9.11 expand V_{eff} around unstable maximum,

$$V_{eff}(r) = V_{eff}(r_{max}) + \frac{1}{2} \underbrace{\left.\frac{d^2 V_{eff}}{dr^2}\right|_{r_{max}}}_{\equiv -K^2 > 0} (\delta r)^2 + \dots \qquad \delta r = r - r_{max}$$

(9.29) $\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r)$ becomes

$$\frac{1}{2}\left(\frac{d(\delta r)}{d\tau}\right)^2 - \frac{1}{2} K^2 (\delta r)^2 = 0$$

solution $\delta r \propto \exp(\pm K\tau) \Rightarrow \tau_* = 1/K$

explicitly: $V_{eff}(r) = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3}$

$$V'_{eff}(r) = \frac{M}{r^2} - \frac{\ell^2}{r^3} + \frac{3M\ell^2}{r^4}$$

$$V''_{eff}(r) = -\frac{2M}{r^3} + \frac{3\ell^2}{r^4} - \frac{12M\ell^2}{r^5} = \frac{-2Mr^2 + 3\ell^2 r - 12M\ell^2}{r^5}$$

finally: plug in r_{max}, $\left(-V''_{eff}(r_{max})\right)^{1/2} = K$

$$\tau_* = K^{-1} = \left(2 V_{eff}(r_{max})\right)^{-1/2} \qquad r_{max} = \frac{\ell^2}{2M}\left[1 + \sqrt{1 - 12(M/\ell)^2}\right]$$

$$= \left\{ \frac{1}{2} \frac{r_{max}^5}{-2M r_{max}^2 + 3\ell^2 r_{max} - 12 M\ell^2} \right\}^{1/2}$$

~~consider $2M r_{max}^2 =$~~ consider r_{max}^2

$$r_{max}^2 = \frac{\ell^4}{4M^2}\left[1 + 2\sqrt{1 - 12(M/\ell)^2} + 1 - 12(M/\ell)^2\right] =$$

$$= \frac{\ell^2}{2M} 2 r_{max} - 3\ell^2$$

$$-2M r_{max}^2 = -\frac{2\ell^2}{\cancel{}} r_{max} + 6M\ell^2$$

$$\Rightarrow \tau_* = \left\{ \frac{1}{2} \frac{r_{max}}{\ell^2 r_{max} - 6M\ell^2} \right\}^{+1/2} = \left\{ \frac{1}{2} \frac{r_{max}}{\ell^2 (r_{max} - 6M)} \right\}^{+1/2}$$

$\to \infty$ for $r \to 6M$: unstable & stable coalesces.

9.7

An observer with $\underline{u}_{obs}$ sees the particle with energy E and velocity v,

$$E = -\underline{p}\cdot\underline{u}_{obs} = \frac{m}{\sqrt{1-v^2}}.$$

stationary $u^r_{obs} = u^\vartheta_{obs} = u^\varphi_{obs} = 0$, $\underline{u}_{obs}\cdot\underline{u}_{obs} = -1$

$g_{tt} = -\left(1-\frac{2M}{r}\right)$

$$\Rightarrow \quad u^t_{obs} = \left(1-\frac{2M}{r}\right)^{-1/2}$$

~

$$E = -\underbrace{\underline{p}}_{\underline{p}=m\underline{u}}\cdot\underline{u}_{obs} = -m\,\underline{u}\cdot\underline{u}_{obs} = -m\, g_{\alpha\beta}\, u^\alpha u^\beta_{obs}$$

$$= m\left(1-\frac{2M}{r}\right)^{1/2} u^t \overset{!}{=} \frac{m}{\sqrt{1-v^2}}$$

~

$e = -\underline{\xi}\cdot\underline{u} = \left(1-\frac{2M}{r}\right)\underbrace{\frac{dt}{d\tau}}_{u^t}$ is conserved, insert for u^t

$$\Rightarrow \quad \frac{1}{\sqrt{1-v^2}} = \left(1-\frac{2M}{r}\right)^{-1/2} e \quad \Rightarrow \quad \frac{1}{1-v^2} = \left(1-\frac{2M}{r}\right)^{-1} e^2$$

$1-v^2 = \frac{1-2M/r}{e^2}$; $v = \left[1-\frac{1-2M/r}{e^2}\right]^{1/2} \leq 1$ | $v = \frac{1}{e}\left(e^2 - 1 + \frac{2M}{r}\right)^{1/2}$

12.14

geodesics maximize proper time $\Rightarrow$ path of largest proper time is geodesic for $r=2M$ to 0.

"energy equation"
$$\frac{e^2-1}{2} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \underbrace{V_{eff}(r)}_{\frac{1}{2}\left[\left(1-\frac{2M}{r}\right)\left(1+\frac{\ell^2}{r^2}\right)-1\right]}$$

$$\tau = \int dr \, \frac{d\tau}{dr} \qquad \Bigg| \qquad \frac{dr}{d\tau} = \left[e^2 - \left(1-\frac{2M}{r}\right)\left(1+\frac{\ell^2}{r^2}\right)\right]^{1/2}$$

$$= \int_0^{2M} dr \left[e^2 + \underbrace{\left(\frac{2M}{r}-1\right)}_{>}\left(1+\ell^2/r^2\right)\right]^{-1/2} = \int_0^{2M} dr \, \left(f(r)\right)^{-1/2}$$

$f(r)$ becomes minimal for $e, \ell \to 0$

$\Rightarrow$ τ maximal for $e = \ell = 0$

$$\tau = \int_0^{2M} dr \left(\frac{2M}{r}-1\right)^{-1/2} = 2M \int_0^1 d\xi \, \frac{\xi^{1/2}}{\sqrt{1-\xi}} = \pi M$$

"the more you struggle, the shorter your life"

18.10

need to know $\dot{M}_{hor}(t)$

$$d_h(t) = a(t) \int_0^t \frac{dt'}{a(t')}$$

$$\Rightarrow M_h(t) = \frac{4\pi}{3} \underbrace{\rho_m(t)\, a^3(t)}_{\text{const}} \left(\int_0^t \frac{dt'}{a(t')} \right)^3$$

$$\frac{dM_h}{dt} = 4\pi \rho_m a^2 \left(\int_0^t \frac{dt'}{a(t')} \right)^2 = 4\pi \rho_m d_h^2$$

$$d_h = \frac{2}{H_0} \quad \text{(matter)}$$

$$\approx \frac{3.3}{H_0} \quad (\Lambda\text{CDM}) \qquad ; \quad \rho_m = \Omega_m \rho_{cr} = \Omega_m \frac{3H_0^2}{8\pi G}$$

$$\Rightarrow \dot{M}_h = 4\pi \cdot \Omega_m \frac{3H_0^2}{8\pi G} \cdot \frac{10}{H_0^2} \approx 15\, \Omega_m / G \quad \text{[independent of } H\text{!]}$$

$$\Delta t_{gal} = \frac{M_{gal}}{\dot{M}_h} \approx \frac{10^{12} M_\odot \cdot G}{15 \cdot 0.2 \cdot c^3} \approx \frac{10^{12} \cdot 2\times10^{33}\, g \times 6.7\cdot10^{-8}\, cm^3\, g^{-1}\, s^{-2}}{3 \cdot (3\cdot10^{10}\, cm/s)^3}$$

$$\approx 1.6\times10^{6}\, s \approx 0.05\, yr$$

18.11 Eq. (18.69) are given

$$a(\eta) = \frac{\Omega}{2H_0(\Omega-1)^{3/2}}(1-\cos\eta) \qquad k+1,\ \Omega \geq 1.$$

$$t(\eta) = \quad '' \quad (\eta - \sin\eta)$$

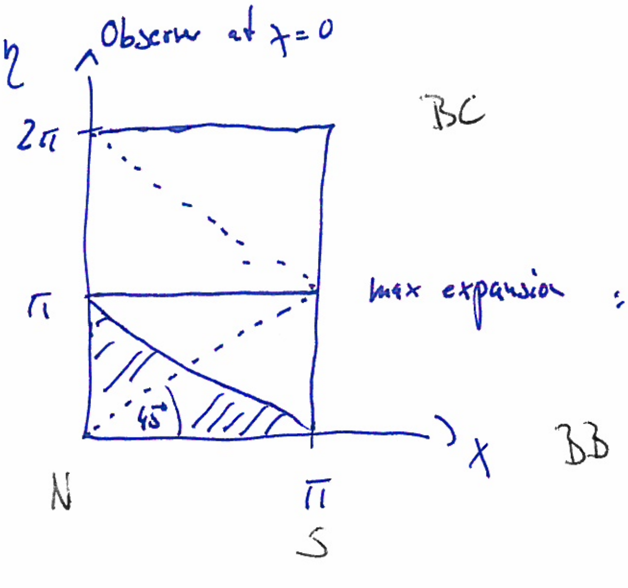
a.) $\Rightarrow$ ~~dt~~ $a(\eta) = \frac{dt}{d\eta}$

b) $a(\eta=0) = 0$ $\qquad \chi \in [0:\pi]$

$a(\eta=\pi) = a_{max}$

$a(\eta=2\pi) = 0$

c.)



: 0 observes whole universe (and later too)

light ray .... just travels from north ($\chi=0$) to south pol ($\chi=\pi$) and back between big bang & big crunch

$\Rightarrow$ observer with ~~xxx~~ $v<c$ cannot