Exercise sheet 3b: Additional exercises

1. Proper distance, area and volumes.

Consider the metric

$$ds^2 = -(1 - Ar^2)^2 dt^2 + (1 - Ar^2)^2 dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta \mathrm{d}\phi^2)$$

i) Calculate the proper distance from $r = 0$ to $r = R$.
ii) Calculate the area of a sphere with coordinate radius R.
iii) Calculate the three-volume of a sphere with coordinate radius R.
iv) Calculate the four-volume of a sphere with coordinate radius R bounded by two t=const planes separated by time difference T.

2. Newtonian gravity as a spacetime phenomenon

Since Newtonian gravity is a special case of Einstein gravity, it should be possible to replace the Newtonian gravitational force by a deformation of Minkowski space. Confirm that the metric describing gravitational effects in the Newtonian limit can be chosen as

$$\mathrm{d}s^2 = \left(1 + 2\Phi/c^2\right) c^2 \mathrm{d}t^2 - \left(1 - 2\Phi/c^2\right) \mathrm{d}l^2 \tag{1}$$

by showing that the resulting action of a point particle is equivalent to the the one using the standard Lagrangian

$$L = \frac{1}{2} m v^2 - m\Phi = T + V \tag{2}$$

in the limit $v/c \to 0$. Here, Φ is the Newtonian gravitational potential and $\mathrm{d}l^2 = dx^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ is the Euclidean line-element.

3. Maximal velocity of a comet

A comet starts at $r \to \infty$, approaches a star of mass M and disappears again to $r \to \infty$. What is the maximal velocity v of the comet measured by a stationary observer at the radius R of nearest approach?
(2 ways of solution: 1. find v as function of R, l and b, connect then l to b and R. 2. Use the orthonormal basis associated with the observer.)