Show that the scalar product of two time-like vectors can be expressed as \( a \cdot b = ab \cosh \eta \), where \( \eta \) is the rapidity connecting the two frames where \( a^\mu \equiv (a, 0) \) and \( \tilde{b}^\mu \equiv (b, 0) \) are valid.

2. Relativity of simultaneity.
Draw a space-time diagram (in \( d = 2 \)) for two inertial frames connected by a boost with velocity \( \beta \): What are the angles between the axes \( t \) and \( t' \), \( x \) and \( x' \)? Draw lines of constant \( t \) and \( t' \) and convince yourself that the time order of two space-like events is not invariant.

3. Uniformly accelerated observer.
Consider a particle moving on the \( x \) axis along a world-line parametrised by
\[
t(\sigma) = \frac{1}{a} \sinh \sigma, \quad x(\sigma) = \frac{1}{a} \cosh \sigma.
\]
a.) Find the connection between \( \sigma \) and proper-time \( \tau \); express the world-line as function of \( \tau \).
b.) Calculate the four-velocity \( u^\alpha \) and the three-velocity \( v^1 \) of the particle. Check their normalisation.
c.) Calculate the four-acceleration \( a^\alpha \) of the particle.
d.) Draw a space-time diagram.

4. Infinitesimal Lorentz transformation.
Symmetry transformations form groups; continuous transformations in physics depend analytically on their parameters (e.g. as \( \cos \vartheta \) and \( \sin \vartheta \) on the rotation angle \( \vartheta \)). An element \( g \) of such a group (called “Lie group”) can be expanded as a power series,
\[
g(\vartheta) = 1 + \sum_{a=1}^{n} i \vartheta^a T^a + \mathcal{O}(\vartheta^2) \equiv 1 + i \vartheta^a T^a + \mathcal{O}(\vartheta^2).
\]
The linear transformation in the arbitrary direction \( \vartheta^a \) is called an infinitesimal transformation, the \( T^a \) the (infinitesimal) generators of the transformation. The generators \( T^a \) can be obtained by differentiation, \( T^a = -i \frac{dg(\vartheta)}{d\vartheta^a}|_{\vartheta=0} \). Conversely, analyticity implies that the group element \( g(\vartheta) \) can be obtained by exponentiation,
\[
g(\vartheta) = \lim_{n \to \infty} [1 + i \vartheta^a T^a/n]^n
\]
a.) Calculate the generators of Lorentz transformations.
b.) Determine their “Lie algebra”, i.e. calculate the real numbers \( f^{abc} \) called structure constants in
\[
[T^a, T^b] = if^{abc}T^c.
\]
Solutions are discussed Friday, 28.01.22