Exercise sheet 2

1. Scalar product of time-like vectors.
Show that the scalar product of two time-like vectors can be expressed as $\boldsymbol{a}\cdot\boldsymbol{b} = ab\cosh\eta$, where η is the rapidity connecting the two frames where $a^\mu \equiv (a,\mathbf{0})$ and $\tilde{b}^\mu \equiv (b,\mathbf{0})$ are valid.

2. Action of free relativistic particle.
Consider $S = -\alpha\int \mathrm{d}\tau$ as action for a free relativistic particle.
a.) Determine the constant α requiring the correct non-relativistic limit.
b.) Does a classically allowed path maximise or minimise the action?

3. Uniformly accelerated observer.
Consider a particle moving on the x axis along a world-line parametrised by

$$t(\sigma) = \frac{1}{a}\sinh\sigma, \qquad x(\sigma) = \frac{1}{a}\cosh\sigma.$$

a.) Find the connection between σ and proper-time τ; express the world-line as function of τ.
b.) Calculate the four-velocity u^α and the three-velocity v^1 of the particle. Check their normalisation.
c.) Calculate the four-acceleration a^α of the particle.
d.) Draw a spacetime diagram including $x^\mu(\sigma)$.

4. Infinitesimal Lorentz transformation.
Symmetry transformations form groups; continuous transformations in physics depend analytically on their parameters (e.g. as $\cos\vartheta$ and $\sin\vartheta$ on the rotation angle ϑ). An element g of such a group (called "Lie group") can be therefore expanded as a power series,

$$g(\vartheta) = 1 + \sum_{a=1}^{n} \mathrm{i}\vartheta^a T^a + \mathcal{O}(\vartheta^2) \equiv 1 + \mathrm{i}\vartheta^a T^a + \mathcal{O}(\vartheta^2). \tag{1}$$

The linear transformation in the arbitrary direction ϑ^a is called an infinitesimal transformation, the T^a the (infinitesimal) generators of the transformation. The generators T^a can be obtained by differentiation, $T^a = -\mathrm{i}\,\mathrm{d}g(\vartheta)/\mathrm{d}\vartheta^a|_{\vartheta=0}$. Conversely, analyticity implies that the group element $g(\vartheta)$ (more precisely, thus connected to the unity element) can be obtained by exponentiation,

$$g(\vartheta) = \lim_{n\to\infty}\left[1 + \mathrm{i}\vartheta^a T^a/n\right]^n \tag{2}$$

a.) Calculate the generators of Lorentz transformations.
b.) Determine their "Lie algebra", i.e. calculate the real numbers f^{abc} called structure constants in

$$[T^a, T^b] = \mathrm{i}f^{abc}T^c. \tag{3}$$

Solutions are discussed Thursday, 25.01.24