Exercise sheet 1

1. Units.
a.) The four fundamental constants $\hbar$ (Planck's constant), c (velocity of light), G_N (gravitational constant) and k_B (Boltzmann constant) can be combined to obtain the dimension of a length, time, mass, energy and temperature. Find the four relations and calculate the numerical values of two of these so-called "Planck units". Do you have a guess of their physical meaning?
b.) Natural units: We will set in the these lectures $c = \hbar = 1$. Then the famous Einstein relation becomes $E = m$. Show how one can restore usual units.

2. Transformation between inertial frames.
Consider two inertial frames K and K' with parallel axes at $t = t' = 0$ that are moving with the relative velocity v in the x direction.
a.) Show that the linear transformation between the coordinates in K and K' is given by

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} At + Bx \\ Dt + Ex \\ y \\ z \end{pmatrix} = \begin{pmatrix} At + Bx \\ A(x - vt) \\ y \\ z \end{pmatrix} \tag{1}$$

b.) Show that requiring the invariance of

$$\Delta s^2 \equiv c^2 t^2 - x^2 - y^2 - z^2 = c^2 t'^2 - x'^2 - y'^2 - z'^2 \tag{2}$$

leads to Lorentz transformations.

3. Doppler effect.
The photon is a massless particle. Consider a Lorentz transformation of its wave four-vector $k^\mu = p^\mu/\hbar = (\omega, \boldsymbol{k})$ to obtain the relativistic Doppler formula.

4. Index gymnastics.
a.) Splitt the arbitrary tensor $T^{\mu\nu}$ into its symmetric part $S^{\mu\nu} = S^{\nu\mu}$ and its anti-symmetric part $A^{\mu\nu} = -A^{\nu\mu}$.
b.) Show that this splitting is invariant under Lorentz transformations.
c.) Show that $S^{\mu\nu} A_{\mu\nu} = 0$.

5. Lorentz group.
Lorentz transformations $\Lambda \in O(1,3)$ are all those coordinate transformations $x^\mu \to \tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu$ that keep the norm $\eta_{\mu\nu} x^\mu x^\nu$ of space-time points x^μ invariant. Apart from rotations and boosts which are continuously connected to the unit element, there are other, unconnected pieces of the Lorentz group. Determine these unconnected pieces by considering the relation $\eta = \Lambda^T \eta \Lambda$. [Hint: A relation like $[f(\Lambda)]^2 = 1$ shows that the Lorentz group consists of at least two disconnected pieces, one with $f = 1$, another one with $f = -1$.]

Solutions are discussed Monday, 15.01.24