already converted into helium.
- As shown by the failure of using fusion for the energy production on Earth and longevity of stars fusion is a non-trivial process.

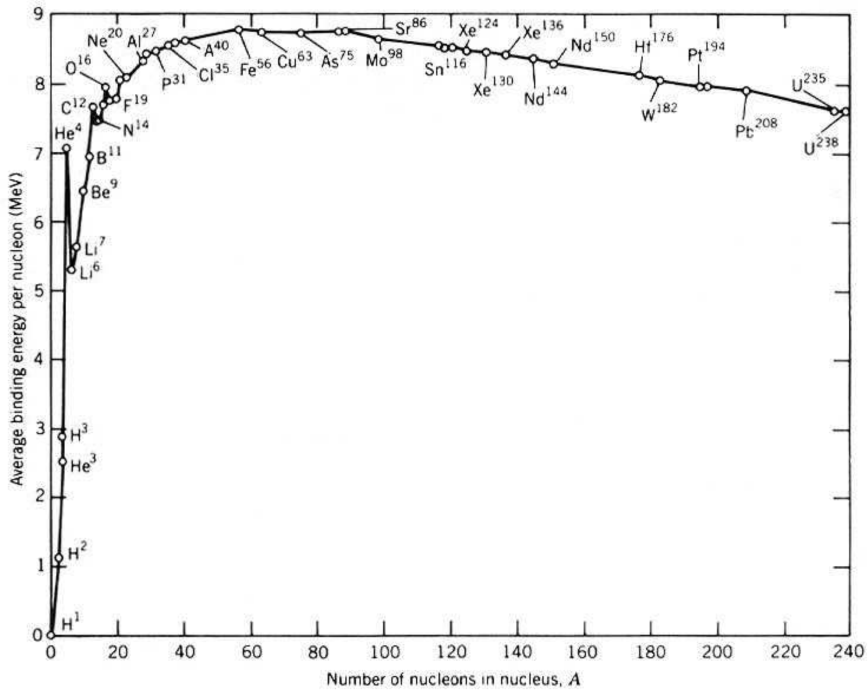


Figure 1.3: The binding energy per nucleon $E_b/A$ as function of the nucleon number $A$.

- All four interactions are involved in the energy release by nuclear fusion: The strong interaction leads to the binding of nucleons in nuclei, the Coulomb repulsion has to be overcome to combine them, and the weak interactions convert half of the protons into neutrons. Finally, gravitation is responsible for the confinement of matter, heating it up the proto-star to the "start temperature" and serving then as a heat regulator.

### 1.4.2 Thermonuclear reactions and Gamov peak

**Coulomb barrier-classically** Nuclear forces are strong but of short range. The simplest model for the forces between two nuclei is therefore an attractive square potential well (Topf?) with radius equal to the size of the nuclei plus the Coulomb force dominatng outside over the strong force. Thus a nuclei should have classically the energy $V \approx Z_1 Z_2 e^2/r_N$ to cross the "Coulomb barrier" and to reach another nucleus of size $r_N$. For a thermal plasma of particles with reduced mass $\mu$ and charges $Z_i e$, this condition reads

$$\frac{1}{2}\mu\langle v^2\rangle = \frac{3}{2}kT = \frac{Z_1 Z_2 e^2}{r}\,. \tag{1.64}$$

Specifically, we obtain for protons with $Z_1 = Z_2 = 1$ and size $r_N \approx 10^{-15}$ m that the temperature should be above

$$T \gtrsim \frac{2e^2}{3kr_0} \approx 10^{10}\,\mathrm{K}\,. \tag{1.65}$$

On the other hand, we have estimated the central temperature of the Sun as $T_c \approx 10^7$ K. Hence we should expect that only the tiny fraction of protons with $v^2 \gtrsim 1000\langle v^2\rangle$ is able to cross the Coulomb barrier.

**Gamov factor and tunneling** Quantum mechanically, tunneling though the Coulomb barrier is possible. The wave-function of a particle with $E - V < 0$ is non-zero, but exponentially suppressed. In order to avoid a too strong suppression, we require that $\lambda = h/p \approx r_N$,

$$\frac{h^2}{2\mu\lambda^2} = \frac{Z_1 Z_2 e^2}{\lambda}\,. \tag{1.66}$$

Inserting $\lambda = h^2/(2Z_1 Z_2 e^2 \mu)$ for $r_N$ in Eq. (1.65) gives

$$T \gtrsim \frac{4Z_1^2 Z_2^2 e^4 \mu}{3h^2 k} \approx 10^7\mathrm{K}\,. \tag{1.67}$$

Hence we expect that quantum effects lead to a not too strong suppression of fusion rates.

Next we want to make this statement more precise: We can estimate the tunneling probability using the WKB approximation as

$$P_0 \propto \exp\left(-2\int_R^{r_0} \mathrm{d}r \sqrt{\frac{2m}{\hbar^2}[V(r) - E]}\right) \equiv \exp(-I)\,. \tag{1.68}$$

Here $R$ is the range of the nuclear force, i.e. the point where the strong force becomes stronger than the the Coulomb force, and $r_0 = 2\mu Z_1 Z_2 e^2/(\hbar k)^2$ is the classical turning point, $V(r_0) = E = (\hbar k)^2/(2m)$, for a particle with energy $E$ and wave-vector $k$ at infinity. The suppression is smallest for s-wave scattering, when the cetrifugal barrier in the potential $V$ is absent. Then

$$I = 2k \int_R^{r_0} \mathrm{d}r \sqrt{\frac{r_0}{R} - 1} \tag{1.69}$$

and substitutumg $\xi = (r/r_0)^{1/2}$, we obtain

$$I = 4kr_0 \int_{\sqrt{R/r_0}}^1 \mathrm{d}\xi \sqrt{1 - \xi^2}\,. \tag{1.70}$$

We now use that $R \ll r_0$ to rewrite the integral first as $\int_0^1 - \int_0^{\sqrt{R/r_0}}$ and then to expand the second term,

$$I = 4kr_0 \left\{ \int_0^1 \mathrm{d}\xi \sqrt{1 - \xi^2} - \int_0^{\sqrt{R/r_0}} \mathrm{d}\xi \left(1 - \frac{1}{2}\xi^2 + \ldots\right) \right\} = 4kr_0 \left\{ \frac{\pi}{4} - \sqrt{R/r_0} + \ldots \right\} \tag{1.71}$$

Thus luckily the leading term does not depend on poorly known details of nuclear physiscs like the range $R$ of the nuclear potential. Neglecting the subleading terms, we can write

$$I = \frac{\pi Z_1 Z_2 e^2}{\hbar} \sqrt{\frac{2\mu}{E}} \equiv B/\sqrt{E} \tag{1.72}$$

**Reaction rates** We introduced the interaction depth $\tau = nl\sigma$ as the probability that a particle interacts travelling the distance $l$ through targets with density $n$. The rate $\Gamma$ of such interactions, i.e. the number of reactions per time follows then with $l = vt$ as $\Gamma \equiv n\sigma v$, if

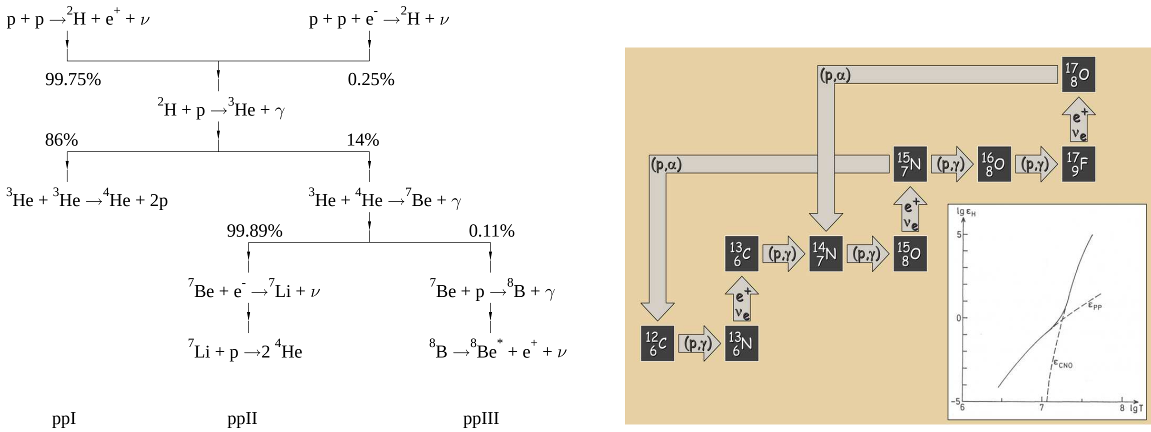


Figure 1.4: Left: proton-proton chains Right: CNO cycle.

the particle moves with velocity $v$. Since the energies of the particles are not uniform, but distributed according a non-relativistic Maxwell-Boltzmann distribution,

$$n_v dv = \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{mv^2}{2kT}\right) 4\pi v^2 \mathrm{d}v \propto \mathrm{e}^{-E/kT} E^{1/2} \mathrm{d}E\,, \tag{1.73}$$

we should average over the distribution $n_v$.

Cross section for strong interactions are of geometrical nature, $\sigma(E) \approx \pi\lambda^2$ and with $\lambda = h/p$ it follows $\sigma(E) \propto 1/E$. Thus the rate is

$$\Gamma = \langle n\sigma v\rangle \propto \int dE\, \sigma(E) E \mathrm{e}^{-b/E^{1/2}} \mathrm{e}^{-E/kT} = \int dE\, S(E) \mathrm{e}^{-b/E^{1/2}} \mathrm{e}^{-E/kT}\,. \tag{1.74}$$

In the last step we introduced the so-called $S$-factor $S(E) = E\sigma(E)$ of the reaction. If the cross-section behaves indeed as $\sigma(E) \propto 1/E$, then $S(E)$ is a slowly varying function The reminder of the integrand is sharply peaked ("Gamov peak") in the region around 10 keV.

## 1.5 Main nuclear burning reactions

### 1.5.1 Hydrogen burning: pp-chains and CNO-cycle

The pp-chains are shown in detail in in the left panel of Fig. 1.4. Its main chain uses three steps:
Step 1: $p + p \to d + e^+ + \nu_e$
Step 2: $p + d \to^3 \mathrm{He} + \gamma$
Step 2: ${}^3\mathrm{He} +^3 \mathrm{He} \to^4 \mathrm{He} + p + p$

The CNO-cyle us shown in detail in in the right panel of Fig. 1.4. The small inlet compared the temperature dependence of the pp-chain and the CNO-cycle: For solar temperatures, the contribution of the pp-chains to the solar energy production is four order of magnitudes more important then the CNO-cycle.

**Radius-mass relation of MS stars** Hydrogen is burned at nearly fixed temperature $T$. Via the virial theorem, also the gravitational potential is nearly the same for all MS stars and

| Burning Phase | Dominant Process | $T_c$ [keV] | $\rho_c$ [g/cm³] | $L_\gamma$ [$10^4$ $L_{sun}$] | $L_\nu/L_\gamma$ | Duration [years] |
|---|---|---|---|---|---|---|
| Hydrogen | H → He | 3 | 5.9 | 2.1 | - | $1.2\times10^7$ |
| Helium | He → C, O | 14 | $1.3\times10^3$ | 6.0 | $1.7\times10^{-5}$ | $1.3\times10^6$ |
| Carbon | C → Ne, Mg | 53 | $1.7\times10^5$ | 8.6 | 1.0 | $6.3\times10^3$ |
| Neon | Ne → O, Mg | 110 | $1.6\times10^7$ | 9.6 | $1.8\times10^3$ | 7.0 |
| Oxygen | O → Si | 160 | $9.7\times10^7$ | 9.6 | $2.1\times10^4$ | 1.7 |
| Silicon | Si → Fe, Ni | 270 | $2.3\times10^8$ | 9.6 | $9.2\times10^5$ | 6 days |

Figure 1.5: Burning Phases of a 15 $M_\odot$ Star.

thus $GM/R \approx$ const. As a result, the radius of MS stars increases approximate linearly with the stellar mass.

### 1.5.2 Later phases

The increasing Coulomb barrier for heavier nuclei means that the fusion of heavier nuclei requires higher and higher temperatures. Therefore the different fusion phases – hydrogen, helium, carbon,... burning – never coexist, but follow each-other. Since the temperatures decreases outwards, fraction of the core participating in fusion becomes smaller in each new burning phase, cf. Fig. 1.5.

- Hydrogen burning 4p+2$e^-$ $\to^4$He+2$\nu_e$
  - proceeds by pp chains and CNO cycle
  - no heavier elements formed because no stable isotopes with mass number $A = 8$
  - neutrinos from $p \to n$ conversion
  - typical temperature $10^7$ K ($\sim 1$ keV)
- Helium burning $^4$He+$^4$He+$^4$He $\leftrightarrow^8$Be+$^4$He $\to^{12}$C
  - triple alpha reaction builds up Be with concentration $\sim 10^9$
    $^{12}$C+$^4$He→$^{16}$O
    $^{16}$O+$^4$He→$^{20}$Ne
  - typical temperature $10^8$ K ($\sim 10$ keV)
- Carbon burning
  - many reactions like $^{12}$C+$^{12}$C→$^{20}$Ne+$^4$He etc.
  - typical temperature $10^9$ K ($\sim 100$ keV)

## 1.6 Solar neutrinos

**Solar neutrinos flux** From $L_\odot = 4 \times 10^{33}$erg/s= $2 \times 10^{39}$MeV/s and the energy release of 26.2 MeV per reaction, the minimal number of neutrinos produced is $\dot{N}_\nu = 2 \times 10^{38}$/s. As we have seen, photon perform a random-walk. Neutrinos have much smaller interactions,

$$\sigma_{\nu_e e} = 10^{-43}\text{cm}^2 \frac{E_\nu}{\text{MeV}} \tag{1.75}$$

and thus they can escape from the Sun: The interaction depth for a neutrino in the Sun is approximately

$$\tau = \sigma_{\nu_e e} n_e R_\odot = 10^{-9}\,. \tag{1.76}$$

At Earth this corresponds to the flux of

$$\phi_\nu = \frac{\dot{N}_\nu}{4\pi D^2} = 7 \times 10^{10} \frac{1}{\text{cm}^2\text{s}}\,. \tag{1.77}$$

(or directly via $\phi_\nu = 2S/(26.2MeV)$ with Solar constant.)

Weak interactions in the Sun produce always electron neutrinos, i.e. $p \to n + e^+\nu_e$, but not $p \to n + \mu^+\nu_\mu$ or $p \to n + \tau^+\nu_\tau$, because the energy released in nuclear reaction and the temperature is too small too produce a $\mu$ or $\tau$. Similarly, only $\nu_e$ neutrinos are detected in radiochemical reactions via "inverse beta-decay", while all type of neutrinos can be detected in elastic scattering on electrons.

**Solar neutrino experiments** Radiochemical experiments detect neutrinos by "inverse beta-decay" in suitable nuclei. The historically first isotope used was chlorine, $\nu_e+^{37}\text{Cl} \to^{37} \text{Ar}+e^-$, i.e. changing a neutron inside a $^{37}$Cl nuclei into a proton, thereby converting it into a $^{37}$Ar. The disadvantage of this reaction is its high energy threshold, $E_\nu \geq 0.814$MeV, only sensitive to 9% of all solar neutrinos.

The experiment consists of 615 tons of $C_2Cl_4$ solutions in a mine 1500 m underground. After exposure of a 2,3 months, a few Ar atoms are produced. They are chemically extracted and counted by their subsequent decays (halftime 35 days). Starting from first data in 1968, a deficit appeared relative to theoretical expected fluxes: only 30% of predicted event number is measured! This deficit was dubbed "solar neutrino problem." Finding the solution to this problem required more than 30 years of intensive experimental and theoretical work.

Starting from 1991, 2 Gallium experiments $\nu_e +^{71}\text{Ga} \to^{71} \text{Ge} + e^-$ with threshold $E_\nu \geq$ 233keV took data. They found 55% of the expected neutrino flux, corresponding to 9 atoms of $^{71}$Ge in 30 tons of solution containing 12 tons $^{71}$Ga, after three weeks of run time.

What are plausible solutions to the solar neutrino problem?

- Experiments might be wrong (difficult chemistry, no calibration of Ar cross section,...)

- Nuclear physics (cross sections measured at higher energies are extrapolated to the Gamov peak).
- Our model of the Sun (crucial $T_c$)
- Particle physics: does a $\nu_e$ survive the travel to the Earth?

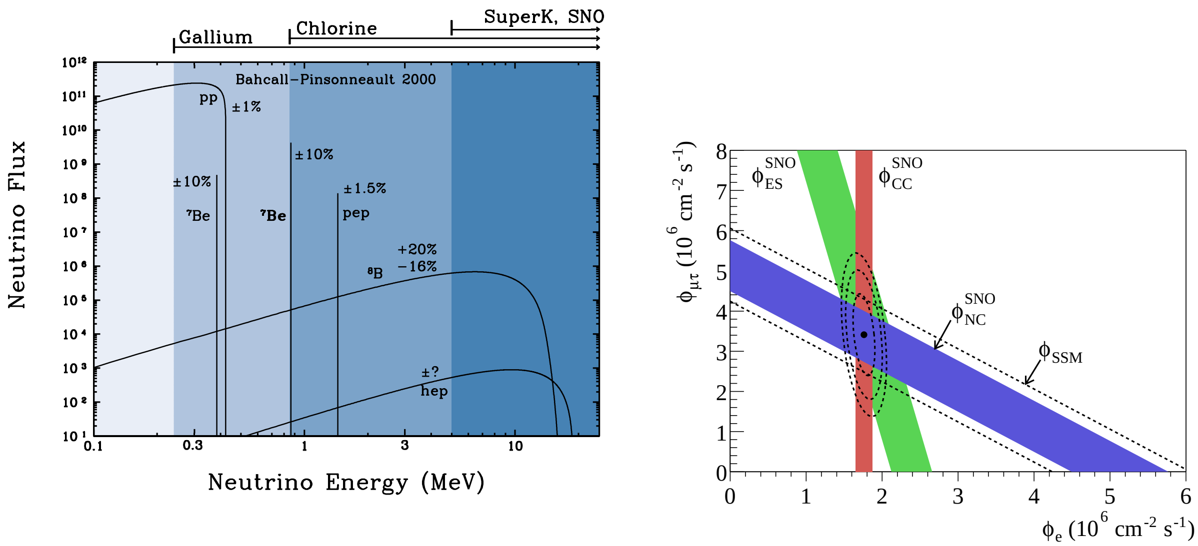


Figure 1.6: Left: The solar standard model predicts the neutrino flux and thus also the number of events that should be measured. Right: Results of the SNO experiment.

As latter experiments showed concuclsively, the latter reason is the correct one: An energy-dependent fraction of electron neutrinos is tranformed ("neutrino oscillations") into a combination of muon and tau neutrinos. The red band in the right panel of Fig. 1.6 shows the flux of $\nu_e$, measured in the SNO experiment by inverse-beta decay reactions. This flux is just 36% of the expected flux from the solar standard model. However, the experiment was able to measure also the flux of $\nu_\mu$ and $\nu_\tau$. Summing all three up, one obtains the value predicted by the solar standard model. Hence neutrinos oscillate, changing their flavor.