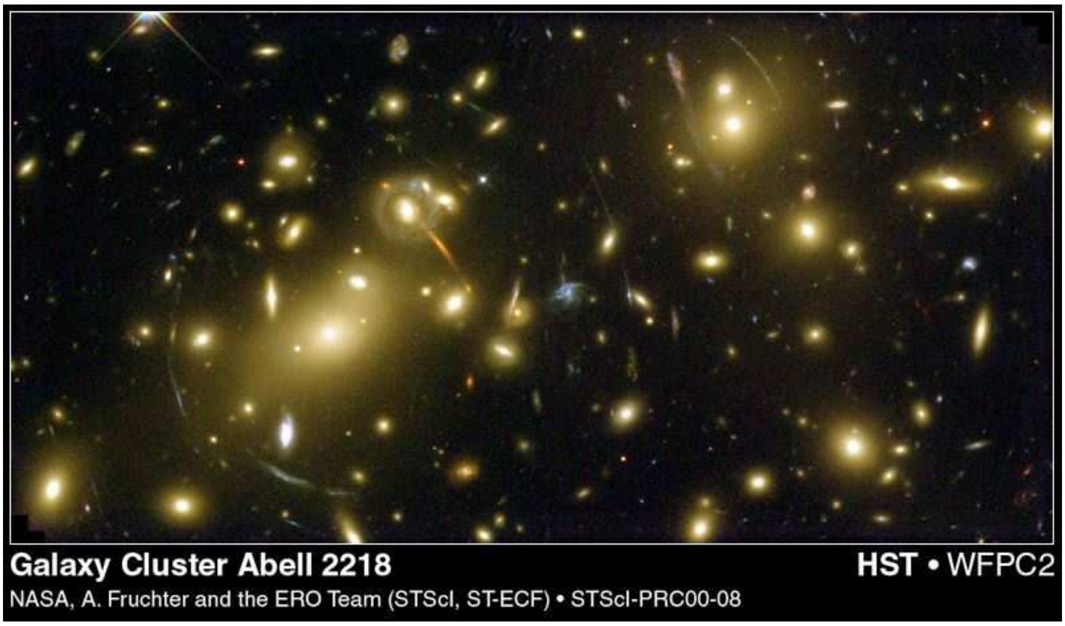
Lecture Notes for FY3452 Gravitation and Cosmology

M. Kachelrieß



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Watch out for errors, most was written late in the evening.
Corrections, feedback and any suggestions always welcome!

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Preface

These notes summarise the lectures for FY3452 Gravitation and Cosmology I gave first in 2009 and 2010. After a break, I restarted to teach the lectures in 2020 and 2022ff. As the title indicates, the course focused in 2009 on cosmology as main application. Since then, black holes and gravitational waves have gained much more popularity. Asked to which of these three more advanced topics more time should be devoted, students in later years voted therefore, not surprisingly, mostly for black holes and/or gravitational waves. As a result, the notes contain more material than manageable in an one semester course. In 2025, we will have to make a similar decision, and there will be a vote in the first weeks of the lecturing period. Moreover, it seems that gauge theories are not longer discussed in the FY3464 QFT lectures. To compensate for this, and in order to make the similarity between gauge fields and gravity clearer, I have rearranged therefore the chapter on classical field theory compared to earlier versions.

There are various differing sign conventions in general relativity possible – all of them are in use. One can classify these choices as follows

$$\eta^{\alpha\beta} = S_1 \times [-1,+1,+1,+1], \tag{0.1a}$$
$$R^{\alpha}{}_{\beta\rho\sigma} = S_2 \times [\partial_\rho \Gamma^{\alpha}{}_{\beta\sigma} - \partial_\sigma \Gamma^{\alpha}{}_{\beta\rho} + \Gamma^{\alpha}{}_{\kappa\rho}\Gamma^{\kappa}{}_{\beta\sigma} - \Gamma^{\alpha}{}_{\kappa\sigma}\Gamma^{\kappa}{}_{\beta\rho}], \tag{0.1b}$$
$$G_{\alpha\beta} = S_3 \times 8\pi G\, T_{\alpha\beta}, \tag{0.1c}$$
$$R_{\alpha\beta} = S_2 S_3 \times R^{\rho}{}_{\alpha\rho\beta}. \tag{0.1d}$$

We choose these three signs as $S_i = \{-,+,+\}$. The convention of few other authors are summarised in the following table:

	HEL	dI,R	MTW, H	W
$[S_1]$	-	-	+	+
$[S_2]$	+	+	+	-
$[S_3]$	-	-	+	-

Some useful books:

H: J. B. Hartle. Gravity: An Introduction to Einstein's General Relativity (Benjamin Cummings)

HEL: Hobson, M.P., Efstathiou, G.P., Lasenby, A.N.: General relativity: an introduction for physicists. Cambridge University Press 2006. [On a somewhat higher level than Hartle.]

- Robert M. Wald: General Relativity. University of Chicago Press 1986. [Uses a modern mathematical language]

- Landau, Lev D.; Lifshitz, Evgenij M.: Course of theoretical physics 2 - The classical theory of fields. Pergamon Press Oxford, 1975.

MTW: Misner, Charles W.; Thorne, Kip S.; Wheeler, John A.: Gravitation. Freeman New York, 1998. [Entertaining and nice description of differential geometry - but lengthy.]

- Schutz, Bernard F.: A first course in general relativity. Cambridge Univ. Press, 2004.

- Stephani, Hans: Relativity: an introduction to special and general relativity. Cambridge Univ. Press, 2004.

W: Weinberg, Steven: Gravitation and cosmology. Wiley New York, 1972. [A classics. Many applications; outdated concerning cosmology.]

- Weyl, Hermann: Raum, Zeit, Materie. Springer Berlin, 1918 (Space, Time, Matter, Dover New York, 1952). [The classics.]

Finally: If you find typos (if not, you havn't read carefully enough) in the part which is already updated, conceptional errors or have suggestions, send me an email!

1 Special relativity

1.1 Newtonian mechanics and gravity

Inertial frames and the principle of relativity Newton presented his mechanics in an axiomatic form. His *Lex Prima* (or the Galilean law of inertia) states: *Each force-less mass point stays at rest or moves on a straight line at constant speed.* Distinguishing between straight and curved lines requires an affine structure of space, while measuring velocities relies on a metric structure that allows one to measure distances. In addition, we have to be able to compare time measurements made at different space points. Thus, in order to apply Newton's first law, we have to add some assumptions on space and time. Implicitly, Newton assumed an Euclidean structure for space, and thus the distance between two points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ in a Cartesian coordinate system is

$$\Delta l_{12}^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 \tag{1.1}$$

or, for infinitesimal distances,

$$\mathrm{d}l^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \,. \tag{1.2}$$

Moreover, he assumed the existence of an absolute time t on which all observers can agree.

In a Cartesian *inertial* coordinate system, Newton's lex prima becomes then

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = 0 \,. \tag{1.3}$$

Most often, we call such a coordinate system just an *inertial frame*. Newton's first law is not just a trivial consequence of its second one, but may be seen as a practical definition of those reference frames for which his following laws are valid.

Which are the transformations which connect these inertial frames or, in other words, which are the symmetries of empty space and time? We know that translations $\boldsymbol{a}$ and rotations R are symmetries of Euclidean space: This means that using two different Cartesian coordinate systems, say a primed and an unprimed one, to label the points P_1 and P_2, their distance defined by Eq. (1.3) remains invariant, cf. with Fig. 1.1. The condition that the norm of the distance vector $\boldsymbol{l}_{12}$ is invariant, $\boldsymbol{l}_{12} = \boldsymbol{l}'_{12}$, implies

$$\boldsymbol{l}'^T \boldsymbol{l}' = \boldsymbol{l}^T R^T R \boldsymbol{l} = \boldsymbol{l}^T \boldsymbol{l} \tag{1.4}$$

or $R^T R = 1$. Thus rotations acting on a three-vector $\boldsymbol{x}$ are represented by orthogonal matrices, $R \in O(3)$. In addition to rotations, this group contains reflexions $x^i \to -x^i$. All frames connected by $\boldsymbol{x}' = R\boldsymbol{x} + \boldsymbol{a}$ to an inertial frame are inertial frames too. Finally, there may be transformations which connect inertial frames which move with a constant relative velocity. In order to determine them, we consider two frames with relative velocity v along the x

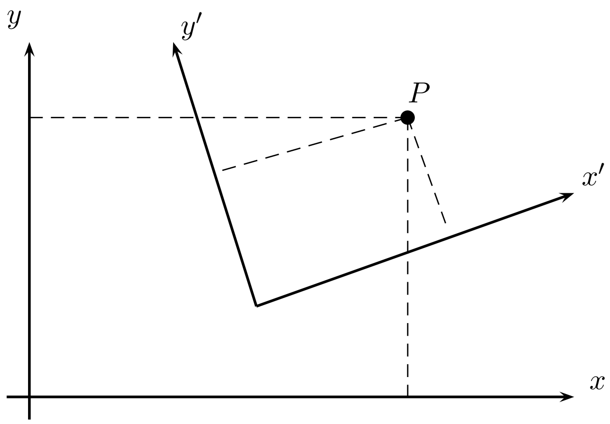


Figure 1.1: The point P is invariant, but its coordinates (x, y) and (x', y') differ in the two coordinate systems.

direction: The most general linear[1] transformation between these two frames is given by

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} At + Bx \\ Dt + Ex \\ y \\ z \end{pmatrix} = \begin{pmatrix} At + Bx \\ A(x - vt) \\ y \\ z \end{pmatrix}. \tag{1.5}$$

In the second step, we used that the transformation matrix depends only on two constants, as you should show in Ex. 1.3.

Newton assumed the existence of an absolute time, $t = t'$, and thus $A = 1$ and $B = 0$. Then proper Galilean transformations $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{v}t$ connect inertial frames moving with relative speed $\boldsymbol{v}$. Taking a time derivative leads to the classical addition law for velocities, $\dot{\boldsymbol{x}}' = \dot{\boldsymbol{x}} + \boldsymbol{v}$. Time differences Δt_{12} and space differences $\Delta \boldsymbol{l}_{12}$ are separately invariant under these transformations.

The *Principle of Relativity* states that *identical experiments performed in different inertial frames give identical results.* Galilean transformations keep (1.3) invariant, hence Newton's first law does not allow to distinguish between different inertial frames. Before the advent of special relativity, it was thought that this principle applies only to mechanical experiments. In particular, it was thaught that electrodynamic waves require a medium (the "aether") to propagate: thence the rest frame of the aether could be used to single out a preferred frame.

Newton's *Lex Secunda* states that *observed from an inertial reference frame, the net force on a particle is proportional to the time rate of change of its linear momentum,*

$$\boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} \tag{1.6}$$

where $\boldsymbol{p} = m_{\mathrm{in}}\boldsymbol{v}$ and m_{in} denotes the inertial mass of the body.

[1] A non-linear transformation would destroy translation invariance, for a formal proof that the transformation has to be linear see exercise 3.1

Newtonian gravity Newton's gravitational law as well as Coulomb's law are examples for an instantaneous force,

$$\boldsymbol{F}(\boldsymbol{x}) = \sum_i K_i \frac{\boldsymbol{x} - \boldsymbol{x}_i}{|\boldsymbol{x} - \boldsymbol{x}_i|^3} \,. \tag{1.7}$$

The force $\boldsymbol{F}(\boldsymbol{x}, t)$ depends on the distance $\boldsymbol{x}(t) - \boldsymbol{x}_i(t)$ to all sources i (electric charges or masses) at the same time t, i.e. the force needs no time to be transmitted from $\boldsymbol{x}_i$ to $\boldsymbol{x}$. The factor K in Newton's law is $-Gm_{\mathrm{g}}M_{\mathrm{g}}$, where we introduced analogue to the electric charge in the Coulomb law the gravitational "charge" m_{g} characterizing the strength of the gravitational force between different particles. Surprisingly, one finds $m_{\mathrm{in}} = m_{\mathrm{g}}$ and we can drop the two indices.

Since the gravitational field is conservative, $\boldsymbol{\nabla} \times \boldsymbol{F} = 0$, we can introduce a potential ϕ via

$$\boldsymbol{F} = -m\boldsymbol{\nabla}\phi \tag{1.8}$$

with

$$\phi(x) = -\frac{GM}{|\boldsymbol{x} - \boldsymbol{x}'|} \,. \tag{1.9}$$

Analogue to the electric field $\boldsymbol{E} = -\boldsymbol{\nabla}\phi$ we can introduce a gravitational field, $\boldsymbol{g} = -\boldsymbol{\nabla}\phi$. We then obtain $\boldsymbol{\nabla} \cdot \boldsymbol{g}(\boldsymbol{x}) = -4\pi G\rho(\boldsymbol{x})$ and as Poisson equation,

$$\Delta\phi(\boldsymbol{x}) = 4\pi G\rho(\boldsymbol{x}) \,, \tag{1.10}$$

where ρ is the mass density, $\rho = \mathrm{d}m/\mathrm{d}^3x$. Similiarly as the full Maxwell equations reduce in the $v/c \to 0$ to the electrostatic Poisson equation, a relativistic generalisation of Newtonian gravity should exist.

1.2 Minkowski space

Light cone and metric tensor A light-signal emitted at the $\boldsymbol{x}_1$ at the time t_1 propagates along a cone defined by

$$(ct_1 - ct_2)^2 - (x_1 - x_2)^2 - (y_1 - y_2)^2 - (z_1 - z_2)^2 = 0. \tag{1.11}$$

In special relativity, we postulate that the speed of light is universal, i.e. that all observers measure $c = c'$. A condition which guarantees this and generalizes Eq. (1.11) is that the squared distance in an inertial frame

$$\Delta s^2 \equiv (ct_1 - ct_2)^2 - (x_1 - x_2)^2 - (y_1 - y_2)^2 - (z_1 - z_2)^2 \tag{1.12}$$

between two spacetime events $x_1^\mu = (ct_1, \boldsymbol{x}_1)$ and $x_2^\mu = (ct_2, \boldsymbol{x}_2)$ is invariant. Hence the symmetry group of space and time is given by all those coordinate transformations $x^\mu \to \tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu$ that keep Δs^2 invariant. Since these transformation mix space and time, we speak about spacetime or, to honor the inventor of this geometrical interpretation, about Minkowski space.

The distance of two infinitesimally close spacetime events is called the line-element $\mathrm{d}s$ of the spacetime. In Minkowski space, it is given by

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 \tag{1.13}$$

using a Cartesian inertial frame. We can define the line-element d$\boldsymbol{s}$ i as the norm of the displacement vector

$$\mathrm{d}\boldsymbol{s} = \mathrm{d}s^\mu \boldsymbol{e}_\mu . \tag{1.14}$$

Choosing as basis the coordinate vectors $x^\mu = (ct, \boldsymbol{x})$, its components are

$$\mathrm{d}s^\mu = \mathrm{d}x^\mu = (c\mathrm{d}t, \mathrm{d}\boldsymbol{x}) . \tag{1.15}$$

We compare now our physical requirement on the distance of spacetime events, Eq. (1.13), with the general result for the scalar product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. If these vectors have the coordinates a^i and b^i in a certain basis $\boldsymbol{e}_i$, then we can write

$$\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{\mu,\nu=0}^{3} (a^\mu \boldsymbol{e}_\mu) \cdot (b^\nu \boldsymbol{e}_\nu) = \sum_{\mu,\nu=0}^{3} a^\mu b^\nu (\boldsymbol{e}_\mu \cdot \boldsymbol{e}_\nu) . \tag{1.16}$$

Thus we can evaluate the scalar product between any two vectors, if we know the symmetric matrix g composed of the products of the basis vectors at all spacetime points x^μ,

$$g_{\mu\nu}(x) = \boldsymbol{e}_\mu(x) \cdot \boldsymbol{e}_\nu(x) = g_{\nu\mu}(x) . \tag{1.17}$$

This symmetric matrix $g_{\mu\nu}$ is called the *metric tensor*.

Applying this now for the displacement vector, we obtain

$$\mathrm{d}s^2 = \mathrm{d}\boldsymbol{s} \cdot \mathrm{d}\boldsymbol{s} = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu \stackrel{!}{=} c^2 \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 . \tag{1.18}$$

Hence the metric tensor $g_{\mu\nu}$ becomes diagonal for the special case of a Cartesian inertial frame in Minkowski space with elements

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \equiv \eta_{\mu\nu} . \tag{1.19}$$

Introducing Einstein's summation convention (cf. the box for details), we can rewrite the scalar product of two vectors with coordinates a^μ and b^μ as

$$a \cdot b \equiv \eta_{\mu\nu} a^\mu b^\nu = a_\nu b^\nu = a^\mu b_\mu . \tag{1.20}$$

In the last part of Eq. (1.20), we "lowered an index:" $a_\nu = \eta_{\mu\nu} a^\mu$ or $b_\mu = \eta_{\mu\nu} b^\nu$. Alternatively, we can introduce the opposite operation of rasing an index by $a^\mu = \eta^{\mu\nu} a_\nu$. Since raising and lowering are inverse operations, we have $\eta_{\mu\nu}\eta^{\nu\sigma} = \delta_\mu^\sigma$. Thus the elements of $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$ form inverse matrices. For the special case of a Cartesian intertial coordinate frame in Minkowski space, the two matrices $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$ have the same elements.

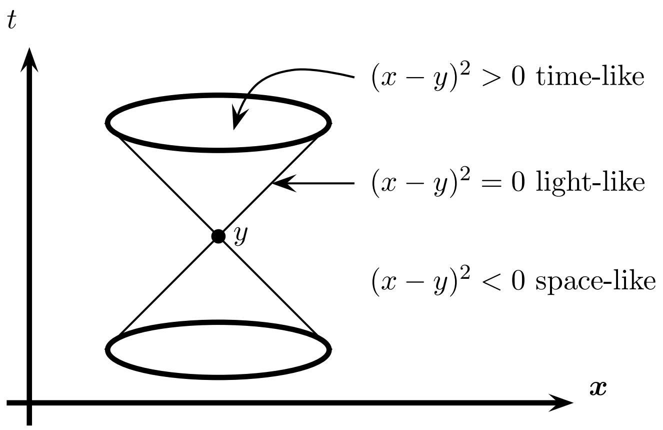


Figure 1.2: Light-cone at the point y generated by light-like vectors. Contained in the light-cone are the time-like vectors, outside the space-like ones.

Einstein's summation convention:

1. Two equal indices, of which one has to be an upper and one an lower index, imply summation. We use Greek letters for indices from zero to three, $\mu = 0, 1, 2, 3$, and Latin letters for indices from one to three, $i = 1, 2, 3$. Thus

$$a_\mu b^\mu \equiv \sum_{\mu=0}^{3} a_\mu b^\mu = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3 = a^0 b^0 - \boldsymbol{a} \cdot \boldsymbol{b} = a^0 b^0 - a^i b^i \,.$$

2. Summation indices are dummy indices which can be freely exchanged; the remaining free indices of the LHS and RHS of an equation have to agree. Hence

$$8 = a^\mu_\mu = c_{\mu\nu} d^{\mu\nu} = c_{\mu\sigma} d^{\mu\sigma}$$

is okay, while $a_\mu = b^\mu$ or $a^\mu = b^{\mu\nu}$ compares apples to oranges.

Since the metric $\eta_{\mu\nu}$ is indefinite, the norm of a vector a^μ can be

$$a_\mu a^\mu > 0, \qquad \text{time-like,} \tag{1.21}$$

$$a_\mu a^\mu = 0, \qquad \text{light-like or null-vector,} \tag{1.22}$$

$$a_\mu a^\mu < 0, \qquad \text{space-like.} \tag{1.23}$$

The cone of all light-like vectors starting from a point P is called *light-cone*, cf. Fig. 1.2. The time-like region inside the light-cone consists of two parts, past and future. Only events inside the past light-cone can influence the physics at point P, while P can influence only its future light-cone.

The line describing the position of an observer is called *world-line*. The *proper-time* τ is the time displayed by a clock moving with the observer. How can we determine the correct definition of τ? First, we ask that in the rest system of the observer, proper- and coordinate-time agree, $\mathrm{d}\tau = \mathrm{d}t$. But for a clock at rest, it is $\mathrm{d}s^\mu/c = (\mathrm{d}t, \mathbf{0})$ and thus $\mathrm{d}s/c = \mathrm{d}t$. Since the RHS of $\mathrm{d}\tau = \mathrm{d}s/c$ is an invariant expression, it has to be valid in any frame and thus also for a moving clock. For finite times, we have to integrate the line-element,

$$\tau_{12} = \int_1^2 \mathrm{d}\tau = \int_1^2 [\mathrm{d}t^2 - (\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)/c^2]^{1/2} \tag{1.24}$$

$$= \int_1^2 \mathrm{d}t\, [1 - (1/c^2)((\mathrm{d}x/\mathrm{d}t)^2 + (\mathrm{d}y/\mathrm{d}t)^2 + (\mathrm{d}z/\mathrm{d}t)^2)]^{1/2} \tag{1.25}$$

$$= \int_1^2 \mathrm{d}t\, [1 - v^2/c^2]^{1/2} < t_2 - t_1\,. \tag{1.26}$$

to obtain the proper-time. The last part of this equation, where we introduced the three-velocity $v^i = \mathrm{d}x^i/\mathrm{d}t$ of the clock, shows explicitly the relativistic effect of time dilation, as well as the connection between coordinate time t and the proper-time τ of a moving clock, $\mathrm{d}\tau = (1 - (v/c)^2)^{1/2}\mathrm{d}t \equiv \mathrm{d}t/\gamma$.

Lorentz transformations If we replace t by $-\mathrm{i}t$ in Δs^2, the difference between two spacetime events becomes (minus) the normal Euclidean distance. Thus we expect that a close correspondence exists between rotations R_{ij} in Euclidean space which leave $\Delta \boldsymbol{x}^2$ invariant and Lorentz transformations $\Lambda^\mu{}_\nu$ which leave Δs^2 invariant. Similarly, the replacement $t \to -\mathrm{i}t$ makes the velocity imaginary. This suggests that boosts are similar to a rotation by an imaginary angle $\eta = \mathrm{i}\alpha$. Then the identity $\cos^2\alpha + \sin^2\alpha = 1$ transforms for the imaginary angle $\eta = \mathrm{i}\alpha$ into $\cosh^2\eta - \sinh^2\eta = 1$. We try therefore as a guess for a boost along the x direction

$$\tilde{ct} = ct\cosh\eta + x\sinh\eta\,, \tag{1.27}$$

$$\tilde{x} = ct\sinh\eta + x\cosh\eta\,, \tag{1.28}$$

with $\tilde{y} = y$ and $\tilde{z} = z$. Direct calculation shows then that Δs^2 is invariant as desired. Consider now in the system $\tilde{K}$ the origin of the system K. Then $x = 0$ and

$$\tilde{x} = ct\sinh\eta \quad \text{and} \quad \tilde{ct} = ct\cosh\eta\,. \tag{1.29}$$

Dividing the two equations gives $\tilde{x}/\tilde{ct} = \tanh\eta$. Since $\beta = \tilde{x}/c\tilde{c}$ is the relative velocity of the two systems measured in units of c, the imaginary "rotation angle η" equals the rapidity

$$\eta = \operatorname{arctanh}\beta\,. \tag{1.30}$$

Note that the rapidity η is a more natural variable than v or β to characterise a Lorentz boost, because η is additive: Boosting a particle with rapidity $\boldsymbol{\eta}_1$ by $\boldsymbol{\eta}$ leads to the rapidity $\boldsymbol{\eta}_2 = \boldsymbol{\eta}_1 + \boldsymbol{\eta}$. Using the following identities,

$$\cosh\eta = \frac{1}{\sqrt{1-\tanh^2\eta}} = \frac{1}{\sqrt{1-\beta^2}} \equiv \gamma \tag{1.31}$$

$$\sinh\eta = \frac{\tanh\eta}{\sqrt{1-\tanh^2\eta}} = \frac{\beta}{\sqrt{1-\beta^2}} = \gamma\beta \tag{1.32}$$

in (1.27) gives the standard form of the Lorentz transformations,

$$\tilde{x} = \frac{x + vt}{\sqrt{1-\beta^2}} = \gamma(x + \beta ct) \tag{1.33}$$

$$\tilde{ct} = \frac{ct + vx/c}{\sqrt{1-\beta^2}} = \gamma(ct + \beta x)\,. \tag{1.34}$$

The inverse transformation is obtained by replacing $v \to -v$ and exchanging quantities with and without tilde.

In addition to boosts parametrised by the rapidity $\boldsymbol{\eta}$, rotations parametrised by the angle $\boldsymbol{\alpha}$ keep the spacetime distance invariant and are thus Lorentz transformations. For the special case of a boost along and a rotation around the x^1 axis, they are given in matrix form by

$$\Lambda^\mu{}_\nu(\eta_x) = \begin{pmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad \Lambda^\mu{}_\nu(\alpha_x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\alpha & \sin\alpha \\ 0 & 0 & -\sin\alpha & \cos\alpha \end{pmatrix}. \tag{1.35}$$

Four-vectors and tensors In Minkowski space, we call a four-vector any four-tupel V^μ that transforms as $\tilde{V}^\mu = \Lambda^\mu{}_\nu V^\nu$. By convention, we associate three-vectors with the spatial part of vectors with upper indices, e.g. we set $x^\mu = \{ct, x, y, z\}$ or $A^\mu = \{\phi, \boldsymbol{A}\}$. Lowering then the index by contraction with the metric tensor results in a minus sign of the spatial components of a four-vector, $x_\mu = \eta_{\mu\nu}x^\nu = \{ct, -x, -y, -z\}$ or $A_\mu = \{\phi, -\boldsymbol{A}\}$. Summing over a pair of Lorentz indices, always one index occurs in an upper and one in a lower position. Additionally to four-vectors, we will meet tensors $T^{\mu_1\cdots\mu_n}$ of rank n which transform as $\tilde{T}^{\mu_1\cdots\mu_n} = \Lambda^{\mu_1}{}_{\nu_1}\cdots\Lambda^{\mu_n}{}_{\nu_n}T^{\nu_1\cdots\nu_n}$. Every tensor index can be raised and lowered, using the metric tensors $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$.

A scalar is a tensor of rank $n = 0$; it is a single number which transforms trivially under Lorentz transformation. The simplest example is the mass of a particle. Example for a scalar field $\phi(x)$ is the pion or the Higgs field.

Special tensors are the Kronecker delta, $\delta_\mu^\nu = \eta_\mu^\nu$ with $\delta_\mu^\nu = 1$ for $\mu = \nu$ and 0 otherwise, and the Levi–Civita tensor $\varepsilon_{\mu\nu\rho\sigma}$. The latter tensor is completely antisymmetric and has in four dimensions the elements $+1$ for an even permutation of ε_{0123}, -1 for odd permutations and zero otherwise. In three dimensions, we define the Levi–Civita tensor by $\varepsilon^{123} = \varepsilon_{123} = 1$.

Next consider differential operators. Forming the differential of a function $f(x^\mu)$ defined on Minkowski space,

$$\mathrm{d}f = \frac{\partial f}{\partial t}\,\mathrm{d}t + \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z = \frac{\partial f}{\partial x^\mu}\,\mathrm{d}x^\mu\,, \tag{1.36}$$

we see that an upper index in the denominator counts as lower index, and vice versa. We define the four-dimensional nabla operator as

$$\frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \equiv \partial_\mu\,.$$

Note the "missing" minus sign in the spatial components, which is consistent with the rule for the differential in Eq. (1.36). The notations $\partial_\mu = \frac{\partial}{\partial x^\mu}$ makes it explicit that the index μ counts as an lower one. Finally, note that $\partial_\mu \mathrm{d}x^\nu = \delta_\mu^\nu$.

The d'Alembert or wave operator is

$$\Box \equiv \eta_{\mu\nu}\partial^\mu\partial^\nu = \partial_\mu\partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\,. \tag{1.37}$$

This operator is a scalar, i.e. all the Lorentz indices are contracted, and thus invariant under Lorentz transformations.

1.3 Relativistic mechanics

From now on, we use natural units and set $c = \hbar = k_B = 1$. In the next chapters, we set also $G_N = 1$, implying that we measure all quantities in Planck units. Starting from chapter 7, we keep $G_N \neq 1$. This implies that a single dimensionfull unit, which we choose typically as a mass scale, can be used to characterise all dimensionfull quantities.

Four-velocity and four-momentum What is the relativistic generalization of the three-velocity $\boldsymbol{v} = \mathrm{d}\boldsymbol{x}/\mathrm{d}t$? The nominator $\mathrm{d}\boldsymbol{x}$ has already the right behaviour to become part of a four-vector, if the denominator would be invariant. We use therefore instead of $\mathrm{d}t$ the invariant proper time $\mathrm{d}\tau$ and write

$$u^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\,. \tag{1.38}$$

The four-velocity is thus the tangent vector to the world-line $x^\alpha(\tau)$ parametrised by the proper-time τ of a particle. Written explicitly, we have

$$u^0 = \frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{1-v^2}} = \gamma \tag{1.39}$$

and

$$u^i = \frac{\mathrm{d}x^i}{\mathrm{d}\tau} = \frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{v^i}{\sqrt{1-v^2}} = \gamma v^i\,. \tag{1.40}$$

Hence the four-velocity is $u^\alpha = (\gamma, \gamma\boldsymbol{v})$ and its norm is

$$\boldsymbol{u}\cdot\boldsymbol{u} = u^0u^0 - u^iu^i = \gamma^2 - \gamma^2v^2 = \gamma^2(1-v^2) = 1\,. \tag{1.41}$$

Thus the four-velocity is a unit tangent vector and contains only three independent elements.

Energy and momentum After having constructed the four-velocity, the simplest guess for the four-momentum is

$$p^\alpha = mu^\alpha = (\gamma m, \gamma m\boldsymbol{v})\,. \tag{1.42}$$

For small velocities, $v \ll 1$, we obtain

$$p^i \;=\; \left(1 + \frac{v^2}{2} - \ldots\right) mv^i \tag{1.43}$$

$$p^0 \;=\; m + \frac{mv^2}{2} - \ldots = m + E_{\mathrm{kin,nr}} + \ldots \tag{1.44}$$

Thus we can interpret the components as $p^\alpha = (E, \boldsymbol{p})$. The norm follows with (1.41) immediately as

$$\boldsymbol{p}\cdot\boldsymbol{p} = m^2\,. \tag{1.45}$$

Solving for the energy, we obtain

$$E = \pm\sqrt{m^2 + p^ip^i} \tag{1.46}$$

including the famous $E = mc^2$ as special case for a particle at rest. Note that (1.46) predicts the existence of solutions with negative energy—undermining the stability of the universe. According Feynman, we should view these negative energy solutions as positive energy solutions moving backward in time, $\exp(-\mathrm{i}(-\sqrt{m^2+\boldsymbol{p}^2})t) = \exp[-\mathrm{i}(+\sqrt{m^2+\boldsymbol{p}^2})(-t)]$.

Four-forces We postulate now that in relativistic mechanics Newton's law becomes

$$f^\alpha = \frac{\mathrm{d}p^\alpha}{\mathrm{d}\tau} \tag{1.47}$$

where we introduced the four-force f^α. Since both u^α and p^α consist of only three independent components, we expect that there exists also a constraint on the four-force f^α. We form the scalar product

$$\boldsymbol{u}\cdot\boldsymbol{f} = \boldsymbol{u}\cdot\frac{\mathrm{d}(m\boldsymbol{u})}{\mathrm{d}\tau} = \boldsymbol{u}\cdot\boldsymbol{u}\frac{\mathrm{d}m}{\mathrm{d}\tau} + m\boldsymbol{u}\cdot\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}\tau} = \frac{\mathrm{d}m}{\mathrm{d}\tau}\,. \tag{1.48}$$

In the last step we used twice that $\boldsymbol{u}\cdot\boldsymbol{u} = 1$. Since all electrons ever observed have the same mass, no force should exist which changes m. As a consequence, we have to ask that all physical acceptable force-laws satisfy $\boldsymbol{u}\cdot\boldsymbol{f} = 0$; such forces are called *pure forces*. Moreover, $\boldsymbol{u}\cdot\boldsymbol{f} = 0$ implies $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{u})$ and thus all four-forces have to be velocity dependent.

1.A Appendix: Practising with tensors

How to guess physical tensors Classical electrodynamics is typically taught using a formulation which is valid in a specific frame. Thus one uses scalars like the charge density ρ, vectors like the electric and magnetic field strengths $\boldsymbol{E}$ and $\boldsymbol{B}$ and tensors like Maxwell's stress tensor σ_{ij}, defining their transformation properties with respect to rotations in three-dimensional space. This leads to the question how we can guess how the four-dimensional tensors are composed out of their three-dimensional relatives.

In the simplest cases, we may guess this by considering quantities which are related by a physical law. An example is current conservation,

$$\partial_t\rho + \boldsymbol{\nabla}\cdot\boldsymbol{j} = 0. \tag{1.49}$$

We know that any 4-vector a^μ has $4 = 3+1$ components, which transform as a 3-scalar (a^0) and a 3-vector ($\boldsymbol{a}$) under rotations. This suggests to combine $(\rho, \boldsymbol{j}) = j^\mu$ and $\partial_\mu = (\partial_t, \boldsymbol{\nabla})$ into four-vectors (consistent with our definition of the nabla operator), leading to $\partial_\mu j^\mu = 0$. Similarly, we combine the scalar potential ϕ and the vector potential $\boldsymbol{A}$ into a four-vector $A^\mu = (\phi, \boldsymbol{A})$. If we move to tensors of rank two, i.e. 4×4 matrices, it is useful to formalise the splitting of such a tensor.

Reduicible and irreduicible tensors An object which contains invariant subgroups with respect to a symmetry operation is called reducible. In our case at hand, we want to determine the irreducible subgroups of a tensor of rank n with respect to spatial rotations. We start with an arbitrary tensor $T^{\mu\nu}$ of rank two. First, we note that we can split any tensor $T^{\mu\nu}$ into a symmetric and an antisymmetric part, $T^{\mu\nu} = S^{\mu\nu} + A^{\mu\nu}$ with $S^{\mu\nu} = S^{\nu\mu}$ and $A^{\mu\nu} = -A^{\nu\mu}$, writing

$$T_{\mu\nu} = \frac{1}{2}\left(T_{\mu\nu} + T_{\nu\mu}\right) + \frac{1}{2}\left(T_{\mu\nu} - T_{\nu\mu}\right) \equiv T_{\{\mu\nu\}} + T_{[\mu\nu]} \equiv S_{\mu\nu} + A_{\mu\nu}. \tag{1.50}$$

This splitting is invariant under general coordinate transformations, and thus also under rotations, cf. with exercise. 12. Physically this expected, since our equations tell us that some quantities are antisymmetric (e.g. the field-strength tensor $F^{\mu\nu}$), while others are symmetric (e.g. Maxwell's stress tensor σ_{ij}) and all observers should agree on this.

Thus we can examine the symmetric and antisymmetric tensors seperately, and we start with the former. We can split $S^{\mu\nu}$ into a scalar S^{00}, a three-vector S^{0i} and a three-tensor S^{ij},

$$S^{\mu\nu} = \begin{pmatrix} S^{00} & S^{0i} \\ S^{i0} & S^{ij} \end{pmatrix}. \tag{1.51}$$

To show that the three pieces have the claimed transformation properties under rotations, calculate simply the effect of a rotation, $\tilde{S}_{\mu\nu} = \Lambda_\mu^{\ \rho}\Lambda_\nu^{\ \sigma} S_{\rho\sigma}$, or in matrix notation $S' = \Lambda S \Lambda^T$, where for a rotation

$$\Lambda_\mu^{\ \nu} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0}^T & R \end{pmatrix}. \tag{1.52}$$

The tensor S^{ij} is again reducible, since its trace is a scalar. Thus we can decompose S^{ij} into its trace $s = S^{ii}$ and its traceless part $S^i_j - s\delta^i_j/(d-1)$.

An antisymmetric tensor $A_{\mu\nu}$ has $3+2+1=6$ components, i.e. such a tensor combines two 3-vectors, or more precisely a pure vector like $\boldsymbol{E}$ and an axial vector like $\boldsymbol{B}$ (where we use names motivated by the electrodynamics),

$$A^{\mu\nu} = \begin{pmatrix} 0 & -E^i \\ E^i & \mathcal{B}^{ij} \end{pmatrix} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -\mathcal{B}^{12} & \mathcal{B}^{13} \\ E_y & \mathcal{B}^{12} & 0 & -\mathcal{B}^{23} \\ E_z & -\mathcal{B}^{13} & \mathcal{B}^{23} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}. \tag{1.53}$$

Here, we used that we can map an antisymmetric tensor in $d=3$ on an axial vector,

$$B_i = \frac{1}{2}\varepsilon_{ijk}\mathcal{B}^{jk}.$$

(Anti-) symmetrisation Finally let us note some useful relations for contractions involving symmetric and antisymmetric tensors. First, they are "orthogonal" in the sense that the contraction of a symmetric tensor $S_{\mu\nu}$ with an antisymmetric tensor $A_{\mu\nu}$ gives zero,

$$S_{\mu\nu}A^{\mu\nu} = 0. \tag{1.54}$$

This allows one to (anti-) symmetrize the contraction of an arbitrary tensor $C_{\mu\nu}$ with an (anti-) symmetric tensor: First split $C_{\mu\nu}$ into symmetric and antisymmetric parts,

$$C_{\mu\nu} = \frac{1}{2}\left(C_{\mu\nu} + C_{\nu\mu}\right) + \frac{1}{2}\left(C_{\mu\nu} - C_{\nu\mu}\right) \equiv C_{\{\mu\nu\}} + C_{[\mu\nu]}. \tag{1.55}$$

Then

$$S_{\mu\nu}C^{\mu\nu} = S_{\mu\nu}C^{\{\mu\nu\}} \quad \text{and} \quad A_{\mu\nu}C^{\mu\nu} = A_{\mu\nu}C^{[\mu\nu]}. \tag{1.56}$$

Index gymnastics We are mainly concerned with vectors and tensors of rank two. In this case we can express all equations as matrix operations. For instance, lowering the index of a vector, $A_\mu = \eta_{\mu\nu}A^\nu$, becomes

$$A_\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} = \begin{pmatrix} A^0 \\ -A^1 \\ -A^2 \\ -A^3 \end{pmatrix}.$$

Raising and lowering indices is the inverse, and thus $\eta_{\mu\nu}\eta^{\nu\sigma} = \delta_\mu^\sigma$. In matrix notation,

$$\eta\eta^{-1} = \mathbf{1}.$$

We can view $\eta_{\mu\nu}\eta^{\nu\sigma} = \delta_\mu^\sigma$ as the operation of raising an index of $\eta_{\mu\nu}$ (or lowering an index of $\eta^{\mu\nu}$): in both cases, we see that the Kronecker delta corresponds to the metric tensor with mixed indices, $\delta_\mu^\sigma = \eta_\mu^\sigma$.
The expression for the line-element becomes

$$\begin{aligned} \mathrm{d}s^2 &= \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = \mathrm{d}x^\mu\eta_{\mu\nu}\mathrm{d}x^\nu = \left(\mathrm{d}x^0, \mathrm{d}x^1, \mathrm{d}x^2, \mathrm{d}x^3\right)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} \mathrm{d}x^0 \\ \mathrm{d}x^1 \\ \mathrm{d}x^2 \\ \mathrm{d}x^3 \end{pmatrix} \\ &= (\mathrm{d}x^0)^2 - (\mathrm{d}x^1)^2 - (\mathrm{d}x^2)^2 - (\mathrm{d}x^3)^2. \end{aligned}$$

For a second-rank tensor, raising one index gives

$$T_{\mu\nu} = \eta_{\mu\rho}T^\rho{}_\nu = T^\rho{}_\nu\eta_{\mu\rho} \neq \eta_{\mu\rho}T_\nu{}^\rho = T_{\nu\mu}$$

Note that the order of tensors does not matter, but the order of indices does. If we move to matrix notation, we have to restore the right order. Raising next the second index,

$$T_{\mu\nu} = \eta_{\mu\rho}\eta_{\nu\sigma}T^{\rho\sigma}$$

we have to re-order it as $T_{\mu\nu} = \eta_{\mu\rho}T^{\rho\sigma}\eta_{\sigma\nu}$ in matrix notation (using that η is symmetric).
We apply this to the field-strength tensor: Starting from $F^{\mu\nu}$, we want to construct $F_{\mu\nu} = \eta_{\mu\rho}F^{\rho\sigma}\eta_{\sigma\nu}$,

$$\begin{aligned} F_{\mu\nu} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}. \end{aligned} \tag{1.57}$$

Note the general behaviour: The F^{00} element and the 3-tensor F^{ik} are multiplied by 1^2 and $(-1)^2$, respectively and do not change sign. The 3-vector F^{0k} is multiplied by $(-1)(+1)$ and does change sign.

Next we want to construct a Lorentz scalar out of $F^{\mu\nu}$. A Lorentz scalar has no indices, so we contract the two indices, $\eta_{\mu\nu}F^{\mu\nu} = F_\mu{}^\mu$. This is invariant, but zero (and thus not useful) because $F^{\mu\nu}$ is antisymmetric. As next try, we construct a Lorentz scalar S using two F's: Multiplying the two matrices $F_{\mu\nu}$ and $F^{\mu\nu}$, and taking then the trace, gives

$$S = F_{\mu\nu}F^{\mu\nu} = -\mathrm{tr}\{F_{\mu\nu}F^{\nu\rho}\} = -\mathrm{tr}\begin{pmatrix} \boldsymbol{E}\cdot\boldsymbol{E} & & & \\ & E_x^2 - B_z^2 - B_y^2 & & \\ & & E_y^2 - B_z^2 - B_x^2 & \\ & & & E_z^2 - B_y^2 - B_x^2 \end{pmatrix}$$

i.e. $S = -2(\boldsymbol{E}\cdot\boldsymbol{E} - \boldsymbol{B}\cdot\boldsymbol{B})$. Note the minus, since we have to change the order of indices in the second F.

Note also that S has to be a bilinear in $\boldsymbol{E}$ and $\boldsymbol{B}$ and invariant under rotations. Thus the only possible terms entering S are the scalar products $\boldsymbol{E}\cdot\boldsymbol{E}$, $\boldsymbol{B}\cdot\boldsymbol{B}$ and $\boldsymbol{E}\cdot\boldsymbol{B}$. Since $\boldsymbol{B}$ is a polar (or axial) vector, $P\boldsymbol{B} = \boldsymbol{B}$, the last term is a pseudo-scalar and cannot enter the scalar S.

Now we become more ambitious, looking at a tensor with 4 indices, the Levi–Civita or completely antisymmetric tensor $\varepsilon^{\alpha\beta\gamma\delta}$ in four dimensions, with

$$\varepsilon_{0123} = +1, \tag{1.58}$$

and all even permutations, -1 for odd permutations and zero otherwise. We lower its indices,

$$\varepsilon^{\alpha\beta\gamma\delta} = \varepsilon_{\bar\alpha\bar\beta\bar\gamma\bar\delta}\eta^{\bar\alpha\alpha}\eta^{\bar\beta\beta}\eta^{\bar\gamma\gamma}\eta^{\bar\delta\delta}$$

and consider the 0123 element using that the metric is diagonal,

$$\varepsilon^{0123} = +1\eta^{00}\eta^{11}\eta^{22}\eta^{33} = -1. \tag{1.59}$$

Thus in 4 dimensions, $\varepsilon^{\alpha\beta\gamma\delta}$ and $\varepsilon_{\alpha\beta\gamma\delta}$ have opposite signs.

We can use the Levi-Civita tensor to define the dual field-strength tensor

$$\tilde F^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}.$$

How to find the elements of this? Using simply the definitions,

$$\tilde F_{01} = \frac{1}{2}\Big(\underbrace{\varepsilon_{0123}}_{1}\underbrace{F^{23}}_{-B_x} + \varepsilon_{0132}F^{32}\Big) = -B_x$$

$$\tilde F_{12} = \frac{1}{2}\Big(\varepsilon_{1203}F^{03} + \varepsilon_{1230}F^{30}\Big) = -E_z$$

etc., gives

$$\tilde F_{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & -E_z & E_y \\ B_y & E_z & 0 & -E_x \\ B_z & -E_y & E_x & 0 \end{pmatrix} \quad \text{and} \quad \tilde F^{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{pmatrix}.$$

The dual field-strength tensor is useful, because the homogeneous Maxwell equation

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0 \tag{1.60}$$

becomes simply

$$\partial_\alpha \tilde F^{\alpha\beta} = 0\,. \tag{1.61}$$

Inserting the potential, we obtain zero,

$$\partial_\alpha \tilde F^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}\partial_\alpha F_{\gamma\delta} = \varepsilon^{\alpha\beta\gamma\delta}\partial_\alpha\partial_\gamma A_\delta = 0\,, \tag{1.62}$$

because we contract a symmetric tensor ($\partial_\alpha\partial_\gamma$) with an anti-symmetric one ($\varepsilon^{\alpha\beta\gamma\delta}$).

Having $F^{\mu\nu}$ and $\tilde{F}_{\mu\nu}$, we can form another (pseudo-) scalar, $A = \tilde{F}_{\mu\nu}F^{\mu\nu}$. Multiplying the two matrices $\tilde{F}_{\mu\nu}$ and $F^{\mu\nu}$, and taking then the trace, gives

$$\tilde{F}_{\mu\nu}F^{\mu\nu} = -\mathrm{tr}\{\tilde{F}_{\mu\nu}F^{\nu\rho}\} = \mathrm{tr}\begin{pmatrix} \boldsymbol{B}\cdot\boldsymbol{E} & & & \\ & \boldsymbol{B}\cdot\boldsymbol{E} & & \\ & & \boldsymbol{B}\cdot\boldsymbol{E} & \\ & & & \boldsymbol{B}\cdot\boldsymbol{E} \end{pmatrix}$$

i.e. $\tilde{F}_{\mu\nu}F^{\mu\nu} = 4\boldsymbol{E}\cdot\boldsymbol{B}$. We know that $\boldsymbol{E}\cdot\boldsymbol{B}$ is a pseudo-scalar. This tells us that including the Levi-Civita tensor converts a tensor into a pseudo-tensor, which does not change sign under a parity transformation $P\boldsymbol{x} = -\boldsymbol{x}$. (This analogous to $B_i = \varepsilon_{ijk}\partial_j A_k$, which converts two pure vectors into an axial one.)

1.B Appendix: Some transformation properties

While the transformation properties of some quantities like the charge or energy density under Lorentz boosts are rather obvious, the behavior of other observables like the intensity or the emissivity of radiation under Lorentz transformations is less trivial to determine. Other quantities like the relative velocity loose even their intuitive meaning we are used from our non-relativistic experience. In this appendix, we discuss therefore the transformation properties of some quantities appearing often in applications of astrophysics and high-energy physics.

Integration measure

Radiation

Scattering 2-2 scattering, Mandelstam variables, relative velocity
threshold energies

Problems

1.1 Transformation between inertial frames Consider two inertial frames K and K' with parallel axes at $t = t' = 0$ that are moving with the relative velocity v in the x direction.
a.) Show that the linear transformation between the coordinates in K and K' can be expressed by Eq. (1.5). b.) Show that requiring the invariance of

$$\Delta s^2 \equiv c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2 \tag{1.63}$$

leads to Lorentz transformations.

1.2 Splitting into (anti-) symmetric parts Show that the splitting of the arbitrary tensor $T^{\mu\nu}$ into its symmetric part $S^{\mu\nu} = S^{\nu\mu}$ and its anti-symmetric part $A^{\mu\nu} = -A^{\nu\mu}$ is invariant under Lorentz transformations.

2 Lagrangian mechanics and symmetries

We review briefly the Lagrangian formulation of classical mechanics and it connection to symmetries.

2.1 Calculus of variations

A map $F[f(x)]$ from a certain space of functions $f(x)$ into $\mathbb{R}$ is called a functional. We will consider functionals from the space $C_2[a:b]$ of (at least) twice differentiable functions between fixed points a and b. Extrema of functionals are obtained by the calculus of variations. Let us consider as functional the action S defined by

$$S[L(q^i,\dot{q}^i,t)] = \int_a^b \mathrm{d}t\, L(q^i,\dot{q}^i,t)\,, \tag{2.1}$$

where L is a function of the $2n$ independent functions q^i and $\dot{q}^i = \mathrm{d}q^i/\mathrm{d}t$ as well as of the parameter t. In classical mechanics, we call L the Lagrange function of the system, q are its generalised coordinates, $\dot{q}^i$ the corresponding velocities and t is the time. The extremum of this action gives the paths from a to b which are solutions of the equations of motion for the system described by L. We discuss in the next section how one derives the correct L given a set of interactions and constraints.

The calculus of variations shows how one find those paths that extremize such functionals: Consider an infinitesimal variation of the path, $q^i(t) \to q^i(t) + \delta q^i(t)$ with $\delta q^i(t) = \varepsilon\eta^i(t)$ that keeps the endpoints fixed, but is otherwise arbitrary. Thus we do *not* vary the velocities $\dot{q}^i$ independently, considering the action as a functional of only the coordinates, $S = S[q^i]$. The resulting variation of the functional is

$$\delta S[q^i] = \int_a^b \mathrm{d}t\, \delta L(q^i,\dot{q}^i,t) = \int_a^b \mathrm{d}t\, \left(\frac{\partial L}{\partial q^i}\delta q^i + \frac{\partial L}{\partial \dot{q}^i}\delta \dot{q}^i\right)\,. \tag{2.2}$$

We can eliminate the dependent variation $\delta\dot{q}^i$ of the velocities, integrating the second term by parts using $\delta(\dot{q}^i) = \mathrm{d}/\mathrm{d}t(\delta q^i)$,

$$\delta S[q^i] = \int_a^b \mathrm{d}t\, \left[\frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right)\right]\delta q^i + \left[\frac{\partial L}{\partial \dot{q}^i}\delta q^i\right]_a^b\,. \tag{2.3}$$

The boundary term vanishes, because we required that the variations δq^i are zero at the endpoints a and b. Since the variations are otherwise arbitrary, the terms in the first bracket have to be zero for an extremal curve, $\delta S = 0$. Paths that satisfy $\delta S = 0$ are classically allowed. The equations resulting from the condition $\delta S = 0$ are called the Euler-Lagrange equations of the action S,

$$\boxed{\frac{\delta S}{\delta q^i} = \frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i} = 0\,,} \tag{2.4}$$

and give the equations of motion of the system specified by L. Physicists call these equations often simply Lagrange equations or, especially in classical mechanics, Lagrange equations of the second kind.

The Lagrangian L is not uniquely fixed: Adding a total time-derivative, $L' = L + \mathrm{d}f(q,t)/\mathrm{d}t$ does not change the resulting Lagrange equations,

$$S' = S + \int_a^b \mathrm{d}t\, \frac{\mathrm{d}f}{\mathrm{d}t} = S + f(q(b), t_b) - f(q(a), t_a)\,, \tag{2.5}$$

since the last two terms vanish varying the action with the restriction of fixed endpoints a and b.

Remark 2.1: Infinitesimal variation—If you are worried about the meaning of "infinitesimal" variations, the following definition may help: Consider an one-parameter family of paths,

$$q^i(t,\varepsilon) = q^i(t,0) + \varepsilon\eta^i(t).$$

Then the "infinitesimal" variation δq corresponds to the change linear in ε,

$$\delta q \equiv \lim_{\varepsilon\to 0} \frac{q(t,\varepsilon) - q(t,0)}{\varepsilon} = \left.\frac{\partial q(t,\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0}, \tag{2.6}$$

and similarly for functions and functionals of q. Moreover, it is obvious from Eq. (2.6) that the assumption of time-independent ε implies that the variation δ and the time-derivative $\mathrm{d}/\mathrm{d}t$ acting on q commute,

$$\delta(\dot{q}) = \left.\frac{\partial\dot{q}(t,\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0} = \frac{\mathrm{d}}{\mathrm{d}t}(\delta q)\,,$$

2.2 Hamilton's principle and the Lagrange function

The observation that the solution of the equations of motion can be obtained as the extrema of an appropriate functional ("the action S") of the Lagrangian L subject to the conditions $\delta q^i(a) = \delta q^i(b) = 0$ is called *Hamilton's principle* or the *principle of least action*. Note that the last name is a misnomer, since the the extremum can be also a maximum or saddle-point of the action.

We derive now the Lagrangian L of a free non-relativistic particle from the Galilean principle of inertia. More precisely, we use that the homogeneity of space and time forbids that L depends on $\boldsymbol{x}$ and t, while the isotropy of space implies that L depends only on the norm of the velocity vector, but not on its direction,

$$L = L(v^2).$$

Let us consider two inertial frames moving with the infinitesimal velocity $\boldsymbol{\varepsilon}$ relative to each other. Then a Galilean transformation connects the velocities measured in the two frames as $\boldsymbol{v}' = \boldsymbol{v} + \boldsymbol{\varepsilon}$. The Galilean principle of relativity requires that the laws of motion have the same form in both frames, and thus the Langrangians can differ only by a total time-derivative. Expanding the difference δL in ε gives with $\delta v^2 = 2\boldsymbol{v}\boldsymbol{\varepsilon}$

$$\delta L = \frac{\partial L}{\partial v^2}\,\delta v^2 = 2\boldsymbol{v}\boldsymbol{\varepsilon}\frac{\partial L}{\partial v^2}\,. \tag{2.7}$$

The difference has to be a total time-derivative. Since $v = \dot{q}$, the derivative term $\partial L/\partial v^2$ has to be independent of $\boldsymbol{v}$. Hence, $L \propto v^2$ and we call the proportionality constant $m/2$, and the total expression kinetic energy T,

$$L = T = \frac{1}{2}mv^2 . \tag{2.8}$$

Example 2.1: Check the relativity principle for finite relative velocities:
Evaluating

$$L' = \frac{1}{2}mv'^2 = \frac{1}{2}m(\boldsymbol{v} + \boldsymbol{V})^2 = \frac{1}{2}mv^2 + m\boldsymbol{v}\cdot\boldsymbol{V} + \frac{1}{2}mV^2,$$

we can write

$$L' = L + \frac{\mathrm{d}}{\mathrm{d}t}\left(m\boldsymbol{x}\cdot\boldsymbol{V} + \frac{1}{2}mV^2t\right) .$$

Thus the difference is indeed a total time derivative.

We can write the velocity with $\mathrm{d}l^2 = dx^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ as

$$v^2 = \frac{\mathrm{d}l^2}{\mathrm{d}t^2} = g_{ik}\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^k}{\mathrm{d}t}, , \tag{2.9}$$

where the quadratic form g_{ik} is the metric tensor of the configuration space $\{q^i\}$. For instance, in spherical coordinates $\mathrm{d}l^2 = \mathrm{d}r^2 + r^2\sin^2\vartheta\mathrm{d}\phi^2 + r^2\mathrm{d}\vartheta^2$ and thus

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\sin^2\vartheta\dot{\phi}^2 + r^2\dot{\vartheta}^2\right) . \tag{2.10}$$

Choosing the appropriate coordinates, we can account for constraints: The kinetic energy of a particle moving on sphere with radius R would be simply given by $T = mR^2(\sin^2\vartheta\dot{\phi}^2 + \dot{\vartheta}^2)/2$.

For a system of non-interacting particles, L is additive, $L = \sum_a \frac{1}{2}m_a v_a^2$. If there are interactions (assumed for the moment to dependent only on the coordinates), then we subtract a function $V(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots)$ called potential energy.

We can now derive the equations of motion for a system of n interacting particles,

$$L = \sum_{a=1}^{n}\frac{1}{2}m_a v_a^2 - V(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_n) . \tag{2.11}$$

using the Lagrange equations,

$$m_a\frac{\mathrm{d}\boldsymbol{v}_a}{\mathrm{d}t} = -\frac{\partial V}{\partial \boldsymbol{r}_a} = \boldsymbol{F}_a . \tag{2.12}$$

We can change from Cartesian coordinates to arbitrary (or "generalized") coordinates for the n particles,

$$x^a = f^a(q^1, \ldots, q^n), \qquad \dot{x}^a = \frac{\partial f^a}{\partial q^k}\dot{q}^k . \tag{2.13}$$

Substituting gives

$$L = \frac{1}{2}a_{ik}\dot{q}^i\dot{q}^k - V(q^i) , \tag{2.14}$$

where the matrix $a_{ik}(q)$ is a quadratic function of the velocities $\dot{q}^i$ that is apart from the factors m_a identical to the metric tensor on the configuration space $\{q^i\}$. Finally, we define the canonically conjugated momentum p_i as

$$p_i = \frac{\partial L}{\partial \dot{q}^i} \,. \tag{2.15}$$

A coordinate q_i that does not appear explicitly in L is called cyclic. The Lagrange equations imply then $\partial L/\partial \dot{q}_i = \text{const.}$, so that the corresponding canonically conjugated momentum $p_i = \partial L/\partial \dot{q}^i$ is conserved.

Remark 2.2: Feynman's approach to quantum theory—The whole information about a quantum mechanical system is contained in its time-evolution operator $U(t,t')$. Its matrix elements $K(x',t';x,t)$ (propagator or Green's function) in the coordinate basis relate wavefunctions at different times as

$$\psi(x',t') = \int \mathrm{d}^3x K(x',t';x,t)\psi(x,t) \,.$$

Following a pretty vague suggestion by Dirac, Feynman proposed the following connection between the propagator K and the classical action S,

$$K(x',t';x,t) = N \int_x^{x'} \mathcal{D}q \, \exp(\mathrm{i}S) \,,$$

where $\mathcal{D}q$ denotes the "integration over all paths." Hence the difference between the classical and quantum world is that in the former only paths extremizing the action S are allowed while in the latter all paths weighted by $\exp(\mathrm{i}S)$ contribute.

For a readable introduction see R. P. Feynman, A. R. Hibbs: *Quantum mechanics and path integrals* or R. P. Feynman (editor: Laurie M. Brown), *Feynman's thesis : a new approach to quantum theory.*

2.3 Symmetries and conservation laws

Quantities that remain constant during the evolution of a mechanical system are called integrals of motion. Seven of them that are connected to the fundamental symmetries of space and time are of special importance: These are the conserved quantities energy, momentum and angular momentum.

Energy The Lagrangian of a closed system depends, because of the homogeneity of time, not on time. Its total time derivative is

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\partial L}{\partial q^i}\,\dot{q}^i + \frac{\partial L}{\partial \dot{q}^i}\,\ddot{q}^i \,. \tag{2.16}$$

Replacing $\partial L/\partial q^i$ by $(\mathrm{d}/\mathrm{d}t)\partial L/\partial \dot{q}^i$, it follows

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \dot{q}^i\,\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i} + \frac{\partial L}{\partial \dot{q}^i}\,\ddot{q}^i = \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{q}^i\,\frac{\partial L}{\partial \dot{q}^i}\right) \,. \tag{2.17}$$

Hence the quantity

$$E \equiv \dot{q}^i\,\frac{\partial L}{\partial \dot{q}^i} - L \tag{2.18}$$

remains constant during the evolution of a closed system. This holds also more generally, e.g. in the presence of static external fields, as long as the Lagrangian is not time-dependent.

We have still to show that E coincides indeed with the usual definition of energy. Using as $L = T(q,\dot{q}) - V(q)$, where T is quadratic in the velocities, we have

$$\dot{q}^i \frac{\partial L}{\partial \dot{q}^i} = \dot{q}^i \frac{\partial T}{\partial \dot{q}^i} = 2T. \tag{2.19}$$

Thus $E = 2T - L = T + V$ and the total energy E is the sum of kinetic and potential energy.

Momentum Homogeneity of space implies that a translation by a constant vector of a closed system does not change its properties. Thus an infinitesimal translation from $\boldsymbol{r}$ to $\boldsymbol{r}+\boldsymbol{\varepsilon}$ should not change L. Since velocities are unchanged, we have (summation over a particles)

$$\delta L = \sum_a \frac{\partial L}{\partial \boldsymbol{r}_a} \cdot \delta \boldsymbol{r}_a = \boldsymbol{\varepsilon} \cdot \sum_a \frac{\partial L}{\partial \boldsymbol{r}_a} \,. \tag{2.20}$$

The condition $\delta L = 0$ is true for arbitrary $\boldsymbol{\varepsilon}$, if

$$\sum_a \frac{\partial L}{\partial \boldsymbol{r}_a} = 0 \,. \tag{2.21}$$

Using again Lagrange's equations, we obtain

$$\sum_a \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \boldsymbol{v}_a} = \frac{\mathrm{d}}{\mathrm{d}t} \sum_a \frac{\partial L}{\partial \boldsymbol{v}_a} = 0 \,. \tag{2.22}$$

Hence, in a closed mechanical system the *total momentum* vector of the system

$$\boldsymbol{p}_{\mathrm{tot}} = \sum_a \frac{\partial L}{\partial \boldsymbol{v}_a} = \sum_a m_a \boldsymbol{v}_a = \text{const.} \tag{2.23}$$

is conserved.

The condition (2.21) signifies with $\partial L/\partial \boldsymbol{r}_a = -\partial V/\partial \boldsymbol{r}_a$ that the sum of forces on all particles is zero, $\sum_a \boldsymbol{F}_a = 0$. For the particular case of a two-particle system, $\boldsymbol{F}_a = -\boldsymbol{F}_b$, we have thus derived Newton's third law, the equality of action and reaction.

Isotropy We consider now the consequences of the isotropy of space, i.e. search the conserved quantity that follows from a Lagrangian invariant under rotations. Under an infinitesimal rotation by $\delta\boldsymbol{\phi}$ both coordinates and velocities change,

$$\begin{aligned} \delta \boldsymbol{r} &= \delta\boldsymbol{\phi} \times \boldsymbol{r} \,, & (2.24) \\ \delta \boldsymbol{v} &= \delta\boldsymbol{\phi} \times \boldsymbol{v} \,. & (2.25) \end{aligned}$$

Inserting the expression into

$$\delta L = \sum_a \left(\frac{\partial L}{\partial \boldsymbol{r}_a} \delta \boldsymbol{r}_a + \frac{\partial L}{\partial \boldsymbol{v}_a} \delta \boldsymbol{v}_a \right) = 0 \tag{2.26}$$

gives, using also the definition $\boldsymbol{p}_a = \partial L/\partial \boldsymbol{v}_a$ as well as the Lagrange equation $\dot{\boldsymbol{p}}_a = \partial L/\partial \boldsymbol{r}_a$,

$$\delta L = \sum_a (\dot{\boldsymbol{p}}_a \cdot \delta\boldsymbol{\phi} \times \boldsymbol{r}_a + \boldsymbol{p}_a \cdot \delta\boldsymbol{\phi} \times \boldsymbol{v}_a) = 0\,. \tag{2.27}$$

Permuting the factors and extracting $\delta\boldsymbol{\phi}$ gives

$$\delta\boldsymbol{\phi} \cdot \sum_a (\boldsymbol{r}_a \times \dot{\boldsymbol{p}}_a + \boldsymbol{v}_a \times \boldsymbol{p}_a) = \delta\boldsymbol{\phi} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \sum_a \boldsymbol{r}_a \times \boldsymbol{p}_a = 0\,. \tag{2.28}$$

Thus the total *angular momentum*

$$\boldsymbol{M} = \sum_a \boldsymbol{r}_a \times \boldsymbol{p}_a = \text{const.} \tag{2.29}$$

is conserved.

General formulation We can derive a general condition for the existence of conserved quantities for a Lagrangian system. Under a continous symmetry, the Lagrangian can change at most by a total derivative, and thus

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\partial L}{\partial q^i}\,\dot{q}^i + \frac{\partial L}{\partial \dot{q}^i}\,\ddot{q}^i = \frac{\mathrm{d}L'}{\mathrm{d}t} + \frac{\mathrm{d}f}{\mathrm{d}t}. \tag{2.30}$$

Now we use that Lagrange equation and apply the Leibniz product rule to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_k}\delta q_k - f\right) = 0. \tag{2.31}$$

This is Noether's theorem for a Lagrangian system with a finite number of degrees of freedom.

2.4 Two-body problem

General two-body problem We consider now for illustration the important example of two bodies interacting via a potential,

$$L = \frac{1}{2}m_1\dot{\boldsymbol{r}}_1^2 + \frac{1}{2}m_2\dot{\boldsymbol{r}}_2^2 - V(\boldsymbol{r}_1, \boldsymbol{r}_2). \tag{2.32}$$

Thus the Lagrangian L depends on six variables. Introducing the center-of-mass (CoM) coordinate

$$\boldsymbol{R} = \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{m_1 + m_2} \tag{2.33}$$

and the total mass $M = m_1 + m_2$, it follows

$$\boldsymbol{r}_1 = \boldsymbol{R} + \frac{m_2}{M}\boldsymbol{r} \quad \text{and} \quad \boldsymbol{r}_2 = \boldsymbol{R} - \frac{m_1}{M}\boldsymbol{r}. \tag{2.34}$$

Differenting these expressions, we can eleminate $\dot{\boldsymbol{r}}_1$ and $\dot{\boldsymbol{r}}_2$, obtaining

$$L = \frac{1}{2}M\dot{\boldsymbol{R}}^2 + \frac{1}{2}\mu\dot{\boldsymbol{r}}^2 - V(\boldsymbol{r}, \boldsymbol{R}) \tag{2.35}$$

with the reduced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = \frac{m_1 m_2}{M}. \tag{2.36}$$

In the special case of a central potential, $V(\boldsymbol{r}_1, \boldsymbol{r}_2) = V(r)$, the CoM coordinate $\boldsymbol{R}$ is cyclic and thus the CoM momentum $\boldsymbol{P} = \partial L/\partial\dot{\boldsymbol{R}}$ is clearly conserved. In general, $\boldsymbol{R}$ is however not cyclic, $V(\boldsymbol{r}_1, \boldsymbol{r}_2)$ breaks translation invariance and thus the total momentum is not conserved. We will now show that in the general case (2.32) the invariance under Galilean transformations leads to $\dot{\boldsymbol{R}} = \text{const}$.

Example 2.2: We apply Noether's theorem (2.31) to the case of Gallilean transformations, $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{V}t$. Then

$$\frac{\partial L}{\partial \dot{q}_k}\delta q_k = (m_1\dot{\boldsymbol{r}}_1 + m_2\dot{\boldsymbol{r}}_2)\boldsymbol{V}t = M\dot{\boldsymbol{R}}\cdot\boldsymbol{V}t.$$

We know already that the kinetic energy changes as $\delta T = (m\dot{\boldsymbol{r}}_1 + m_2\dot{\boldsymbol{r}}_{\boldsymbol{2}})\boldsymbol{V} = M\dot{\boldsymbol{R}}\cdot\boldsymbol{V}$. In contrast, the potential energy is invariant,

$$\delta V = \left(\frac{\partial V}{\partial \boldsymbol{r}_1} + \frac{\partial V}{\partial \boldsymbol{r}_2}\right)\boldsymbol{V}t = 0,$$

since $\boldsymbol{F}_a = -\boldsymbol{F}_b$. Combining the two terms, we obtain

$$\frac{\partial L}{\partial \dot{q}_k}\delta q_k - f = M\dot{\boldsymbol{R}}\boldsymbol{V}t - M\boldsymbol{R}\boldsymbol{V} = \text{const}. \tag{2.37}$$

Hence $\dot{\boldsymbol{R}}t - \boldsymbol{R} = \text{const.}$ or $\boldsymbol{R} = \dot{\boldsymbol{R}}t + \boldsymbol{R}_0$, implying that the velocity of the center-of-mass $\dot{\boldsymbol{R}}$ is constant.

Two-body problem with central forces Next we specialise the potential to the important example of two bodies interacting via a central force. In this case, we can introduce a potential $V(r)$ which depends only on the relative coordinate $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$,

$$L = \frac{1}{2}m_1\dot{\boldsymbol{r}}_1^2 + \frac{1}{2}m_2\dot{\boldsymbol{r}}_2^2 - V(r) = \frac{1}{2}M\dot{\boldsymbol{R}}^2 + \frac{1}{2}\mu\dot{\boldsymbol{r}}^2 - V(r). \tag{2.38}$$

Moving to the CoM frame, it is $\dot{\boldsymbol{R}} = 0$. Hence we have the two-body problem reduced to the one-body problem of a mass μ moving in the potential $V(r)$. We exploit now that the Lagrangian depends only on $r = |\boldsymbol{r}|$ and is thus invariant under rotations. We introduce therefore spherical coordinates,

$$L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\sin^2\vartheta\dot{\phi}^2 + r^2\dot{\vartheta}^2\right) - V(r). \tag{2.39}$$

The conservation of angular momentum $\boldsymbol{L}$ allows us to choose $\boldsymbol{L}\|\boldsymbol{e}_z$. Then the orbit is contained in the xy plane and $\vartheta = \pi/2$. The Lagrange equations become with $l = L/m$

$$r^2\dot{\phi} = l\,, \quad \text{and} \quad \ddot{r} = \frac{l}{r^3} + V'\,. \tag{2.40}$$

We use now energy conservation, $E = T + V$, and employ $\dot{\phi} = l/r^2$ to eleminate the angular dependence. This reduces the radial equation to an one-dimesnional motion in the radial direction with an effective potential,

$$E = \frac{1}{2}\mu\left(\dot{r}^2 + \frac{l^2}{r^2}\right) + V(r) = \frac{1}{2}\mu\dot{r}^2 + V_{\text{eff}} \tag{2.41}$$

with

$$V_{\text{eff}} = V(r) + \frac{l^2}{2r^2}. \tag{2.42}$$

Kepler problem We choose now as potential the Newtonian $V(r) = -G\mu M/r$, where we used $\mu M = m_1 m_2$ to eleminate the individual masses m_i. Then the Lagrange equations follow from $L = (1/2)\mu(\dot{r}^2 + r^2\dot{\phi}^2) + G\mu M/r$ as

$$r^2\dot{\phi} = l\,, \quad \text{and} \quad \ddot{r} = \frac{l}{r^3} - \frac{GM}{r^2}\,. \tag{2.43}$$

In order to derive the orbital equation $r = r(\phi)$, we eliminate the time t by

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\phi}\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\phi}\frac{l}{r^2} \equiv r'\,\frac{l}{r^2} \tag{2.44}$$

and introduce $u = 1/r$,

$$u'' + u = \frac{GM}{l^2}\,. \tag{2.45}$$

Inserting as trial solution the equation of a conic section,

$$u = \frac{1}{r} = \frac{1 + e\cos\phi}{a(1-e^2)}, \tag{2.46}$$

we find

$$u'' + u = \frac{1 + e\cos\phi - e\cos\phi}{a(1-e^2)} \stackrel{!}{=} \frac{GM\mu^2}{L^2}\,. \tag{2.47}$$

Thus we obtain as constraint for the angular momentum

$$L = \mu\sqrt{GMa(1-e^2)}\,. \tag{2.48}$$

The effective potential becomes for the Newtonian potential

$$V_{\text{eff}} = -\frac{G\mu M}{r} + \frac{l^2}{2r^2}. \tag{2.49}$$

For small r, the positive centrifugal potential dominates. Thus a particle with finite angular momentum, $l > 0$, cannot reach $r = 0$. The orbits of particles with negative energy, $E < 0$, are bounded, $r_{\text{min}} < r < r_{\text{max}}$, while they are unbound for $E > 0$.

Example 2.3: Total energy of a binary system.
Derive the velocity of the reduced mass μ at perihelion ($\vartheta = 0$) and at aphelion ($\vartheta = \pi$). Show that the total energy in the center-of-mass system can be expressed as $E = -GM\mu/(2a) = \langle V\rangle/2$.
At both points, v and r are perpendicular and thus $L = \mu r v$. Inserting L into Eq. (2.48) at perihelion and at aphelion, respectively, results in

$$v_p^2 = GM(1+e)/r_p \quad \text{and} \quad v_a^2 = GM(1-e)/r_a.$$

Using then (2.46) for an ellipse, it follows

$$v_p^2 = \frac{GM}{a}\left(\frac{1-e}{1+e}\right), \qquad v_a^2 = \frac{GM}{a}\left(\frac{1+e}{1-e}\right).$$

Evaluating the energy expression

$$E = \frac{1}{2}\mu v^2 - \frac{Gm\mu}{r}$$

for r_p and v_p gives

$$E = \frac{GM\mu}{a}\left[\frac{1}{2}\frac{1+e}{1-e} - \frac{1}{1-e}\right] = -\frac{GM\mu}{2a}. \tag{2.50}$$

Since $\langle 1/r\rangle = \langle u\rangle = 1/a$, the total energy corresponds to half the time-averaged potential energy, as expected from the virial theorem.

2.5 Free relativistic particle

Massive particles We introduced the proper-time τ to measure the time along the worldline of a massive particle,

$$\tau_{12} = \int_1^2 \mathrm{d}\tau = \int_1^2 [\mathrm{d}t^2 - (\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)]^{1/2} \tag{2.51}$$

$$= \int_1^2 [\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu]^{1/2}. \tag{2.52}$$

If we use a different parameter σ, e.g. such that $\sigma(\tau = 1) = 0$ and $\sigma(\tau = 2) = 1$, then

$$\tau_{12} = \int_0^1 \mathrm{d}\sigma \left[\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}\right]^{1/2}. \tag{2.53}$$

Note that τ_{12} is invariant under a reparameterisation $\sigma' = f(\sigma)$.

We check now if the choice

$$L = \left[\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}\right]^{1/2} \tag{2.54}$$

is sensible for a free particle[1]: L is Lorentz-invariant with $x^\mu = (\boldsymbol{x}, t)$ as dynamical variables, while σ plays the role of the parameter time t in the non-relativistic case. The Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\frac{\partial L}{\partial(\mathrm{d}x^\alpha/\mathrm{d}\sigma)} = \frac{\partial L}{\partial x^\alpha}. \tag{2.55}$$

Consider e.g. the x^1 component, then

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\frac{\partial L}{\partial(\mathrm{d}x^1/\mathrm{d}\sigma)} = \frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{1}{L}\frac{\mathrm{d}x^1}{\mathrm{d}\sigma}\right) = 0. \tag{2.56}$$

Since $L = \mathrm{d}\tau/\mathrm{d}\sigma$, it follows after multiplication with $\mathrm{d}\sigma/\mathrm{d}\tau$

$$\frac{\mathrm{d}^2 x^1}{\mathrm{d}\tau^2} = 0 \tag{2.57}$$

[1] We neglect the unimportant mass term

and the same for the other coordinates.

An alternative Lagrangian which we use latter more often is

$$L = \eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \tag{2.58}$$

with $\dot{x}^\mu = \mathrm{d}x^\nu/\mathrm{d}\tau$. Since this Lagrangian is the square-root of the one defined in Eq. (2.54) for the special choice $\sigma = \tau$, is it clear that the same equation of motion result. While this Lagrangian is more useful in calculations, it is invariant only under affine transformations, $\tau \to A\tau + B$.

Massless particles The energy-momentum relation of massless particles like the photon becomes $\omega = |\boldsymbol{k}|$. Thus their four-velocity and four-momenta are light-like, $\boldsymbol{u}^2 = \boldsymbol{p}^2 = 0$, and light signals form the future light-cone of the emission point P. Since $\mathrm{d}s = \mathrm{d}\tau = 0$ on the light-cone, we cannot use the Lagrangians (2.54) or (2.58).

To find an alternative, consider how we can parameterise the curve $x = t$. Choosing $u^\alpha = (1, 1, 0, 0)$, we can set

$$x^\alpha(\lambda) = \lambda u^\alpha. \tag{2.59}$$

Then the four-velocity becomes the tangent vector $u^\alpha = \mathrm{d}x^\alpha(\lambda)/\mathrm{d}\lambda$, similar to the definition (1.38) for massive particles. With the choice (2.59), the four-velocity for a massless particle satisfies

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}\lambda} = 0. \tag{2.60}$$

Such parameters are called *affine*, and the set of these parameters are invariant under affine transformations, $\lambda \to A\lambda + B$. In this case, we can use the same equations of motion for massive and massless particles, only replacing $\boldsymbol{u} \cdot \boldsymbol{u} = 1$ with $\boldsymbol{u} \cdot \boldsymbol{u} = 0$.

Problems

2.1 Oscillator with friction. Consider a one-dimensional system described by the Lagrangian $L = \exp(2\alpha t)L_0$ and $L_0 = \frac{1}{2}m\dot{q}^2 - V(q)$. a.) Show that the equation of motion corresponds to an oscillator with friction term. b). Derive the energy lost per time $\mathrm{d}E/\mathrm{d}t$ of the oscillator, with $E = \frac{1}{2}m\dot{q}^2 + V(q)$. c.) Show that the result in b.) agrees with the one obtained from the Lagrange equations of the first kind, $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = Q$, where the generalised force Q perform the work $\delta A = Q\delta q$.

2.2 Classical driven oscillator. Consider an harmonic oscillator satisfying $\ddot{q}(t) - \Omega^2 q(t) = 0$ for $0 < t < T$ and $\ddot{q}(t) + \omega^2 q(t) = 0$ otherwise, with ω and Ω as real constants. a.) Show that for $q(t) = A_1 \sin(\omega t)$ for $t < 0$ and $\Omega T \gg 1$, the solution $q(t) = A_2 \sin(\omega_0 t + \alpha)$ with $\alpha = \mathrm{const.}$ satisfies $A_2 \approx \frac{1}{2}(1 + \omega^2/\Omega^2)^{1/2}\exp(\Omega T)$. b.) If the oscillator was in the ground-state at $t < 0$, how many quanta are created?

2.3 Higher derivatives.
a.) Find the Lagrange equation for a Lagrangian containing higher derivatives, $L = L(q, \dot{q}, \ddot{q}, \ldots)$.
b.) Consider $L = L(q, \dot{q}, \ddot{q})$ choosing as canonical variables $Q_1 = q$, $Q_2 = \dot{q}$, $P_1 = \frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \ddot{q}}$ and $P_2 = \frac{\partial L}{\partial \ddot{q}}$ and defining as Hamiltonian $H(Q_1, Q_2, P_1, P_2) = \sum_{i=1}^2 P_i q^{(i)} - L$. Show that the resulting Hamilton equations give the correct time evolution and that H corresponds to the energy. Show that H is unbounded from below, i.e. that it describes a unstable system.

3 Basic differential geometry

3.1 Gravitation as a spacetime phenomenon

We motivate this chapter about differential geometry by giving some arguments why a relativistic theory of gravity should replace Minkowski space by a curved manifold. Let us start by reviewing three basic properties of gravitation.

1.) The idea underlying the equivalence principle emerged in the 16th century, when among others Galileo Galilei found experimentally that the acceleration g of a test mass in a gravitational field is universal. Because of this universality, the gravitating mass $m_g = F/g$ and the inertial mass $m_i = F/a$ are identical in classical mechanics, a fact that puzzled already Newton. While $m_i = m_g$ can be achieved for one material by a convenient choice of units, there should be in general deviations for test bodies with differing compositions.

   Knowing more forces, this puzzle becomes even stronger: Contrast the acceleration of a particle in a gravitational field to the one in a Coulomb field. In the latter case, two independent properties of the particle, namely its charge q determining the strength of the electric force acting on it and its mass m_i, i.e. the inertia of the particle, are needed as input in the equation of motion. In the case of gravity, the "gravitational charge" m_g coinicides with the inertial mass m_i.

   The equivalence of gravitating and inertial masses has been tested already by Newton and Bessel, comparing the period P of pendula of different materials,

$$P = 2\pi\sqrt{\frac{m_i l}{m_g g}}\,, \tag{3.1}$$

   but finding no measurable differences. The first precision experiment giving an upper limit on deviations from the equivalence principle was performed by Loránd Eötvös in 1908 using a torsion balance. Current limits for departures from universal gravitational attraction for different materials are $|\Delta g_i/g| < 10^{-12}$.

2.) Newton's gravitational law postulates as the latter Coulomb law an instantaneous interaction. Such an interaction is in contradiction to special relativity. Thus, as interactions of currents with electromagnetic fields replace the Coulomb law, a corresponding description should be found for gravity. Moreover, the equivalence of mass and energy found in special relativity requires that, in a loose sense, not only mass but all forms of energy should couple to gravity: Imagine a particle-antiparticle pair falling down a gravitational potential well, gaining energy and finally annihilating into two photons moving the gravitational potential well outwards. If the two photons would not loose energy climbing up the gravitational potential well, a perpetuum mobile could be constructed. If all forms of energy act as sources of gravity, then the gravitational field itself is gravitating. Thus the theory is non-linear and its mathematical structure is much more complicated than the one of electrodynamics.

3.) Gravity can be switched-off locally, just by cutting the rope of an elevator: Inside a freely falling elevator, one does not feel any gravitational effects except for tidal forces. The latter arise if the gravitational field is non-uniform and tries to stretch the elevator. Inside a sufficiently small freely falling system, also tidal effects plays no role. This allows us to perform experiments like the growing of crystalls in "zero-gravity" on the International Space Station which is orbiting around the Earth at an altitude of only 300 km.

Motivated by 2.), Einstein used 1.), the principle of equivalence, and 3.) to derive general relativity, a theory that describes the effect of gravity as a deformation of the space-time known from special relativity.

In general relativity, the gravitational force of Newton's theory that accelerates particles in an Euclidean space is replaced by a curved spacetime in which particles move force-free along geodesic lines. In particular, as in special relativity, photons still move along curves satisfying $\mathrm{d}s^2 = 0$, while all effects of gravity are now encoded in the non-Euclidean geometry of spacetime which is determined by the line element $\mathrm{d}s$ or the metric tensor $g_{\mu\nu}$,

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \,. \tag{3.2}$$

Switching on a gravitational field, the metric tensor $g_{\mu\nu}$ can be transformed only locally by a coordinate change into the form $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. Thus we should develop the tools necessary to undertake analysis on a curved manifold $\mathscr{M}$ which geometry is described by the metric tensor $g_{\mu\nu}$.

Remark 3.1: Information from quantum theory—Quantum theory provides three additional pieces of information on the possible form of the gravitational interaction. The first one comes from the low-energy interactions of massless particles with spin $s = 0, 1, 2, \ldots$ in Minkowski space [8]. In the low-energy limit, these particles see a classical current and, as a result, the probability amplitude for the emission or absorption of such a particle factorises into an universal factor and a remainder. While for massless spin $s = 0$ particles no constraint arises, massless spin $s = 1$ particles have to couple to a current j^μ which is conserved, $\partial_\mu j^\mu = 0$. Massless spin $s = 2$ particles have to couple again to a current $T^{\mu\nu}$ which is conserved, $\partial_\mu T^{\mu\nu} = 0$. In addition, now the coupling strength has to be universal. This implies the equivalence of inertial and gravitational mass. Finally, no consistent theory of interacting massless particles with spin $s \geq 3$ is possible.

Second, one can derive the potential energy between two static sources due to the exchange of particles with spin $s = 0, 1, 2, \ldots$. One finds that this energy is positiv for equal charges for odd spin; this corresponds to the fact that the Coulomb force between equal charges is repulsive. In contrast, the energy is negative for the exchange of particles with even spin. Since gravity is attractive, the force has to be mediated by particles with even spin. Finally, a long-range force $\propto 1/r^2$ implies that the exchange particle is massless.

Combining these pieces of information, we can conclude that the classical gravitational field is mediated by a massless spin-0 or spin-2 field. The equivalence principle allows us to inteprete gravity as a spacetime phenomen, suggesting to identify the gravitational field with the metric tensor $g_{\mu\nu}$.

3.2 Manifolds and tensor fields

Manifolds A manifold $\mathscr{M}$ is any set that can be continuously parameterised. The number of independent parameters needed to specify uniquely any point of $\mathscr{M}$ is its dimension n, the parameters $x = \{x^1, \dots, x^n\}$ are called coordinates. Locally, a manifold with dimension n can be approximated by $\mathbb{R}^n$. Examples for manifolds are the group of rotations in $\mathbb{R}^3$ (with three Euler angles, dim = 3), the configuration space q^i (with dim = n) or the phase space (q^i, p_i) (with dim = $2n$) of classical mechanics, and spacetime in general relativity. We require the manifold to be smooth. The transitions from one set of coordinates to another one, $x^i = f(\tilde{x}^i, \dots, \tilde{x}^n)$, should be C^∞. In general, it is impossible to cover all $\mathscr{M}$ with one coordinate system that is well-defined on all $\mathscr{M}$. An example are spherical coordinate (ϑ, ϕ) on a sphere S^2, where ϕ is ill-defined at the poles. Instead one has to cover the manifold with patches of different coordinates that partially overlap.

Vector fields A vector field $\boldsymbol{V}(x^\mu)$ on (a subset $\mathscr{S}$ of) $\mathcal{M}$ is a set of vectors associating to each spacetime point $x^\mu \in \mathscr{S}$ exactly one vector. The paradigm for such a vector field is the four-velocity $\boldsymbol{u}(\tau) = \mathrm{d}\boldsymbol{x}/\mathrm{d}\tau$ which is the tangent vector to the world-line $x(\tau)$ of a particle. Since the differential equation $\mathrm{d}\boldsymbol{x}/\mathrm{d}\sigma = \boldsymbol{X}(\sigma)$ has locally always a solution, we can find for any given $\boldsymbol{X}$ a curve $x(\sigma)$ which has $\boldsymbol{X}$ as tangent vector, and vice versa. Although the definition $\boldsymbol{u}(\tau) = \mathrm{d}\boldsymbol{x}/\mathrm{d}\tau$ coincides with the one familiar from Minkowski space, there is an important difference: In a general manifold, we cannot imagine a vector $\boldsymbol{V}$ as an "arrow" $\overrightarrow{PP'}$ pointing from a certain point P to another point P' of the manifold. Instead, the vectors $\boldsymbol{V}$ generated by all smooth curves through P span an n-dimensional vector space at the point P called tangent space $T_P\mathscr{M}$. We can visualise the tangent space for the case of a two-dimensional manifold embedded in $\mathbb{R}^3$: at any point P, the tangent vectors lie in a plane $\mathbb{R}^2$ which we can associate with T_P. In general, $T_P\mathscr{M} \neq T_{P'}\mathscr{M}$ and we cannot simply move a vector $\boldsymbol{V}(x^\mu)$ to another point $\tilde{x}^\mu$. This implies in particular that we cannot add the vectors $\boldsymbol{V}(x^\mu)$ and $\boldsymbol{V}(\tilde{x}^\mu)$, if the points x^μ and $\tilde{x}^\mu$ differ. Therefore we cannot differentiate a vector field without introducing an additional mathematical structure which allows us to transport a vector from one tangent space to another.

If we want to decompose the vector $\boldsymbol{V}(x^\mu)$ into components $V^\nu(x^\mu)$, we have to introduce a basis $\boldsymbol{e}_\mu$ in the tangent space. There are two natural choices for such a basis: First, we can use the Cartesian inertial system associated with an world-line passing through x^μ. Such a choice, which can model a local observer, will be discussed in section 3.4 when we discuss how one connects the components of tensor fields to measurements. In general, we use as basis vectors instead the tangential vectors along the coordinate lines x^μ in $\mathscr{M}$,

$$\boldsymbol{e}_\mu = \frac{\partial}{\partial x^\mu} \equiv \partial_\mu. \tag{3.3}$$

Here the index μ counts the different basis vector: Their only non-zero entry is the partial derivative w.r.t. to the μ.th coordinate, $\boldsymbol{e}_\mu = (0, \dots, \partial/\partial x^\mu, \dots 0)$. Using this basis, a vector can be decomposed as

$$\boldsymbol{V} = V^\mu \boldsymbol{e}_\mu = V^\mu \partial_\mu. \tag{3.4}$$

A coordinate change

$$x^\mu = f(\tilde{x}^1, \dots, \tilde{x}^n), \tag{3.5}$$

or more briefly $x^\mu = x^\mu(\tilde{x}^\nu)$, changes the basis vectors as

$$\boldsymbol{e}_\mu = \frac{\partial}{\partial x^\mu} = \frac{\partial \tilde{x}^\nu}{\partial x^\mu}\frac{\partial}{\partial \tilde{x}^\nu} = \frac{\partial \tilde{x}^\nu}{\partial x^\mu}\,\tilde{\boldsymbol{e}}_\nu. \tag{3.6}$$

Therefore the vector $\boldsymbol{V}$ will be invariant under general coordinate transformations,

$$\boldsymbol{V} = V^\mu \partial_\mu = \tilde{V}^\mu \tilde{\partial}_\mu = \tilde{\boldsymbol{V}}, \tag{3.7}$$

if its components transform opposite to the basis vectors $\boldsymbol{e}_\mu = \partial_\mu$, or

$$V^\mu = \frac{\partial x^\mu}{\partial \tilde{x}^\nu}\,\tilde{V}^\nu. \tag{3.8}$$

If x^μ and $\tilde{x}^\mu$ are two inertial frames in Minkowski space, we came back to Lorentz transformations $\partial x^\mu/\partial \tilde{x}^\nu = \Lambda^\mu{}_\nu$ as a special case of general coordinate transformations.

Remark 3.2: Vector fields as differential operators—The notation $\boldsymbol{V} = V^\mu\partial_\mu$ suggests that we can interprete a vector field as a differential operator. This allows us to define the commutator $[\boldsymbol{X}, \boldsymbol{Y}]$ of two vector fields by their action on scalar functions ϕ as

$$[\boldsymbol{X}, \boldsymbol{Y}]\phi = (\boldsymbol{XY} - \boldsymbol{YX})\phi. \tag{3.9}$$

Consider as an example the three space-like vector fields given by $\boldsymbol{J}_1 = y\partial_z - z\partial_y$ and its cyclic permutations. Simple calculation gives

$$\begin{aligned}(\boldsymbol{J}_1\boldsymbol{J}_2 - \boldsymbol{J}_2\boldsymbol{J}_1)\phi &= (\underline{y\partial_z z\partial_y} - y\partial_z x\partial_z - y\partial_z x\partial_z + z\partial_y x\partial_z)\phi \\ &\quad - (z\partial_x y\partial_z - z\partial_x z\partial_y - x\partial_z y\partial_z + \underline{x\partial_z z\partial_y})\phi = (y\partial_x - x\partial_y)\phi = -\boldsymbol{J}_3\phi.\end{aligned} \tag{3.10}$$

Here we used that partial derivatives commute and thus all second order terms cancel. As a result, only the terms linear in derivatives arising from the underlined terms survive. The other comutation relations are computed in the same way, leading to

$$[\boldsymbol{J}_i, \boldsymbol{J}_j] = -\varepsilon_{ijk}\boldsymbol{J}_k, \tag{3.11}$$

The replacement $\boldsymbol{J}_i \to \tilde{\boldsymbol{J}} = \mathrm{i}\boldsymbol{J}_i$ (which makes them hermitian operators) converts this into the usual commutatation relations for the angular momentum operators in quantum mechanics. There we learnt that the angular momentum operators are the infinitesimal generators of rotations, i.e. they transport the wave-function $\phi(x)$ along the vector field $\boldsymbol{J}$. This connection will become clearer in Example 4.1.

Covectors or one-forms In quantum mechanics, we use Dirac's bracket notation to associate to each vector $|a\rangle$ a dual vector $\langle a|$ and to introduce a scalar product $\langle a|\,b\rangle$. If the vectors $|n\rangle$ form a basis, then the dual basis $\langle n|$ is defined by $\langle n|\,n'\rangle = \delta_{nn'}$. Similarly, we define a basis $\boldsymbol{e}^\mu$ dual to the basis $\boldsymbol{e}_\mu$ in $T_P\mathscr{M}$ by

$$\boldsymbol{e}^\mu(\boldsymbol{e}_\nu) = \delta^\mu_\nu. \tag{3.12}$$

This basis can be used to form a new vector space $T_P^*\mathscr{M}$ called the cotangent space which is dual to $T_P\mathscr{M}$. Its elements $\boldsymbol{\omega}$ are called covectors or one-forms, and can be expanded as

$$\boldsymbol{\omega} = \omega_\mu \boldsymbol{e}^\mu. \tag{3.13}$$

Combining a vector and a one-form, we obtain a map into the real numbers,

$$\boldsymbol{\omega}(\boldsymbol{V}) = \omega_\mu V^\nu \boldsymbol{e}^\mu(\boldsymbol{e}_\nu) = \omega_\mu V^\mu. \tag{3.14}$$

The last equality shows that we can calculate $\boldsymbol{\omega}(\boldsymbol{V})$ in component form without reference to the basis vectors. In order to simplify notation, we will use therefore in the future simply $\omega_\mu V^\mu$; we also write $\boldsymbol{e}^\mu \boldsymbol{e}_\nu$ instead of $\boldsymbol{e}^\mu(\boldsymbol{e}_\nu)$.

Using a coordinate basis, the duality condition (3.12) is obviously satisfied, if we choose $\boldsymbol{e}^\mu = \mathrm{d}x^\mu$. Then the one-form $\boldsymbol{\omega}$ becomes

$$\boldsymbol{\omega} = \omega_\mu \mathrm{d}x^\mu \,. \tag{3.15}$$

Thus the familiar "infinitesimals" $\mathrm{d}x^\mu$ are actually the finite basis vectors of the cotangent space $T_P^*\mathscr{M}$. We require again that the transformation of the components ω_μ of a covector cancels the transformation of the basis vectors,

$$\omega_\mu = \frac{\partial \tilde{x}^\nu}{\partial x^\mu}\, \tilde{\omega}_\nu \,. \tag{3.16}$$

This condition guarantees that the covector itself is an invariant object, since

$$\boldsymbol{\omega} = \omega_\mu \mathrm{d}x^\mu = \frac{\partial \tilde{x}^\nu}{\partial x^\mu}\, \tilde{\omega}_\nu \frac{\partial x^\mu}{\partial \tilde{x}^\sigma}\, \mathrm{d}\tilde{x}^\sigma = \tilde{\omega}_\mu \mathrm{d}\tilde{x}^\mu = \tilde{\boldsymbol{\omega}} \,. \tag{3.17}$$

Covariant and contravariant tensors Next we generalise the concept of vectors and covectors. We call a vector $\boldsymbol{X}$ also a contravariant tensor of rank one, while we call a covector also a covariant vector or covariant tensor of rank one. A general tensor of rank (n, m) is a multilinear map

$$\boldsymbol{T} = T^{\mu,\dots,\nu}_{\alpha,\dots,\beta}\, \underbrace{\partial_\mu \otimes \ldots \otimes \partial_\nu}_{n} \otimes \underbrace{\mathrm{d}x^\alpha \otimes \ldots \otimes \mathrm{d}x^\beta}_{m} \tag{3.18}$$

which components transforms as

$$\tilde{T}^{\mu,\dots,\nu}_{\alpha,\dots,\beta}(\tilde{x}) = \underbrace{\frac{\partial \tilde{x}^\mu}{\partial x^\rho} \cdots \frac{\partial \tilde{x}^\nu}{\partial x^\sigma}}_{n}\, \underbrace{\frac{\partial x^\gamma}{\partial \tilde{x}^\alpha} \cdots \frac{\partial x^\delta}{\partial \tilde{x}^\beta}}_{m}\, T^{\rho,\dots,\sigma}_{\gamma,\dots,\delta}(x) \tag{3.19}$$

under a coordinate change.

Metric tensor A (pseudo-) Riemannian manifold is a differentiable manifold containing as additional structure a symmetric tensor field $g_{\mu\nu}$ which allows us to measure distances and angles. We define the scalar product of two vectors $\boldsymbol{a}(x)$ and $\boldsymbol{b}(x)$ which have the coordinates a^μ and b^μ in a certain basis $\boldsymbol{e}_\mu$ as

$$\boldsymbol{a} \cdot \boldsymbol{b} = (a^\mu \boldsymbol{e}_\mu) \cdot (b^\nu \boldsymbol{e}_\nu) = (\boldsymbol{e}_\mu \cdot \boldsymbol{e}_\nu) a^\mu b^\nu = g_{\mu\nu} a^\mu b^\nu. \tag{3.20}$$

Thus we can evaluate the scalar product between any two vectors, if we know the symmetric matrix $g_{\mu\nu}$ composed of the n^2 products of the basis vectors,

$$g_{\mu\nu}(x) = \boldsymbol{e}_\mu(x) \cdot \boldsymbol{e}_\nu(x), \tag{3.21}$$

at any point x of the manifold. This symmetric matrix $g_{\mu\nu}$ is called *metric tensor*. The manifold is called Riemannian if all eigenvalues of $g_{\mu\nu}$ are positive, and thus the scalar product defined by $g_{\mu\nu}$ is positive-definite. If the scalar product is indefinite, as in the case of general relativity, one calls the manifold pseudo-Riemannian.

In the same way, we define for the dual basis $\boldsymbol{e}^\mu$ the metric $g^{\mu\nu}$ via

$$g^{\mu\nu} = \boldsymbol{e}^\mu \cdot \boldsymbol{e}^\nu. \tag{3.22}$$

Now we want to show that the definitions (3.21) and (3.22) together with Eq. (3.12) imply that we can use the metric tensor to raise and lower indices. We set first

$$\boldsymbol{e}^\mu = A^{\mu\nu} \boldsymbol{e}_\nu\,, \tag{3.23}$$

with the tensor $A^{\mu\nu}$ to be determined. Then we form the scalar product with $\boldsymbol{e}^\rho$ and obtain

$$g^{\mu\rho} = \boldsymbol{e}^\mu \cdot \boldsymbol{e}^\rho = A^{\mu\nu} \boldsymbol{e}_\nu \cdot \boldsymbol{e}^\rho = A^{\mu\rho}\,. \tag{3.24}$$

Hence the metric $g^{\mu\nu}$ maps covariant vectors X_μ into contravariant vectors X^μ, while $g_{\mu\nu}$ provides a map into the opposite direction. In the same way, we can use the metric tensor to raise and lower indices of any tensor.

Next we want to determine the relation of $g^{\mu\nu}$ with $g_{\mu\nu}$. We multiply $\boldsymbol{e}^\rho$ with $\boldsymbol{e}_\mu = g_{\mu\nu}\boldsymbol{e}^\nu$, obtaining

$$\delta^\rho_\mu = \boldsymbol{e}^\rho \cdot \boldsymbol{e}_\mu = \boldsymbol{e}^\rho \cdot \boldsymbol{e}^\nu g_{\mu\nu} = g^{\rho\nu} g_{\mu\nu} \tag{3.25}$$

or

$$\delta^\rho_\mu = g_{\mu\nu} g^{\nu\rho}. \tag{3.26}$$

Thus the components of the covariant and the contravariant metric tensors, $g_{\mu\nu}$ and $g^{\mu\nu}$, are inverse matrices of each other. Moreover, the RHS corresponds to the rasing or lowering of an index in the metric tensor. Thus the mixed metric tensor of rank (1,1) is given by the Kronecker delta, $g^\nu_\mu = \delta^\nu_\mu$. Note that this implies that the trace of the metric tensor is not -2, but

$$\mathrm{tr}(g_{\mu\nu}) = g^{\mu\mu} g_{\mu\mu} = \delta^\mu_\mu = 4\,, \tag{3.27}$$

because we have to contract an upper and a lower index.

Example 3.1: Spherical coordinates 1: Calculate for spherical coordinates $x = (r, \vartheta, \phi)$ in $\mathbb{R}^3$,

$$\begin{aligned} x'_1 &= r \sin\vartheta \cos\phi\,, \\ x'_2 &= r \sin\vartheta \sin\phi\,, \\ x'_3 &= r \cos\vartheta\,, \end{aligned}$$

the components of g_{ij} and g^{ij}, and $g \equiv \det(g_{ij})$. From $\boldsymbol{e}_i = \partial x'^j/\partial x^i \boldsymbol{e}'_j$, it follows

$$\begin{aligned}
\boldsymbol{e}_1 &= \frac{\partial x'_j}{\partial r}\,\boldsymbol{e}'_j = \sin\vartheta\cos\phi\,\boldsymbol{e}'_1 + \sin\vartheta\sin\phi\,\boldsymbol{e}'_2 + \cos\vartheta\boldsymbol{e}'_3\,, \\
\boldsymbol{e}_2 &= \frac{\partial x'_j}{\partial\vartheta}\,\boldsymbol{e}'_j = r\cos\vartheta\cos\phi\,\boldsymbol{e}'_1 + r\cos\vartheta\sin\phi\,\boldsymbol{e}'_2 - r\sin\vartheta\boldsymbol{e}'_3\,, \\
\boldsymbol{e}_3 &= \frac{\partial x'_j}{\partial\phi}\,\boldsymbol{e}'_j = -r\sin\vartheta\sin\phi\,\boldsymbol{e}'_1 + r\sin\vartheta\cos\phi\,\boldsymbol{e}'_2\,.
\end{aligned}$$

Since the $\boldsymbol{e}_i$ are orthogonal to each other, the matrices g_{ij} and g^{ij} are diagonal. From the definition $g_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j$ one finds $g_{ij} = \mathrm{diag}(1, r^2, r^2\sin^2\vartheta)$ Inverting g_{ij} gives $g^{ij} = \mathrm{diag}(1, r^{-2}, r^{-2}\sin^{-2}\vartheta)$. The determinant is $g = \det(g_{ij}) = r^4\sin^2\vartheta$. Note that the volume integral in spherical coordinates is given by

$$\int d^3x' = \int d^3x\; J = \int d^3x\;\sqrt{g} = \int \mathrm{d}r\mathrm{d}\vartheta\mathrm{d}\phi\; r^2\sin\vartheta\,,$$

since $g_{ij} = \frac{\partial x^{k\prime}}{\partial\tilde{x}^i}\frac{\partial x^{l\prime}}{\partial\tilde{x}^j}\,g'_{kl}$ and thus $\det(g) = J^2\det(g') = J^2$ with $\det(g') = 1$.

3.3 Covariant derivative and the geodesic equation

Covariant derivative In an inertial system in Minkowski space, taking the partial derivative ∂_μ maps a tensor of rank (n, m) into a tensor of rank $(n, m+1)$. As required for any derivative, this map obeys linearity and the Leibniz product rule. We will see that in general the partial derivative in a curved space does not transform as tensor. We therefore introduce a new derivative ∇_μ called covariant derivative, modified such that it fulfils these rules.

Let us first see how a derivative is connected with the transport of tensors. Performing a Taylor expansion, we can move a scalar from the point x to $x + \mathrm{d}x$ as

$$f(x + \mathrm{d}x) = f(x) + \partial_\mu f(x)\mathrm{d}x^\mu + \ldots \tag{3.28}$$

In the case of a vector, we have to take into account that its components are given by the projection on the basis vectors in the two tangent spaces at x and $x + \mathrm{d}x$. If the basis vectors are rotated or boosted, an additional change Γ results. This change should be proportional to $\boldsymbol{A}$ and $\mathbf{d}\boldsymbol{s}$ and carries therefore two lower indices in addition to the upper index required by the LHS. Thus we introduce the connection (coefficients) ${\Gamma^\mu}_{\sigma\nu}$,

$$A^\mu(x + \mathrm{d}x) = A^\mu(x) + \left[\partial_\nu A^\mu(x) + {\Gamma^\mu}_{\sigma\nu}A^\sigma\right]\mathrm{d}x^\nu + \ldots = A^\mu(x) + \nabla_\nu A^\mu \mathrm{d}x^\nu + \ldots \tag{3.29}$$

and ask that the expression in the square bracket transforms as a tensor of rank $(1, 1)$. Any n^3 numbers ${\Gamma^\mu}_{\rho\sigma}$ which satisfy this requirement are called (affine) connection coefficients or symbols in order to stress that they are not the components of a tensor. Mathematically, an infinite number of choices for ${\Gamma^\mu}_{\sigma\nu}$ is consistent with the requirement that $\nabla_\nu A^\mu$ transforms as a tensor of rank $(1, 1)$ (and that ∇_μ is a derivative). These conditions define the set of affine connections on $\mathcal{M}$ which are briefly discussed in the appendix.

We will impose in the following two additional conditions.

- The length of a vector should remain constant being transported along the manifold. (Think about the four-momentum $g_{\mu\nu}p^\mu p^\nu = m^2$.) This requires that $\nabla_\sigma g_{\mu\nu} = \nabla_\sigma g^{\mu\nu} = 0$; A connection satisfying this condition is called metric compatible. In the derivation below, we will include this requirement by not differentiating the “dot” in a scalar product.

- A vector should not be twisted "unnecessarily" being transported along the manifold. If no such twist arises during parallel transport, the connection is called torsion free. A necessary condition for this is that the connection is symmetric, $\Gamma^{\sigma}{}_{\mu\nu} = \Gamma^{\sigma}{}_{\nu\mu}$.

We start by considering the gradient $\partial_\mu \phi$ of a scalar ϕ. By definition, a scalar quantity does not depend on the coordinate system, $\phi(x) = \tilde{\phi}(\tilde{x})$. Therefore its gradient transforms as

$$\partial_\mu \phi \to \tilde{\partial}_\mu \tilde{\phi} = \frac{\partial x^\nu}{\partial \tilde{x}^\mu} \partial_\nu \phi. \tag{3.30}$$

Thus the gradient is a covariant vector. Similarly, the derivative of a vector $\boldsymbol{V}$ transforms as a tensor,

$$\partial_\mu \boldsymbol{V} \to \tilde{\partial}_\mu \tilde{\boldsymbol{V}} = \frac{\partial x^\nu}{\partial \tilde{x}^\mu} \partial_\nu \boldsymbol{V}, \tag{3.31}$$

because $\boldsymbol{V}$ is an invariant quantity. If we consider however its components $V^\mu = \boldsymbol{e}^\mu \cdot \boldsymbol{V}$, then the moving coordinate basis in curved spacetime, $\partial_\mu \boldsymbol{e}^\nu \neq 0$, leads to an additional term in the derivative,

$$\partial_\mu V^\nu = \boldsymbol{e}^\nu \cdot (\partial_\mu \boldsymbol{V}) + \boldsymbol{V} \cdot (\partial_\mu \boldsymbol{e}^\nu). \tag{3.32}$$

The term $\boldsymbol{e}^\nu \cdot (\partial_\mu \boldsymbol{V})$ transforms as a tensor, since both $\boldsymbol{e}^\nu$ and $\partial_\mu \boldsymbol{V}$ are tensors. This implies that the combination of the two remaining terms has to transform as tensor too, which we define as the covariant derivative

$$\nabla_\mu V^\nu \equiv \boldsymbol{e}^\nu \cdot (\partial_\mu \boldsymbol{V}) = \partial_\mu V^\nu - \boldsymbol{V} \cdot (\partial_\mu \boldsymbol{e}^\nu). \tag{3.33}$$

The first equality tells us that we can view the covariant derivative $\nabla_\mu V^\nu$ as the projection of $\partial_\mu \boldsymbol{V}$ onto the direction $\boldsymbol{e}^\nu$.

We now expand the partial derivatives of the basis vectors as a linear combination of the basis vectors,

$$\partial_\rho \boldsymbol{e}^\mu = -\Gamma^{\mu}{}_{\rho\sigma} \boldsymbol{e}^\sigma \quad \text{and} \quad \partial_\rho \boldsymbol{e}_\mu = \Gamma^{\sigma}{}_{\rho\mu} \boldsymbol{e}_\sigma. \tag{3.34}$$

Comparing with Eq. (3.29), we see that the two definitions agree for a coordinate basis, $\boldsymbol{e}^\mu = \mathrm{d}x^\mu$. The coefficients $\Gamma^{\mu}{}_{\rho\sigma}$ in Eq. (3.34) are a special case of the generic affine connection.

Example 3.2: Show the validity of the RHS of Eq. (3.34):
The duality relation $\boldsymbol{e}_\mu \cdot \boldsymbol{e}^\nu = \delta^\nu_\mu$ leads to $\partial_\rho(\boldsymbol{e}_\mu \cdot \boldsymbol{e}^\nu) = 0$. Inserting the definition of the connection coefficients, it follows indeed

$$\partial_\rho(\boldsymbol{e}_\mu \cdot \boldsymbol{e}^\nu) = (\partial_\rho \boldsymbol{e}_\mu)\boldsymbol{e}^\nu + \boldsymbol{e}_\mu \cdot (\partial_\rho \boldsymbol{e}^\nu) = \Gamma^{\sigma}{}_{\rho\mu} \boldsymbol{e}_\sigma \cdot \boldsymbol{e}^\nu - \boldsymbol{e}_\mu \Gamma^{\nu}{}_{\rho\sigma} \cdot \boldsymbol{e}^\sigma = \Gamma^{\nu}{}_{\rho\mu} - \Gamma^{\nu}{}_{\rho\mu} = 0.$$

Introducing this expansion into (3.33) we can rewrite the covariant derivative of a vector field as

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^{\nu}{}_{\sigma\mu} V^\sigma, \tag{3.35}$$

and of a covector as

$$\nabla_\sigma X_\mu = \partial_\sigma X_\mu - \Gamma^{\nu}{}_{\mu\sigma} X_\nu \tag{3.36}$$

For a general tensor, the covariant derivative is defined by the same reasoning as

$$\nabla_\sigma T^{\mu\dots}_{\nu\dots} = \partial_\sigma T^{\mu\dots}_{\nu\dots} + \Gamma^{\mu}{}_{\rho\sigma} T^{\rho\dots}_{\nu\dots} + \dots - \Gamma^{\rho}{}_{\nu\sigma} T^{\mu\dots}_{\rho\dots} - \dots \tag{3.37}$$

Note that it is the last index of the connection coefficients that is the same as the index of the covariant derivative. The *p*lus sign goes together with upper (su*p*erscripts), the minus with lower indices.

Levi-Civita connection Next we want to find the relation between the connection coefficients and the metric tensor. We first differentiate the definition of the metric tensor, $g_{ab} = \boldsymbol{e}_a \cdot \boldsymbol{e}_b$, with respect to x^c,

$$\partial_c g_{ab} = (\partial_c \boldsymbol{e}_a) \cdot \boldsymbol{e}_b + \boldsymbol{e}_a \cdot (\partial_c \boldsymbol{e}_b) = \Gamma^d_{ac} \boldsymbol{e}_d \cdot \boldsymbol{e}_b + \boldsymbol{e}_a \Gamma^d_{bc} \boldsymbol{e}_d = \tag{3.38}$$

$$= \underline{\Gamma^d_{ac} g_{db}} + \underline{\underline{\Gamma^d_{bc} g_{ad}}} \,. \tag{3.39}$$

We obtain two equivalent expression by a cyclic permutation of the indices $\{a, b, c\}$,

$$\partial_b g_{ca} = \underline{\underline{\Gamma^d_{cb} g_{da}}} + \utilde{\Gamma^d_{ab} g_{cd}} \tag{3.40}$$

$$\partial_a g_{bc} = \utilde{\Gamma^d_{ba} g_{dc}} + \underline{\Gamma^d_{ca} g_{bd}} \,. \tag{3.41}$$

We add the first two terms and subtract the last one. Assuming additionally the symmetries $\Gamma^a_{bc} = \Gamma^a_{cb}$ and $g_{ab} = g_{ba}$, all terms except the single-underlined terms cancel, and dividing by two we obtain

$$\frac{1}{2}(\partial_c g_{ab} + \partial_b g_{ac} - \partial_a g_{bc}) = \Gamma^d{}_{cb} g_{ad} \,. \tag{3.42}$$

Multiplying by g^{ea} and relabeling indices gives as final result

$$\boxed{\Gamma^\mu{}_{\nu\lambda} = \left\{ {\mu \atop \nu\lambda} \right\} = \frac{1}{2} g^{\mu\kappa}(\partial_\nu g_{\kappa\lambda} + \partial_\lambda g_{\nu\kappa} - \partial_\kappa g_{\nu\lambda}).} \tag{3.43}$$

This equation defines the Levi-Civita connection (aka Christoffel symbols $\left\{ {a \atop bc} \right\}$ aka Riemannian connection): It is the unique connection on a Riemannian manifold which is metric compatible and torsion-free (i.e. symmetric). Admitting torsion, on the RHS of Eq. (3.43) three permutations of the torsion tensor T^a_{bc} would appear. Such a connection would be still be a metric connection, but not torsion-free.

We now check our claim that the connection (3.43) is metric compatible. We define[1]

$$\Gamma_{\mu\nu\lambda} = g_{\mu\kappa} \Gamma^\kappa{}_{\nu\lambda}. \tag{3.44}$$

Thus $\Gamma_{\mu\nu\lambda}$ is symmetric in the last two indices. Then it follows

$$\Gamma_{\mu\nu\lambda} = \frac{1}{2}(\partial_\nu g_{\mu\lambda} + \partial_\lambda g_{\nu\mu} - \partial_\mu g_{\nu\lambda}). \tag{3.45}$$

Adding $2\Gamma_{\mu\nu\lambda}$ and $2\Gamma_{\nu\mu\lambda}$ gives

$$\begin{aligned} 2(\Gamma_{\mu\nu\lambda} + \Gamma_{\nu\mu\lambda}) = {} & \partial_\nu g_{\mu\lambda} + \partial_\lambda g_{\nu\mu} - \partial_\mu g_{\nu\lambda} \\ & + \partial_\mu g_{\nu\lambda} + \partial_\lambda g_{\mu\nu} - \partial_\nu g_{\mu\lambda} = 2\partial_\lambda g_{\mu\nu} \end{aligned} \tag{3.46}$$

or

$$\partial_\lambda g_{\mu\nu} = \Gamma_{\mu\nu\lambda} + \Gamma_{\nu\mu\lambda} \,. \tag{3.47}$$

Applying the general rule for covariant derivatives, Eq. (3.37), to the metric,

$$\nabla_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\kappa{}_{\mu\lambda} g_{\kappa\nu} - \Gamma^\kappa{}_{\nu\lambda} g_{\mu\kappa} = \partial_\lambda g_{\mu\nu} - \Gamma_{\nu\mu\lambda} - \Gamma_{\mu\nu\lambda} \,, \tag{3.48}$$

[1] We showed that the metric tensor can be used to raise or to lower tensor indices, but the connection Γ is not a tensor.

and inserting Eq. (3.47) shows that

$$\nabla_\lambda g_{\mu\nu} = \nabla_\lambda g^{\mu\nu} = 0\,. \tag{3.49}$$

Hence ∇_λ commutes with contracting indices,

$$\nabla_\lambda(X^\mu X_\mu) = \nabla_\lambda(g_{\mu\nu}X^\mu X^\nu) = g_{\mu\nu}\nabla_\lambda(X^\mu X^\nu) \tag{3.50}$$

and conserves the norm of vectors as announced. Thus the Christoffel symbols are symmetric and compatible with the metric. These two properties specify uniquely the connection.

Since we can choose for a flat space an Cartesian coordinate system, the connection coefficients are zero and thus $\nabla_\mu = \partial_\mu$. This suggests as general rule that physical laws valid in Minkowski space hold in general relativity, if one replace ordinary derivatives by covariant ones and $\eta_{\mu\nu}$ by $g_{\mu\nu}$.

Geodesic equation In flat space, we know that the solution to the equation of motion of a free particle is a straight line. Such a path is characterised by two properties: it is the shortest curve between the considered initial and final point, and it is the curve whose tangent vector remains constant if it is parallel transported along it. Both conditions can be generalised to curved space and the curves satisfying either one of them are called geodesics. Using the definition of a geodesic as the "straightest" line on a manifold requires as mathematical structure only the possibility to parallel transporting a tensor and thus the existence of an affine connection. In contrast, the concept of an "extremal" (shortest or longest) line between two points on a manifold relies on the existence of a metric. Requiring that these two definitions agree fixes uniquely the connection to be used in the covariant derivative.

We start by defining a geodesics as the extremal curve between two points on a manifold. The Lagrangian of a free particle in Minkowski space, Eq. (2.58), is generalised to a curved spacetime manifold with the metric tensor $g_{\mu\nu}$ by replacing $\eta_{\mu\nu}$ with $g_{\mu\nu}$ (we set also $m = -1$),

$$L = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu. \tag{3.51}$$

The Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L}{\partial(\dot{x}^\lambda)} - \frac{\partial L}{\partial x^\lambda} = 0. \tag{3.52}$$

Only the metric tensor $g_{\mu\nu}$ depends on x^μ and thus $\partial L/\partial x^\lambda = g_{\mu\nu,\lambda}\dot{x}^\mu\dot{x}^\nu$. Here we also introduced the shorthand notation $g_{\mu\nu,\lambda} = \partial_\lambda g_{\mu\nu}$ for partial derivatives. Now we use $\partial\dot{x}^\mu/\partial\dot{x}^\nu = \delta^\mu_\nu$ and apply the chain rule for $g_{\mu\nu}(x(\sigma))$, obtaining first

$$g_{\mu\nu,\lambda}\dot{x}^\mu\dot{x}^\nu = 2\,\frac{\mathrm{d}}{\mathrm{d}\tau}\,(g_{\mu\lambda}\dot{x}^\mu) = 2(g_{\mu\lambda,\nu}\dot{x}^\mu\dot{x}^\nu + g_{\mu\lambda}\ddot{x}^\mu) \tag{3.53}$$

and then

$$g_{\mu\lambda}\ddot{x}^\mu + \frac{1}{2}(2g_{\mu\lambda,\nu} - g_{\mu\nu,\lambda})\dot{x}^\mu\dot{x}^\nu = 0. \tag{3.54}$$

Next we rewrite the second term as

$$2g_{\lambda\mu,\nu}\dot{x}^\mu\dot{x}^\nu = (g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu})\dot{x}^\mu\dot{x}^\nu, \tag{3.55}$$

multiply everything by $g^{\kappa\mu}$ and arrive at our desired result,

$$\ddot{x}^\kappa + \frac{1}{2}g^{\kappa\lambda}(g_{\mu\nu,\lambda} + g_{\mu\lambda,\nu} - g_{\mu\nu,\lambda})\dot{x}^\mu\dot{x}^\nu = \ddot{x}^\kappa + \Gamma^\kappa{}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0\,. \tag{3.56}$$

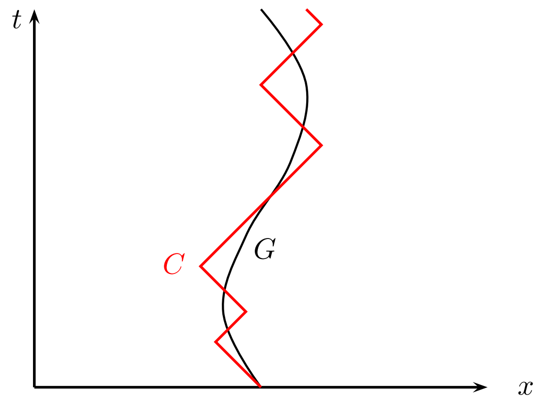


Figure 3.1: Approximation of a time-like geodesics G by adding zigzaging null curves C.

In the last step, we introduced the Levi–Civita connection.

This result justifies the use of a torsionless connection which is metric compatible: Although a star consists of a collection of individual particles carrying spin $\boldsymbol{s}_i$, its total spin sums up to zero, $\sum_i \boldsymbol{s}_i \approx, 0$, because the $\boldsymbol{s}_i$ are uncorrelated. Thus we can describe macrosopic matter in general relativity as a a classical spinless point particle (or fluid, if extended). In such a case, only the symmetric part of the connection influences the geodesic motion of the considered system.

Example 3.3: Calculate the Christoffel symbols of the two-dimensional unit sphere S^2.
The line-element of the two-dimensional unit sphere S^2 is given by $\mathrm{d}s^2 = \mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\phi^2$. A faster alternative to the definition (3.43) of the Christoffel coefficients is the use of the geodesic equation: From the Lagrange function $L = g_{ab}\dot{x}^a\dot{x}^b = \dot{\vartheta}^2 + \sin^2\vartheta\dot{\phi}^2$ we find

$$\frac{\partial L}{\partial \phi} = 0 \quad , \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\phi}} = \frac{\mathrm{d}}{\mathrm{d}t}(2\sin^2\vartheta\dot{\phi}) = 2\sin^2\vartheta\ddot{\phi} + 4\cos\vartheta\sin\vartheta\dot{\vartheta}\dot{\phi}$$

$$\frac{\partial L}{\partial \vartheta} = 2\cos\vartheta\sin\vartheta\dot{\phi}^2 \quad , \quad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\vartheta}} = \frac{\mathrm{d}}{\mathrm{d}t}(2\dot{\vartheta}) = 2\ddot{\vartheta}$$

and thus the Lagrange equations are

$$\ddot{\phi} + 2\cot\vartheta\dot{\vartheta}\dot{\phi} = 0 \qquad \text{and} \qquad \ddot{\vartheta} - \cos\vartheta\sin\vartheta\dot{\phi}^2 = 0\,.$$

Comparing with the geodesic equation $\ddot{x}^\kappa + \Gamma^\kappa{}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$, we can read off the non-vanishing Christoffel symbols as $\Gamma^\phi{}_{\vartheta\phi} = \Gamma^\phi{}_{\phi\vartheta} = \cot\vartheta$ and $\Gamma^\vartheta{}_{\phi\phi} = -\cos\vartheta\sin\vartheta$. (Note that $2\cot\vartheta = \Gamma^\phi{}_{\vartheta\phi} + \Gamma^\phi{}_{\phi\vartheta}$.)

Let us now show that a time-like geodesic G is a local *maximum* of the proper-time: We note first that the path of a light-ray satisfies $\mathrm{d}s^2 = 0$, and thus the proper-time along a light-like geodesics is zero. Next we use that we can approximate G using zig-zaging light-like paths, as shown in Fig. 3.1 by the red curve C. Increasing the number of these paths, the approximation becomes arbitrarly precise, but $\tau(C) = 0 < \tau(G)$. Thus the time-like geodesic cannot be a minimum of the proper-time. This agrees with our knowledge from Minkwoski space that an accelerated clock ticks slower than one at rest.

Parallel transport We say a tensor $\boldsymbol{T}$ is parallel transported along the curve $x(\sigma)$, if its components $T^{\mu\dots}_{\nu\dots}$ stay constant. In flat space, this means simply

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}T^{\mu\dots}_{\nu\dots} = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma}\partial_\alpha T^{\mu\dots}_{\nu\dots} = 0. \tag{3.57}$$

In curved space, we have to replace the normal derivative by a covariant one. We define the directional covariant derivative along $x(\sigma)$ as

$$\frac{D}{\mathrm{d}\sigma} = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma}\nabla_\alpha. \tag{3.58}$$

Then a tensor is parallel transported along the curve $x(\sigma)$, if

$$\frac{D}{\mathrm{d}\sigma}T^{\mu\dots}_{\nu\dots} = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma}\nabla_\alpha T^{\mu\dots}_{\nu\dots} = 0. \tag{3.59}$$

In an inertial system, an alternative definition of a geodesics as the "straightest line" is the path generated by propagating its tangent vector parallel to itself. The tangent vector along the path $x(\sigma)$ is $u^a = \mathrm{d}x^a/d\sigma$. Then the requirement (3.59) of parallel transport for u^a becomes

$$\frac{D}{\mathrm{d}\sigma}\frac{\mathrm{d}x^c}{d\sigma} = \frac{\mathrm{d}^2x^c}{\mathrm{d}\sigma^2} + \Gamma^c{}_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}\sigma}\frac{\mathrm{d}x^b}{\mathrm{d}\sigma} = 0\,. \tag{3.60}$$

Introducing the short-hand $\dot{x}^\alpha = \mathrm{d}x^\alpha/\mathrm{d}\sigma$, we obtain again the geodesic equation in its usual form,

$$\ddot{x}^\sigma + \Gamma^\sigma{}_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0\,. \tag{3.61}$$

(But note that for any connection other than the Levi-Civita connection the two definitions will disagree.)

3.4 Observers and measurements

Observers The world-line $x^\mu(\tau)$ of an observer, or of any massive particle, is time-like: With this we mean not that x^μ as a vector is time-like (a statement not invariant under translations) but that the distance $\mathrm{d}s^2$ between any two points of the world-line is time-like. Equivalently, the four-velocity u^α of a massive particle is a time-like vector. At each instant, we can choose an instantanous Cartesian inertial frame with the four basis vectors $\{\boldsymbol{e}_\mu(\tau)\} = \{\boldsymbol{e}_0(\tau), \boldsymbol{e}_1(\tau), \boldsymbol{e}_2(\tau), \boldsymbol{e}_3(\tau)\}$ in which the observer is at rest. Then the time-like basis vector $\boldsymbol{e}_0(\tau)$ agrees with the four-velocity $\boldsymbol{u}_{\mathrm{obs}}$ of the observer. Moreover, the scalar product of the basis vectors satisfies $\boldsymbol{e}_\mu \cdot \boldsymbol{e}_\nu = \eta_{\mu\nu}$.

Let us consider first an observer in Minkowski space. An observer at rest measures the energy ω and the momenta k_i for a particle with four-momentum $k^\mu = (\omega, \boldsymbol{k})$. We can rewrite these values as tensor equations,

$$\omega = \boldsymbol{k}\cdot\boldsymbol{e}_0 = \boldsymbol{k}\cdot\boldsymbol{u}_{\mathrm{obs}} \quad \text{and} \quad k_i = -\boldsymbol{k}\cdot\boldsymbol{e}_i, \tag{3.62}$$

and thus the RHSs are valid also for a moving observer. Going to a curved spacetime, we have to show that one can introduce an instantanous Cartesian inertial frame at any point—what is equivalent to introducing Riemannian normal coordinates discussed in the next paragraph. Thus measuring the energy and momenta of a particle with four-momentum k^μ corresponds mathematically to the projection of k^μ on the normalised basis vectors of the instantanous Cartesian inertial frame carried by the observer.

Example 3.4: Physical angular momentum:
Consider a particle in $\mathbb{R}^2$ with Lagrangian $L = \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2)$. The angle ϕ is a cyclic coordinate, and thus the canonically conjugated momentum

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = r^2 \dot{\phi}$$

is conserved. We can raise the index,

$$p^\phi = g^{\phi\phi} p_\phi = \dot{\phi},$$

and see that p^ϕ agrees with u^ϕ, as it should for a particle with unit mass. Next we ask what the physical (or measured) angular momentum of the particle is. As discussed, a momentum measurement by an observer corresponds to a projection of this quantity on an unit basis vector of its inertial frame. Rescaling the basis, $\boldsymbol{e}_{\hat{i}} = \boldsymbol{e}_i/\sqrt{g_{ii}}$, the measured angular momentum follows as

$$p_{\hat{\phi}} = p^{\hat{\phi}} = \boldsymbol{p} \cdot \boldsymbol{e}_{\hat{\phi}} = \boldsymbol{p} \cdot \boldsymbol{e}_\phi \frac{1}{r} = p_\phi \frac{1}{r} = r\dot{\phi}.$$

Thus the physical angular momentum $p_{\hat{\phi}} = r\dot{\phi}$ agrees neither with p^ϕ nor with p_ϕ.

Riemannian normal coordinates In a (pseudo-) Riemannian manifold, one can find in each point P a coordinate system, called (Riemannian) normal or geodesic coordinates, with the following properties,

$$\tilde{g}_{ab}(P) = \eta_{ab}, \tag{3.63a}$$

$$\partial_c \tilde{g}_{ab}(P) = 0, \tag{3.63b}$$

$$\tilde{\Gamma}^a_{bc}(P) = 0. \tag{3.63c}$$

We proof this assertion by construction. First, we choose new coordinates $\tilde{x}^a$ centered at P,

$$\tilde{x}^a = x^a - x^a_P + \frac{1}{2}\Gamma^a{}_{bc}(x^b - x^b_P)(x^c - x^c_P). \tag{3.64}$$

Here $\Gamma^a{}_{bc}$ are the connection coefficients in P calculated in the original coordinates x^a. Then we differentiate $\tilde{x}^a$, obtaining

$$\frac{\partial \tilde{x}^a}{\partial x^d} = \delta^a_d + \Gamma^a{}_{db}(x^b - x^b_P)\,. \tag{3.65}$$

Hence $\partial \tilde{x}^a / \partial x^d = \delta^a_d$ at the point P. Differentiating again, we find

$$\frac{\partial^2 \tilde{x}^a}{\partial x^d \partial x^e} = \Gamma^a{}_{db}\delta^b_e = \Gamma^a{}_{de}\,. \tag{3.66}$$

Inserting these results into the transformation law (3.79) of the connection coefficients, where we swap in the second term derivatives of x and $\tilde{x}$,

$$\tilde{\Gamma}^a{}_{bc} = \frac{\partial \tilde{x}^a}{\partial x^d}\frac{\partial x^f}{\partial \tilde{x}^b}\frac{\partial x^g}{\partial \tilde{x}^c}\Gamma^d{}_{fg} - \frac{\partial^2 \tilde{x}^a}{\partial x^d \partial x^f}\frac{\partial x^d}{\partial \tilde{x}^b}\frac{\partial x^f}{\partial \tilde{x}^c} \tag{3.67}$$

gives

$$\tilde{\Gamma}^a{}_{bc} = \delta^a_d\,\delta^f_b\,\delta^g_c\,\Gamma^d{}_{fg} - \Gamma^a{}_{df}\,\delta^d_b\,\delta^f_c = \Gamma^a{}_{bc} - \Gamma^a{}_{bc} \tag{3.68}$$

or

$$\tilde{\Gamma}^a_{de}(P) = 0\,. \tag{3.69}$$

Thus we have found a coordinate system with vanishing connection coefficients at P. By a linear transformation (that does not affect ∂g_{ab}) we can bring finally g_{ab} into the form η_{ab}: As required by the equivalence principle, we can introduce in each spacetime point P a free-falling coordinate system in which physics is described by the known physical laws in the absence of gravity.

Note that the introduction of Riemannian normal coordinates is in general only possible, if the connection is symmetric: Since the antisymmetric part of the connection coefficients, the torsion, transforms as a tensor, it can not be eliminated by a coordinate change. This implies not necessarily a contradiction to the equivalence principle, as long as the torsion is properly generated by source terms in the equations of motion of the matter fields. In particular, the spin current of fermions leads to non-zero torsion. As the elementary spins in macroscopic bodys cancel, torsion is in all relevant astrophysical and cosmological applications negligible. This justifies our choice of a symmetric connection.

Proper distance, area and volume Recall that the n-dimensional volume spanned by n vectors is given by their determinant. Therefore the n-dimensional volume element $\mathrm{d}^n x$ changes by the Jacobian J under a coordinate transformation,

$$\mathrm{d}^n x = \det(\partial x^\mu/\partial\tilde{x}^\sigma)\,\mathrm{d}\tilde{x}^\sigma = J\,\mathrm{d}^n\tilde{x}. \tag{3.70}$$

Taking the determinant of the transformation law of the metric tensor, $g_{ij} = \frac{\partial x^{k\prime}}{\partial \tilde{x}^i}\frac{\partial x^{l\prime}}{\partial \tilde{x}^j}\,g'_{kl}$, implies that $\det(g) = J^2\det(g') = J^2$, if we start from a normalised basis with $\det(g') = 1$. Thus we have to add the factor $\sqrt{|g|}$, where g denotes the determinant of the metric tensor $g_{\mu\nu}$ to obtain an invariant 4-volume element,

$$V_4 = \int_\Omega \mathrm{d}^4 x\sqrt{|g|}. \tag{3.71}$$

In the case of lower-dimensional integrals which are obtained as slices keeping some coordinates fixed, the latter are omitted forming the determinant g. For instance, the proper volume V of a 3-space at a fixed time t is obtained as

$$V_3 = \int \mathrm{d}^3 x\sqrt{g_3}, \tag{3.72}$$

where g_3 is the determinant the metric tensor restricted to g_{ij}.

3.5 Newtonian gravity as a spacetime phenomenon

Since Newtonian gravity is a special case of Einstein gravity, it should be also possible to replace the Newtonian gravitational force by a deformation of Minkowski space. We will show now that the metric describing gravitational effects in the Newtonian limit can be chosen as

$$\mathrm{d}s^2 = \left(1 + 2\Phi/c^2\right)c^2\mathrm{d}t^2 - \left(1 - 2\Phi/c^2\right)\mathrm{d}l^2. \tag{3.73}$$

Here, Φ is the Newtonian gravitational potential and $\mathrm{d}l^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ is the Euclidean line-element. We also kept explicitly factors of c, to facilitate the limit $v/c \to 0$. With this metric, the action of a point particle becomes

$$S = \int_1^2 \mathrm{d}s = \int_1^2 \frac{\mathrm{d}\tau}{c} = \int_1^2 \left[\left(1 + 2\Phi/c^2\right)\mathrm{d}t^2 - \left(1 - 2\Phi/c^2\right)\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right)\right]^{1/2} \tag{3.74a}$$

$$= \int_1^2 \mathrm{d}t \left[1 + \frac{2\Phi}{c^2} - \frac{1}{c^2}\left(1 - 2\Phi/c^2\right)v^2\right]^{1/2}. \tag{3.74b}$$

Expanding the square root and keeping only terms of order $1/c^2$, the action becomes

$$S \simeq \int_1^2 \mathrm{d}t \left[1 - \frac{1}{c^2}\left(\frac{1}{2}v^2 - \Phi\right)\right]. \tag{3.75}$$

The first, constant term does not contribute to the variation δS, so that this action is equivalent to the one using the standard Lagrangian

$$L = \frac{1}{2}mv^2 - m\Phi = T + V \tag{3.76}$$

for a non-relativistic particle with mass m and the gravitational potential energy $V = m\Phi$. Thus the geodesics of the metric (3.73) agree with the classical trajectories in the gravitational potential Φ. Note also that the coefficient of $\mathrm{d}l$ dropped out of Eq. (3.76): Thus an infinite number of spacetimes leads at lowest order in v/c to the same trajectories of non-relativistic particles. Such metrics may however imply different trajectories of relativistic particles like photons.

3.A Appendix: Affine connection and torsion

Our derivation of the covariant derivative $\nabla_a V^b$ in Eq. (3.33) as the projection of $\partial_a \boldsymbol{V}$ onto the direction $\boldsymbol{e}^b$ relied on the existence of a metric to form the scalar product $\boldsymbol{e}^b \cdot (\partial_a \boldsymbol{V})$. In this appendix, we define the more general *affine connection*. Let us consider how the partial derivative of a vector field, $\partial_c X^a$, transforms under a change of coordinates,

$$\partial'_c X'^a = \frac{\partial}{\partial x'^c}\left(\frac{\partial x'^a}{\partial x^b}X^b\right) = \frac{\partial x^d}{\partial x'^c}\frac{\partial}{\partial x^d}\left(\frac{\partial x'^a}{\partial x^b}X^b\right) \tag{3.77a}$$

$$= \frac{\partial x'^a}{\partial x^b}\frac{\partial x^d}{\partial x'^c}\partial_d X^b + \underbrace{\frac{\partial^2 x'^a}{\partial x^b \partial x^d}\frac{\partial x^d}{\partial x'^c}}_{\equiv -\Gamma^a_{bc}} X^b\,. \tag{3.77b}$$

The first term transforms as desired as a tensor of rank (1,1), while the second term—caused by the in general non-linear change of the coordinate basis—destroys the tensorial behavior. If we define a covariant derivative $\nabla_c X^a$ of a vector X^a by requiring that the result is a tensor, we should set

$$\nabla_c X^a = \partial_c X^a + \Gamma^a_{bc} X^b\,. \tag{3.78}$$

Any n^3 quantities Γ^a_{bc} transforming as

$$\Gamma'^a_{\ bc} = \frac{\partial x'^a}{\partial x^d}\frac{\partial x^e}{\partial x'^b}\frac{\partial x^f}{\partial x'^c}\Gamma^d_{ef} + \frac{\partial^2 x^d}{\partial x'^b \partial x'^c}\frac{\partial x'^a}{\partial x^d}\,. \tag{3.79}$$

are called affine connection. From the transformation law (3.79), it is clear that the inhomogeneous term disappears for an antisymmetric combination of the connection coefficients Γ in the lower indices. Thus this combination forms a tensor which is called torsion,

$$T^a{}_{bc} = \Gamma^a{}_{bc} - \Gamma^a{}_{cb} \,. \tag{3.80}$$

Thus the statemement that the connection is symmetric, $\Gamma^a{}_{bc} = \Gamma^a{}_{cb}$, is equivalent to $T^a{}_{bc} = 0$ and thus invariant.

Problems

3.1 Linearity of Lorentz transformation Show that spacetime transformations in Minkoswki space are linear, starting from the general transformation law

$$\tilde{\eta}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu}\,\eta_{\alpha\beta}(x).$$

4 Schwarzschild solution

In the next three chapters, we investigate the solutions of Einsteins field equation which describe the gravitational field outside a spherical mass distribution. The metric valid for a *static* mass distribution was found by Karl Schwarzschild in 1915, only one month after the publication of Einstein's field equation. In this chapter, we will use the Schwarzschild solution to derive the classical tests of general relativity: the perihelion precession of Mercure, the deflection of light by the Sun, and the time-delay of radio signals passing the Sun. All these tests are based on finding the orbits of test particles in the Schwarzschild metric. Similar to the Newtonian case, deriving such orbits is simplified using the conservation laws of energy and angular momentum. Therefore we start by deriving the connection between the symmetries of a curved spacetime and the resulting conservation laws.

4.1 Spacetime symmetries and Killing vector fields

Killing equation A spacetime posseses a symmetry if it looks the same moving from a point P to a different point $\tilde{P}$. More precisely, we mean with "looking the same" that the metric shifted along the vector field ξ^μ from P to the new point $\tilde{P}$ agrees with original metric at this point, $\tilde{g}_{\mu\nu}(\tilde{P}) = g_{\mu\nu}(\tilde{P})$. Such a shift is an *active* coordinate transformation, which should be not confused with the passive coordinate transformations we usually consider. In the latter case, we keep the point P fixed and change the coordinates, while we consider in the former cases two different points.

To simplify the calculations, we restrict ourselves to an infinitesimal shift from P to $\tilde{P}$, described by

$$\tilde{x}^\mu = x^\mu + \varepsilon\xi^\mu(x^\nu) + \mathcal{O}(\varepsilon^2). \tag{4.1}$$

Thus the requirement that the shifted metric and the original metric agree becomes

$$\tilde{g}_{\mu\nu}(x) = g_{\mu\nu}(x) \qquad \text{for all } x. \tag{4.2}$$

Note the difference to the definition of a scalar, $\tilde{\phi}(\tilde{x}) = \phi(x)$, where we consider a passive coordinate transformation. In the latter case, we require that a scalar field has the same value at a point P which in turn changes coordinates from x to $\tilde{x}$.

We start to connect the two metric tensors in (4.2) using the usual transformation law for a tensor of rank two under an arbitrary coordinate transformation,

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu}\, g_{\alpha\beta}(x), \tag{4.3}$$

or, exchanging tilted and untilted quantities,

$$g_{\mu\nu}(x) = \frac{\partial \tilde{x}^\alpha}{\partial x^\mu}\frac{\partial \tilde{x}^\beta}{\partial x^\nu}\, \tilde{g}_{\alpha\beta}(\tilde{x}). \tag{4.4}$$

In order to check the condition (4.2), we have to compare the metric tensors $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ at the same point. We can connect the metric tensor at two infinitesimally separated points performing a Taylor expansion,

$$g_{\mu\nu}(\tilde{x}) = g_{\mu\nu}(x + \varepsilon\xi) = g_{\mu\nu}(x) + \varepsilon\xi^\alpha \partial_\alpha g_{\mu\nu}(x) + \mathcal{O}(\varepsilon^2). \tag{4.5}$$

Evaluating the transformation matrices using (4.1) and inserting the Taylor expansion into (4.2), we obtain

$$\begin{aligned} g_{\mu\nu}(x) &= \frac{\partial \tilde{x}^\alpha}{\partial x^\mu} \frac{\partial \tilde{x}^\beta}{\partial x^\nu}\, g_{\alpha\beta}(\tilde{x}) && \text{(4.6a)} \\ &= (\delta^\alpha_\mu + \varepsilon\partial_\mu \xi^\alpha)(\delta^\beta_\nu + \varepsilon\partial_\nu \xi^\beta)\,[g_{\alpha\beta}(x) + \varepsilon\xi^\rho \partial_\rho g_{\alpha\beta}(x)] + \mathcal{O}(\varepsilon^2) && \text{(4.6b)} \\ &= g_{\mu\nu}(x) + \varepsilon \Big[g_{\alpha\nu}(x)\partial_\mu \xi^\alpha + g_{\mu\beta}(x)\partial_\nu \xi^\beta + \xi^\rho \partial_\rho g_{\alpha\beta}(x) \Big] + \mathcal{O}(\varepsilon^2). && \text{(4.6c)} \end{aligned}$$

Thus the metric is kept invariant, if the condition

$$\delta g_{\mu\nu} = g_{\mu\alpha}\partial_\nu \xi^\alpha + g_{\alpha\nu}\partial_\mu \xi^\alpha + \xi^\alpha \partial_\alpha g_{\mu\nu} = 0 \tag{4.7}$$

is satisfied. Note that we cannot lower the index of ξ^α, since $\partial_\mu \xi^\alpha$ is not a tensor. We eleminate instead the metric tensor in the first two terms by differentiating $\xi_\mu = g_{\mu\alpha}\xi^\alpha$, obtaining

$$g_{\mu\alpha}\partial_\nu \xi^\alpha = \partial_\nu \xi_\mu - \xi^\alpha \partial_\nu g_{\mu\alpha}. \tag{4.8}$$

Using this relation twice results in

$$\delta g_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu + \xi^\alpha [\partial_\alpha g_{\mu\nu} - \partial_\mu g_{\alpha\nu} - \partial_\nu g_{\mu\alpha}]. \tag{4.9}$$

Next we lower also the index of the third ξ^α and recall the definition of the Christoffel symbols,

$$\begin{aligned} \delta g_{\mu\nu} &= \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \xi_\rho g^{\rho\alpha} [\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu}] && \text{(4.10a)} \\ &= \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - 2\xi_\rho \Gamma^\rho{}_{\mu\nu}. && \text{(4.10b)} \end{aligned}$$

Now we can combine the Christoffel symbols with the partial derivatives into covariant derivatives of the vector field $\boldsymbol{\xi}$, obtaining the Killing equation[1]

$$\boxed{\delta g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0.} \tag{4.11}$$

Its solutions $\boldsymbol{\xi}$ are the Killing vector fields of the metric. Moving along a Killing vector field, the metric is kept invariant.

Remark 4.1: Lie derivatives—Since Eq. (4.11) is a tensor equation, the previous Eq. (4.7) is also invariant under arbitrary coordinate transformations, although it contains only partial derivatives. The change $\delta g_{\mu\nu}$ along ξ^μ is a special case of the Lie derivative $\mathscr{L}_{\boldsymbol{\xi}}\boldsymbol{T}$ of a tensor $\boldsymbol{T}$ of rank two along the vector field $\boldsymbol{\xi}$,

$$\mathscr{L}_\xi T_{\mu\nu} = \lim_{\varepsilon\to 0} \frac{T_{\mu\nu}(x+\varepsilon\xi) - T_{\mu\nu}(x)}{\varepsilon} = \xi^\alpha \partial_\alpha T_{\mu\nu} + T_{\mu\alpha}\partial_\nu \xi^\alpha + T_{\alpha\nu}\partial_\mu \xi^\alpha, \tag{4.12}$$

[1] This equation is a much stronger constraint than it looks at a first glance: its solutions are uniquely determined by the equation evaluated at a single point.

while the Lie derivative of a scalar, a vector and co-vector field are given by

$$\mathscr{L}_Y \phi = Y\phi = Y^\nu \partial_\nu \phi, \tag{4.13a}$$

$$\mathscr{L}_Y X^\mu = [\boldsymbol{Y}, \boldsymbol{X}]^\mu = Y^\nu \partial_\nu X^\mu - X^\nu \partial_\nu X^\mu, \tag{4.13b}$$

$$\mathscr{L}_Y X_\mu = Y^\nu \partial_\nu X_\mu + X^\nu \partial_\nu X_\mu. \tag{4.13c}$$

It is a general property of the Lie derivative that the partial derivatives can be exchanged against covariant derivatives, if the connection is torsion free. Then Eq. (4.12) leads directly to the Killing equation setting $T_{\mu\nu} = g_{\mu\nu}$.

Example 4.1: Killing vectors of $\mathbb{R}^3$:
Choosing Cartesian coordinates, $\mathrm{d}l^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$, makes it obvious that translations correspond to Killing vectors $\xi_1 = (1,0,0)$, $\xi_2 = (0,1,0)$, and $\xi_3 = (0,0,1)$. We find the Killing vectors describing rotational symmetry by writing for an infinitesimal rotation around, e.g., the z axis,

$$\begin{aligned} x' &= \cos\alpha x - \sin\alpha y \approx x - \alpha y\,, \\ y' &= \sin\alpha x + \cos\alpha y \approx y + \alpha x\,, \\ z' &= z\,. \end{aligned}$$

Hence $\xi_z = (-y, x, 0)$ and ξ_x, ξ_y follow by cyclic permutation. One of them, ξ_z, we could have also identified by rewriting the line-element in spherical coordinates and noting that $\mathrm{d}l$ does not contain ϕ dependent terms.

Conserved quantities along geodesics Assume that the metric is independent from one coordinate, e.g. x^0. Then there exists a corresponding Killing vector, $\xi = (1,0,0,0)$, and x^0 is a cyclic coordinate, $\partial L/\partial x^0 = 0$. With $L = \mathrm{d}\tau/\mathrm{d}\sigma$, the resulting conserved quantity $\partial L/\partial \dot{x}^0 = \text{const.}$ can be written as

$$\frac{\partial L}{\partial \dot{x}^0} = g_{0\beta}\frac{\mathrm{d}x^\beta}{L\mathrm{d}\sigma} = g_{0\beta}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = \boldsymbol{\xi}\cdot\boldsymbol{u}\,. \tag{4.14}$$

Hence the quantity $\boldsymbol{\xi}\cdot\boldsymbol{u}$ is conserved along the solutions $x^\mu(\sigma)$ of the Lagrange equation, i.e. along geodesics.

In the previous case, the coordinates were adapted to the symmetry. In general, the metric depends on all coordinates, even if there exists Killing vectors. We can check that Eq. (4.11) leads also in this case to a conservation law, multiplying the equation for geodesic motion,

$$\frac{Du^\mu}{\mathrm{d}\tau} = 0, \tag{4.15}$$

first by the Killing vector ξ_μ and using then Leibniz's product rule together with the definition of the absolute derivative (3.58),

$$\xi_\mu \frac{Du^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}}{\mathrm{d}\tau}(\xi_\mu u^\mu) - \nabla_\mu \xi_\nu u^\mu u^\nu = 0. \tag{4.16}$$

The second term vanishes for a Killing vector field ξ^μ, because the Killing equation implies the antisymmetry of $\nabla_\mu \xi_\nu$. Hence the quantity $\xi_\mu u^\mu$ is indeed conserved along any geodesics.

Stationary, static and isotropic spacetimes Spacetimes which are of physical interest are often highly symmetric. For instance, the spacetime around a star is radially symmetric if the influence of other masses can be neglected. Similarly, the universe is homogeneous and isotropic, if one averages over sufficiently large scales. These symmetries allow us in turn to derive exact analytical solutions of the Einstein equation. We define a *stationary* spacetime as a spacetime having a time-like Killing vector field. In appropriate coordinates, the metric tensor is independent of the time coordinate,

$$\mathrm{d}s^2 = g_{00}(\boldsymbol{x})\mathrm{d}t^2 + 2g_{0i}(\boldsymbol{x})\mathrm{d}t\mathrm{d}x^i + g_{ij}(\boldsymbol{x})\mathrm{d}x^i\mathrm{d}x^j\,. \tag{4.17}$$

A stationary spacetime is *static*, if it is invariant under time reversal. Thus the off-diagonal terms g_{0i} have to vanish, and the metric simplifies to

$$\mathrm{d}s^2 = g_{00}(\boldsymbol{x})\mathrm{d}t^2 + g_{ij}(\boldsymbol{x})\mathrm{d}x^i\mathrm{d}x^j. \tag{4.18}$$

An example of a stationary spacetime is the metric around a spherically symmetric mass distribution which rotates with constant velocity. If the mass distribution is at rest then the spacetime becomes static. Finally, an *isotropic* spacetime is invariant under rotations $R \in O(3)$. More precisely, such a spacetime contains three space-like Killing vector fields ξ^μ with closed orbits $\xi^\mu(\sigma)$ which satisfy the commutation relations of the Lie algebra of O(3). Choosing Cartesian coordinates, the Killing vector fields are $\xi^0 = 0$ and $\xi^i = \omega^i{}_j x^j$ with $\omega_{ij} = -\omega_{ji}$. For more details see the appendix.

4.2 Schwarzschild metric

In the appendix, we show that the most general metric of an isotropic and static spacetime can be written as

$$\mathrm{d}s^2 = A(r)\mathrm{d}t^2 - B(r)\mathrm{d}r^2 - r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\phi^2)\,. \tag{4.19}$$

Outside a static radial-symmetric mass distribution, the two functions A and B are given in Schwarzschild coordinates by $A = 1/B = 1 - 2M/r$, or

$$\boxed{\mathrm{d}s^2 = \left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1 - \frac{2M}{r}} - r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\phi^2)\,.} \tag{4.20}$$

Note that the radial coordinate r should not be interpretated as the distance to the center of the mass distribution, since the Schwarzschild metric is not valid inside the star. (Or, if a black hole forms, an event horizon prevents us to measure this distance.) Instead we can measure r indirectly via the area $A = 4\pi r^2$ of a sphere around the mass distribution.

The main properties of this metric are

- symmetries: The metric is time-independent and spherically symmetric. Hence two (out of the four) Killing vectors are $\xi = (1, 0, 0, 0)$ and $\eta = (0, 0, 0, 1)$, where we order coordinates as $\{t, r, \vartheta, \phi\}$.
- asymptotically flat: we recover Minkowski space for $M/r \to 0$.
- the metric is diagonal.
- potential singularities at $r = 2M$ and $r = 0$. The radius $2M$ is called Schwarzschild radius and has the value
$$R_s = \frac{2GM}{c^2} \simeq 3\,\mathrm{km}\,\frac{M}{M_\odot} \tag{4.21}$$
- For $r < R_s$, the coordinate t becomes space-like, while r becomes time-like.

Gravitational redshift According to Eq. (3.62), an observer with four-velocity $\boldsymbol{u}_{\text{obs}}$ measures the frequency

$$\omega = \boldsymbol{k} \cdot \boldsymbol{u}_{\text{obs}} \tag{4.22}$$

of a photon with four-momentum $\boldsymbol{k}$. For an observer at rest,

$$\boldsymbol{u}_{\text{obs}} \cdot \boldsymbol{u}_{\text{obs}} = 1 = g_{tt}(u^t)^2 \,. \tag{4.23}$$

Hence

$$\boldsymbol{u}_{\text{obs}} = (1 - 2M/r)^{-1/2} \boldsymbol{\xi} \,. \tag{4.24}$$

Inserting this relation into Eq. (4.22), we find for the frequency measured by an observer at position r

$$\omega(r) = (1 - 2M/r)^{-1/2} \boldsymbol{\xi} \cdot \boldsymbol{k} \,. \tag{4.25}$$

Since $\boldsymbol{\xi} \cdot \boldsymbol{k}$ is conserved and $\omega_\infty = \boldsymbol{\xi} \cdot \boldsymbol{k}$, we obtain

$$\omega_\infty = \omega(r) \sqrt{1 - \frac{2M}{r}} \,. \tag{4.26}$$

Thus a photon climbing out of the potential wall of the mass M looses energy, in agreement with the principle of equivalence. Any signal sent towards an observer at infinity by a spaceship falling towards $r = 2M$ will be more and more redshifted, with $\omega \to 0$ for $r \to 2M$. Thus the sphere at $r = 2M$ is an *infinite redshift surface.* Such a sphere acts like a membrane hiding all processes inside from the outside.

If $M/r \ll 1$, we can expand the square root. Inserting also G and c, we find

$$\omega_\infty \approx \omega(r) \left(1 - \frac{GM}{rc^2}\right) = \omega(r) \left(1 - \frac{\Phi}{c^2}\right) \,, \tag{4.27}$$

where Φ is the Newtonian potential. Relativistic corrections become small for $\Phi/c^2 \ll 1$. In this limit, the Schwarzschild metric becomes

$$\mathrm{d}s^2 = (1 - 2\Phi)\,\mathrm{d}t^2 - (1 + 2\Phi)\,\mathrm{d}r^2 - r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\phi^2) \,. \tag{4.28}$$

Moreover, in this limit non-linear effects are negligible and the superposition principle is valid. Thus Φ can include the gravitational potential generated by an arbitrary static mass distribution. We have already shown in section 3.5 that a particle in this metric indeed follows the trajectories from classical mechanics.

4.3 Orbits of massive particles

We have discussed the classical Kepler problem in Sec. 2.4. The orbits of massive test particles in the Schwarzschild metric can be solved using similar methods. In particular, we derive the famous precession of Mercury's perihelion.

Radial equation and effective potential for massive particles Spherically symmetry means that the movement of a test particle is contained in a plane. We choose $\vartheta = \pi/2$ and $u_\vartheta = 0$. We replace in the normalization condition $\boldsymbol{u} \cdot \boldsymbol{u} = 1$ written out for the Schwarzschild metric,

$$1 = \left(1 - \frac{2M}{r}\right) \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 - \left(1 - \frac{2M}{r}\right)^{-1} \left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 - r^2 \left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2, \tag{4.29}$$

the velocities u_t and u_r by the conserved quantities

$$e \equiv \boldsymbol{\xi} \cdot \boldsymbol{u} = \left(1 - \frac{2M}{r}\right) \frac{\mathrm{d}t}{\mathrm{d}\tau} \tag{4.30}$$

$$l \equiv -\boldsymbol{\eta} \cdot \boldsymbol{u} = r^2 \sin \vartheta^2 \frac{\mathrm{d}\phi}{\mathrm{d}\tau}. \tag{4.31}$$

Setting then $A = 1 - 2M/r$, we find

$$1 = \frac{e^2}{A} - \frac{1}{A} \left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 - \frac{l^2}{r^2}. \tag{4.32}$$

We want to rewrite this equation in a form similar to the energy equation in the Newtonian case. Multiplying by $A/2$ makes the $\mathrm{d}r/\mathrm{d}\tau$ term similar to a kinetic energy term. Bringing also all constant terms on the LHS and calling them $\mathcal{E} \equiv (e^2 - 1)/2$, we obtain

$$\mathcal{E} \equiv \frac{e^2 - 1}{2} = \frac{1}{2} \left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + V_{\text{eff}} \tag{4.33}$$

with

$$V_{\text{eff}} = -\frac{M}{r} + \frac{l^2}{2r^2} - \frac{Ml^2}{r^3} = V_0 - \frac{Ml^2}{r^3}. \tag{4.34}$$

Hence the energy[2] of a test particle in the Schwarzschild metric can be, as in the Newtonian case, divided into kinetic energy and potential energy. The latter contains the additional term Ml^2/r^3, suppressed by $1/c^2$, that becomes important at small r.

The asymptotic behavior of V_{eff} for $r \to 0$ and $r \to \infty$ is

$$V_{\text{eff}}(r \to \infty) \to -\frac{M}{r} \qquad \text{and} \quad V_{\text{eff}}(r \to 0) \to -\frac{Ml^2}{r^3}, \tag{4.35}$$

while the potential at the Schwarzschild radius, $V(2M) = 1/2$, is independent of M.

We determine the extrema of V_{eff} by solving $\mathrm{d}V_{\text{eff}}/\mathrm{d}r = 0$ and find

$$r_{1,2} = \frac{l^2}{2M} \left[1 \pm \sqrt{1 - 12M^2/l^2}\right] \tag{4.36}$$

Hence the potential has no extrema for $M/l > 1/\sqrt{12}$ and is always negative: A particle can reach $r = 0$ for small enough but finite angular momentum, in contrast to the Newtonian case. By the same argument, there exists a last stable orbit at $r = 6M$, when the two extrema r_1 and r_2 coincide for $l/M = \sqrt{12}$. The effective potential V_{eff} is shown for various values of l/M in Fig. 4.1. Reducing l/M the maximum of V_{eff} decreases, until it disappears for $\sqrt{12} \simeq 3.46$. Similarly, one sees that for $l = 6M$ indeed only one extremum exist.

The orbits can be classified according the relative size of $\mathcal{E}$ and V_{eff} for a given l:

[2] More precisely, e and l are the energy and the angular momentum per unit mass. Thus the -1 in $\mathcal{E}$ corresponds to the rest mass of the test particle.

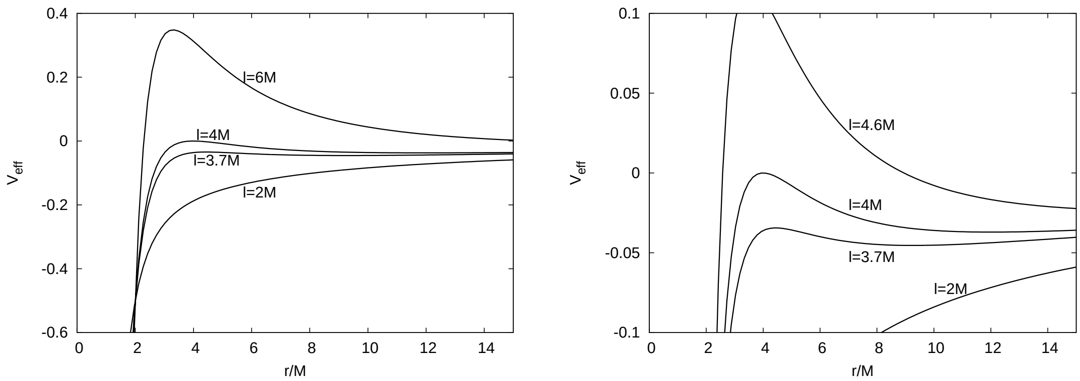


Figure 4.1: The effective potential V_{eff} for various values of l/M as function of distance r/M, for two different scales.

- Bound orbit exists for $\mathcal{E} < 0$. Two circular orbits, one stable at the minimum of V_{eff} and an unstable one at the maximum of V_{eff}; orbits that oscillate between the two turning points.
- Scattering orbit exists for $\mathcal{E} > 0$: If $\mathcal{E} > \max\{V_{\text{eff}}\}$, the particle hits after a finite time the singularity $r = 0$. For $0 < \mathcal{E} < \max\{V_{\text{eff}}\}$, the particle turns at $\mathcal{E} = \max\{V_{\text{eff}}\}$ and escapes to $r \to \infty$.

We derive below a differential equation for $r(\phi)$, from which the orbits in the Schwarzschild metric can be calculated. For the lazy student, several webpages exist where such orbits can be visualised, see e.g. `http://www.fourmilab.ch/gravitation/orbits/`.

Example 4.2: Maximal accretion efficiency a Schwarzschild black hole:
Black holes are the most efficient energy sources known in astrophysics, converting the "potential gravitational energy" of infalling particles into heat via an accretion disk. For a Schwarzschild BH, this efficiency can be calculated as follows: The innermost stable circular orbit is at $r_c = 3R_S = 6M$ with $l_c = \sqrt{3}R_S = 2\sqrt{3}M$. Evaluating the potential (4.34) at r_c, it follows $V_{\text{eff}} = -1/6+12/72-12/216 = -1/18$. Setting $\dot{r} = 0$ in the energy equation (4.33) gives $e = \sqrt{1+2V} = 2\sqrt{2}/3$, while the particle started with $e = 1$ at infinity. Thus the fraction of energy released is $f = 1 - 2\sqrt{2}/3 \simeq 0.06$.

Radial infall We consider the free fall of a particle that is at rest at infinity, $\mathrm{d}t/\mathrm{d}\tau = 1$, $\mathcal{E} = 0$ and $l = 0$. The radial equation (4.33) simplifies to

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = \frac{M}{r} \tag{4.37}$$

and can be integrated by separation of variables,

$$\int_{r_*}^{0} \mathrm{d}r r^{1/2} = \pm\sqrt{2M}\int_{\tau_*}^{\tau} \mathrm{d}\tau. \tag{4.38}$$

Choosing the minus sign appropriate for an infalling particle and $\tau_* = 0$ as start time results in

$$\tau = \frac{2r^{3/2}}{3\sqrt{2M}}\,. \tag{4.39}$$

Hence a freely falling particle needs only a finite proper-time to fall from finite r to $r = 0$. In particular, it passes the Schwarzschild radius $2M$ in finite proper time. Moreover, nothing special happens at $r = 2M$.

We can answer the same question using the coordinate time t by combining Eqs. (4.30) [with $\mathcal{E} = 0$ and thus $e = 1$] and (4.37),

$$\frac{\mathrm{d}t}{\mathrm{d}r} = -\left(\frac{2M}{r}\right)^{-1/2}\left(1-\frac{2M}{r}\right)^{-1}\,. \tag{4.40}$$

Integrating gives

$$\begin{aligned} t - t_0 &= \int_{r_0}^{r} \mathrm{d}r' \left(\frac{2M}{r'}\right)^{-1/2}\left(1-\frac{2M}{r'}\right)^{-1} = \\ &= -2M\left[-\frac{2}{3}\left(\frac{r'}{2M}\right)^{3/2} - 2\left(\frac{r'}{2M}\right)^{1/2} + \ln\left|\frac{\sqrt{r'/2M}+1}{\sqrt{r'/2M}-1}\right|\right]_{r_0}^{r} \to \infty \quad \text{for} \quad r \to 2M. \end{aligned} \tag{4.41}$$

Since the coordinate time t equals the proper-time for an observer at infinity, a freely falling particle reaches the Schwarzschild radius $r = 2M$ only for $t \to \infty$ for such an observer.

The last result can be derived immediately for light-rays. Choosing a light-ray in radial direction with $\mathrm{d}\phi = \mathrm{d}\vartheta = 0$, the metric (4.20) simplifies with $\mathrm{d}s^2 = 0$ to

$$\frac{\mathrm{d}r}{\mathrm{d}t} = 1 - \frac{2M}{r}\,. \tag{4.42}$$

Thus light travelling towards the star, as seen from the outside, will travel slower and slower as it comes closer to the Schwarzschild radius $r = 2M$. The coordinate time is $\propto \ln|1-2M/r|$ and thus for an observer at infinity the signal will reach $r = 2M$ again only asymptotically for $t \to \infty$.

Example 4.3: Two particles fall radially from infinity towards a point mass M, one starting with $e = 1$, the other with $e = 2$. How big is the ratio of their velocities measured by a stationary observer at $r = 6M$?
An observer with $\boldsymbol{u}_{\mathrm{obs}}$ measure as energy E and velocity v

$$E = \boldsymbol{p}\cdot\boldsymbol{u}_{\mathrm{obs}} = \frac{m}{\sqrt{1-v^2}}$$

for a particle with four-momentum p^μ and mass m. If the observer is stationary, $u^r_{\mathrm{obs}} = u^\vartheta_{\mathrm{obs}} = u^\phi_{\mathrm{obs}} = 0$, the normalisation condition $\boldsymbol{u}_{\mathrm{obs}}\cdot\boldsymbol{u}_{\mathrm{obs}} = 1$ gives

$$u^t_{\mathrm{obs}} = \left(1-\frac{2M}{r}\right)^{-1/2}.$$

Thus

$$E = m\boldsymbol{u}\cdot\boldsymbol{u}_{\mathrm{obs}} = mg_{\alpha\beta}u^\alpha u^\beta_{\mathrm{obs}} = m\left(1-\frac{2M}{r}\right)^{1/2}u^t = \frac{m}{\sqrt{1-v^2}}.$$

Now we replace u^t by the conserved energy,

$$e = \left(1 - \frac{2M}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau} = \left(1 - \frac{2M}{r}\right)u^t,$$

to obtain

$$v(e) = \frac{1}{e}\left(e^2 - 1 + \frac{2M}{r}\right)^{1/2}.$$

The ratio of the velocities at $r = 6M$ follows as

$$\frac{v(2)}{v(1)} = \frac{1}{2}\left(\frac{4 - 1 + 1/3}{1 - 1 + 1/3}\right)^{1/2} = \frac{\sqrt{10}}{2}.$$

Perihelion precession We follow the same strategy as in the Kepler problem in the Newtonian limit, starting from Eq. (4.33) for the Schwarzschild metric,

$$\dot{r}^2 + \frac{l^2}{r^2} = e^2 - 1 + \frac{2M}{r} + \frac{2Ml^2}{r^3}\,. \tag{4.43}$$

We eliminate first t and introduce then $u = 1/r$,

$$(u')^2 + u^2 = \frac{e^2 - 1}{l^2} + \frac{2Mu}{l^2} + 2Mu^3\,. \tag{4.44}$$

We can transform this into a linear differential equation differentiating with respect to ϕ. Thereby we eliminate also the constant $(e^2 - 1)/l^2$, and dividing[3] by $2u'$ it follows

$$u'' + u = \frac{M}{l^2} + 3Mu^2 = \frac{GM}{l^2} + \frac{3GM}{c^2}u^2\,. \tag{4.45}$$

In the last step we reintroduced c and G. Hence we see that the Newtonian limit corresponds to $c \to \infty$ ("instantaneous interactions") or $v/c \to 0$ ("static limit"). The latter statement becomes clear, if one uses the virial theorem: $GMu = GM/r \sim v^2$.

In most situations, the relativistic correction is tiny. We use therefore perturbation theory to determine an approximate solution, setting $u = u_0 + \delta u$, where u_0 is the Newtonian solution. Inserting u into Eq. (4.45), we obtain

$$(\delta u)'' + \delta u = \frac{3(GM)^3}{c^2 l^4}(u_0^2 + 2u_0\delta u + \delta u^2)\,. \tag{4.46}$$

Here we used that u_0 solves the Newtonian equation of motion (2.45). Keeping on the RHS only the leading term u_0^2 results in

$$(\delta u)'' + \delta u = \frac{3(GM)^3}{c^2 l^4}(1 + 2e\cos\phi + e^2\cos^2\phi)\,. \tag{4.47}$$

Its solution is

$$\delta u = \frac{3(GM)^3}{c^2 l^4}\left[1 + e\phi\sin\phi + e^2\left(\frac{1}{2} - \frac{1}{6}\cos(2\phi)\right)\right]\,. \tag{4.48}$$

[3] The case $u' = 0$ corresponds to radial infall treated in the previous section.

Example 4.4: The solution of the linear inhomogenous differential equation (4.47) is found by adding the particular solutions of the three inhomogenous terms. With A, B and C being constants, it is

$$u'' + u = A \qquad \Rightarrow \qquad u = A, \tag{4.49}$$

$$u'' + u = B\cos\phi \qquad \Rightarrow \qquad u = \frac{1}{2}B\phi\sin\phi, \tag{4.50}$$

$$u'' + u = C\cos^2\phi \qquad \Rightarrow \qquad u = \frac{1}{2}C - \frac{1}{6}C\cos(2\phi). \tag{4.51}$$

While the first and third term in the square bracket lead only to extremely tiny changes in the orbital parameters, the second term is linear in ϕ and its effect accumulates therefore with time. Thus we include only $\delta u \propto e\phi\sin\phi$ in the approximate solution. Introducing $\alpha = 3(GM)^2/(cl)^2 \ll 1$ and employing

$$\cos[\phi(1-\alpha)] = \cos\phi\cos(\alpha\phi) + \sin\phi\sin(\alpha\phi) \simeq \cos\phi + \alpha\phi\sin\phi, \tag{4.52}$$

we find

$$u = u_0 + \delta u \simeq \frac{GM}{l^2}[1 + e(\cos\phi + \alpha\phi\sin\phi)] \simeq \frac{GM}{l^2}[1 + e\cos(\phi(1-\alpha))]\,. \tag{4.53}$$

Hence the period is $2\pi/(1-\alpha)$, and the ellipse processes with

$$\Delta\phi = \frac{2\pi}{1-\alpha} - 2\pi \simeq 2\pi\alpha = \frac{6\pi(GM)^2}{(lc)^2} = \frac{6\pi GM}{a(1-e^2)c^2}\,. \tag{4.54}$$

The effect increases for orbits with small major axis a and large eccentricity e. Urbain Le Verrier first recognized in 1859 that the precession of the Mercury's perihelion deviates from the Newtonian predicition: Perturbations by other planets lead to $\Delta\phi = 532.3''$/century, compared to the observed value of $\Delta\phi = 574.1''$/century. The main part of the discrepancy is explaind by the effect of Eq. (4.54), predicting a shift of $\Delta\phi = 43.0''$/century. (Tiny additional corrections are induced by the quadrupole moment of the Sun ($0.02''$/century) and the Lense-Thirring effect ($-0.002''$/century)).

4.4 Orbits of photons

We repeat the discussion of geodesics for massive particle for massless ones by changing $\boldsymbol{u}\cdot\boldsymbol{u} = 1$ into $\boldsymbol{u}\cdot\boldsymbol{u} = 0$ and by using an affine parameter λ instead of the proper-time τ. Reordering gives

$$\frac{1}{b^2} \equiv \frac{e^2}{l^2} = \frac{1}{l^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 + W_{\text{eff}} \tag{4.55}$$

with the impact parameter $b = |l/e|$ and

$$W_{\text{eff}} = \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)\,. \tag{4.56}$$

The radial equation (4.55) is invariant under reparametrisations of the affine parameter, $\lambda \to A\lambda + B$, since the change cancels both in b and $l\mathrm{d}\lambda$. Consequently, the orbit of a photon does not depend seperately on the energy e and the angular momentum l, but only on the impact parameter b of the photon.

The maximum of W_{eff} is at $3M$ with height $1/27M^2$. For impact parameters $b > b_0 = 3\sqrt{3}M$, photon orbits have a turning point and photons escape to infinity. For $b < b_0$, they hit $r = 0$, while for $b = b_0$ a (unstable) circular orbit is possible. Thus the absorption cross section of a point mass M for massless particles is $\sigma = \pi b_0^2 = 27\pi M^2$.

Light deflection We transform Eq. (4.55) as in the $m > 0$ case into a differential equation for $u(\phi)$. For small deflections, we use again perturbation theory. In zeroth order in v/c, we can set the RHS of

$$u'' + u = \frac{3GM}{c^2}u^2 \tag{4.57}$$

to zero. The solution u_0 is a straight line,

$$u_0 = \frac{\sin\phi}{b}\,. \tag{4.58}$$

For large r, it is $\phi \simeq b/r$.

Inserting $u = u_0 + \delta u$ gives

$$(\delta u)'' + \delta u = \frac{3GM}{c^2}\frac{\sin^2\phi}{b^2}\,. \tag{4.59}$$

A particular solution is

$$\delta u = \frac{3GM}{2c^2b^2}(1 + 1/3\cos(2\phi))\,. \tag{4.60}$$

Thus the complete approximate solution is

$$u = u_0 + \delta u = \frac{\sin\phi}{b} + \frac{3GM}{2c^2b^2}(1 + 1/3\cos(2\phi))\,. \tag{4.61}$$

Considering the limit $r \to \infty$ or $u \to 0$ of this equation gives half of the deflection angle of a light-ray with impact parameter b to a point mass M,

$$\Delta\phi = \frac{4GM}{c^2b} = \frac{2R_s}{b}\,. \tag{4.62}$$

For a light-ray grazing the Solar surface, $b = R_\odot$, we obtain as numerical estimate

$$\Delta\phi_\odot = \frac{4GM_\odot}{c^2R_\odot} = \frac{2R_s}{R_\odot} \simeq 10^{-5} \approx 2''\,. \tag{4.63}$$

For a recollection of the 1919 results see `https://arxiv.org/abs/2010.13744`.

Shapiro effect Shapiro suggested to use the time-delay of a radar signal as test of general relativity. Suppose we send a radar signal from the Earth to Venus where it is reflected back to Earth. The point r_0 of closest approach to the Sun is characterized by $\mathrm{d}r/\mathrm{d}t|_{r_0} = 0$.

Rewriting Eq. (4.55) as

$$\dot{r}^2 + \frac{l^2}{r^2}\left(1 - \frac{2M}{r}\right) = e^2 \tag{4.64}$$

and introducing the Killing vector $\boldsymbol{e}$ in $\dot{r}^2$,

$$\dot{r}^2 = \left(\frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\lambda}\right)^2 = \frac{e^2}{(1-2M/r)^2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2, \tag{4.65}$$

we find

$$\frac{1}{(1-2M/r)^3}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \frac{l^2}{e^2r^2} - \frac{1}{1-2M/r} = 0\,. \tag{4.66}$$

We now evaluate this equation at the point of closest approach, i.e. for $\mathrm{d}r/\mathrm{d}t|_{r_0} = 0$,

$$\frac{l^2}{e^2} = \frac{r_0^2}{1-2M/r_0}, \tag{4.67}$$

and use this equation to eliminate l^2/e^2 in (4.66). Then we obtain

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \left(1-\frac{2M}{r}\right)\left[1-\frac{r_0^2(1-2M/r)}{r^2(1-2M/r_0)}\right]^{1/2} \tag{4.68}$$

or

$$t(r,r_0) = \int_{r_0}^{r}\frac{\mathrm{d}r}{1-2M/r}\left[1-\frac{r_0^2(1-2M/r)}{r^2(1-2M/r_0)}\right]^{-1/2}. \tag{4.69}$$

Next we expand this expression in $M/r \ll 1$,

$$\begin{aligned} t(r,r_0) &= \int_{r_0}^{r}\mathrm{d}r\frac{r}{(r^2-r_0^2)^{1/2}}\left[1-\frac{2M)}{r}+\frac{Mr_0}{r(r+r_0)}\right] && (4.70)\\ &= \frac{(r^2-r_0^2)^{1/2}}{c} + \frac{2GM}{c^3}\ln\left[\frac{r+(r^2-r_0^2)^{1/2}}{r_0}\right] + \frac{GM}{c^3}\left(\frac{r-r_0}{r+r_0}\right)^{1/2}, && (4.71)\end{aligned}$$

where we restored also G and c in the last step. The first term corresponds to straight line propagation and thus the excess time Δt is given by the second and third term. Finally, we can use that the orbits both of Earth and Venus are much more distant from the Sun than the point of closest approach, $R_E, R_V \gg r_0$. Hence we obtain for the time delay

$$\Delta t = \frac{4GM}{c^3}\left[\ln\left(\frac{4R_ER_V}{r_0^2}\right)+1\right]. \tag{4.72}$$

In Fig. 4.2, one of the first measurements of the Shapiro time-delay is shown together with the prediction using Eq. (4.72); an excellent agreement is visible.

4.5 Post-Newtonian parameters

In order to search for deviations from general relativity one uses the post-Newtonian approximation, i.e. an expansion around Minkowski space. Any spherically symmetric, static spacetime can be expressed as

$$\mathrm{d}s^2 = A(r)\mathrm{d}t^2 - B(r)\mathrm{d}r^2 - r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\phi^2) \tag{4.73}$$

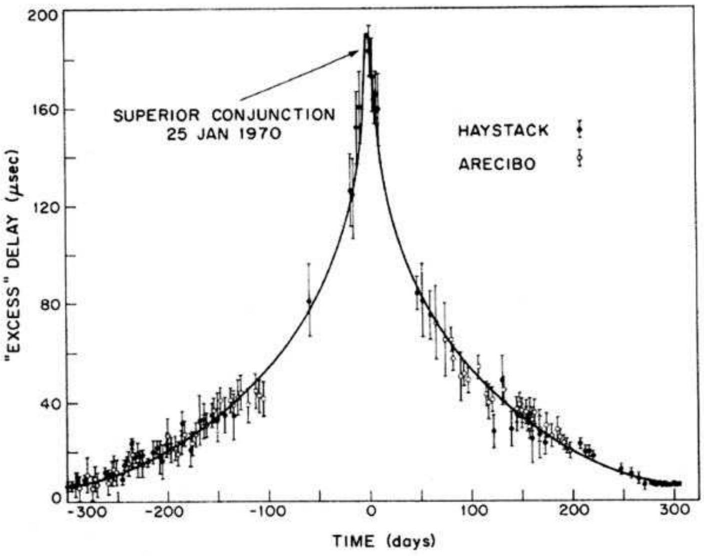


Figure 4.2: Measurement of the Shapiro time-delay compared to the prediction in GR.

with two unknown functions $A(r)$ and $B(r)$. Since the only available length is $R_g = GM/c^2$, A and B can be expanded as power series in $R_g/r \ll 1$,

$$A(r) = 1 + a_1 R_g/r + a_2(R_g/r)^2 + \ldots = 1 - \frac{2GM}{c^2 r} + 2(\beta - \gamma)\left(\frac{2GM}{c^2 r}\right)^2 \ldots \tag{4.74}$$

$$B(r) = 1 + b_1 R_g/r + b_2(R_g/r)^2 + \ldots = 1 + \gamma\,\frac{2GM}{c^2 r} + \ldots \tag{4.75}$$

Agreement with Newtonian gravity is achieved for $A = 1 - 2GM/(rc^2)$ and $B = 1$. Searching for deviations from GR, one keeps therefore $a_1 = 2$ fixed and introduces the "post-Newtonian" parameter β and γ such that agreement with Einstein gravity is achieved for $\gamma = 1$ and $\beta = 1$. The predictions for the three classical tests of GR we have discussed can be redone using the metric (4.74). Alternative theories of gravity predict the numerical values of the post-Newtonian parameters and can thereby easily compared to experimental results.

4.A Appendix: General stationary spherically symmetric metric

The general form of a stationary isotropic metric in coordinates which are adopted to these symmetries could be easily guessed. Since in more complicated case this may not be possible, we start instead from the Killing equation as illustratation of the more general approach.

Killing equation An isotropic spacetime admits three space-like Killing vector fields satisfying $\xi^i = \omega^{ij} x_j$ with $\omega^{ij} = -\omega^{ji}$ and $\xi^0 = 0$. Considering now

$$\delta g_{00} = \varepsilon^{ij} x_j \partial_i g_{00} = 0 \tag{4.76}$$

or

$$x_i \partial_j g_{00} = x_j \partial_i g_{00} \tag{4.77}$$

or $g_{00} = g_{00}(t, r)$. Evaluating δg_{0i} results in

$$g_{0i}\omega^i_{\ j} + (\partial_i g_{0j})\omega^{ik} x_k = 0 \tag{4.78}$$

showing that

$$g_{0i}\mathrm{d}x^i\mathrm{d}t = a(r,t)x_i\mathrm{d}x^i\mathrm{d}t = f(r,t)\mathrm{d}r\mathrm{d}t. \tag{4.79}$$

Finally, evaluating δg_{ij} results in

$$g_{ik}\omega^k_{\ j} + g_{kj}\omega^k_{\ i} + (\partial_k g_{ij})\omega^{kl}x_l = 0 \tag{4.80}$$

and

$$g_{ij}\mathrm{d}x^i\mathrm{d}x^j = (b(t,r)\delta_{ij} + c(t,r)x_i x_j)\mathrm{d}x^i\mathrm{d}x^j \tag{4.81}$$

Introducing spherical coordinates such that $r^2 \equiv \boldsymbol{x}\cdot\boldsymbol{x}$, $\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x} = r\mathrm{d}r$, it follows

$$\mathrm{d}s^2 = A(t,r)\mathrm{d}t^2 - B(t,r)\mathrm{d}t\mathrm{d}r - C(t,r)\mathrm{d}r^2 - D(t,r)\mathrm{d}\Omega \tag{4.82}$$

A stationary spacetime has a time-like Killing vector field which we can choose as $\xi^\mu = (1,0,0,0)$. Inserting this vector into the Killing equation (4.7) results in

$$\delta g_{\mu\nu} = \xi^\alpha\partial_\alpha g_{\mu\nu} = \partial_0 g_{\mu\nu} = 0. \tag{4.83}$$

Thus all components of the metric tensor, i.e. the function A, B, C and D, are independent of the time t.

Symmetry of the line-element We can guess also directly this form of the line-element: Isotropy requires that line-element depends in addition to t only on $r^2 \equiv \boldsymbol{x}\cdot\boldsymbol{x}$, $\mathbf{d}\boldsymbol{x}\cdot\boldsymbol{x}$, and $\mathbf{d}\boldsymbol{x}\cdot\mathbf{d}\boldsymbol{x}$, Introducing spherical coordinates, $r^2 \equiv \boldsymbol{x}\cdot\boldsymbol{x}$, $\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x} = r\mathrm{d}r$ and $\mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x} = \mathrm{d}r^2 + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\phi^2)$, it is

$$\mathrm{d}s^2 = A(r)\mathrm{d}t^2 - B(r)\mathrm{d}t\mathrm{d}r - C(r)\mathrm{d}r^2 - D(r)\mathrm{d}\Omega. \tag{4.84}$$

We can eliminate the function $D(r)$ by the rescaling $r^2 \to Dr^2$. Next we want to cancel $B(t,r)\mathrm{d}t\mathrm{d}r$. We set

$$\mathrm{d}\tilde{t} = \Phi(t,r)\left[A(t,r)\mathrm{d}t - \frac{1}{2}B(t,r)\mathrm{d}r\right], \tag{4.85}$$

where $\Phi(t,r)$ is the integrating factor which makes $\mathrm{d}t$ an exact differential. Squaring

$$\mathrm{d}\tilde{t}^2 = \Phi^2\left[A^2\mathrm{d}t^2 - AB\mathrm{d}t\mathrm{d}r + \frac{1}{4}B^2\mathrm{d}r^2\right] \tag{4.86}$$

or

$$A\mathrm{d}t^2 - B\mathrm{d}t\mathrm{d}r = \frac{\mathrm{d}\tilde{t}^2}{A\Phi^2} - \frac{B^2}{4A}\mathrm{d}r^2 \tag{4.87}$$

Setting now $A = 1/(A\Phi^2)$ and $B = C + B^2/(4A)$, we obtain

$$\mathrm{d}s^2 = A(t,r)\mathrm{d}t^2 - B(t,r)\mathrm{d}r^2 - r^2\mathrm{d}\Omega. \tag{4.88}$$

A general isotropic metric depends therefore only on two independent functions of t and r. A stationary spacetime has a time-like Killing vector field. In appropriate coordinates, the metric tensor is therefore independent of the time coordinate t,

$$\mathrm{d}s^2 = g_{00}(\boldsymbol{x})\mathrm{d}t^2 + 2g_{0i}(\boldsymbol{x})\mathrm{d}t\mathrm{d}x^i + g_{ij}(\boldsymbol{x})\mathrm{d}x^i\mathrm{d}x^j\,. \tag{4.89}$$

We can therefore set $A(t,r) \to A(r)$ and $B(t,r) \to B(r)$. Then the metric satisfies automatically $\mathrm{d}s^2(-t) = \mathrm{d}s^2(t)$, and thence is static.

Further reading For more on Killing vectors see [5].

5 *** Gravitational lensing ***

One distinguishes three different cases of gravitational lensing, depending on the strength of the lensing effect:

1. Strong lensing occurs when the lens is very massive and the source is close to it: In this case light can take different paths to the observer and more than one image of the source will appear, either as multiple images or deformed arcs of a source. In the extreme case that a point-like source, lens and observer are aligned the image forms an "Einstein ring"; if they ar close to be aligned, pairs of lensed images appear as arcs, cf. with Fig. 5.2.
2. Weak Lensing: In many cases the lens is not strong enough that multiple images or arcs are visible. However, the source can still be distorted and its image may be both stretched (shear) and magnified (convergence). If the sources were well known in size and shape, one could just use the shear and convergence to deduce the properties of the lens.
3. Microlensing: The source cannot be resolved and one observes only a point-like image. However, the additional light bent towards the observer leads to a brightening of the source. Thus microlensing is only observable as a transient phenomenon, when the lens crosses approximately the axis observer-source.

Lens equation We consider the simplest case of a point-like mass M, the lens, between the observer O and the source S as shown in Fig. 5.1. The angle β denotes the (unobservable) angle between the true position of the source and the direction to the lens, while $\vartheta_\pm$ are the angles between the image positions and the source. The corresponding distances $D_{\rm OS}$, $D_{\rm OL}$, and $D_{\rm LS}$ are also depicted in Fig. 5.1 and, since $D_{\rm OS}+D_{\rm LS}=D_{\rm OL}$ does not hold in cosmology, we keep all three distances. Finally, the impact parameter b is as usual the smallest distance between the light-ray and the lens.

Then the lens equation in the "thin lens" ($b \ll D_i$) and weak deflection ($\alpha \ll 1$) limit follows from $\overline{AS}+\overline{SB}=\overline{AB}$ as

$$\beta D_{\rm os} + \alpha D_{\rm ls} = \vartheta D_{\rm os}. \tag{5.1}$$

The thin lens approximation implies $\vartheta \ll 1$, and since $\beta < \vartheta$, also β is small. Solving for β and inserting for the deflection angle $\alpha = 4GM/(c^2 b)$ as well as $b = \vartheta D_{\rm ol}$, we find first

$$\beta = \vartheta - \frac{4GM}{c^2}\frac{D_{\rm ls}}{D_{\rm os}D_{\rm ol}}\frac{1}{\vartheta}\,. \tag{5.2}$$

Multiplying then by ϑ, we obtain a quadratic equation for ϑ,

$$\vartheta^2 - \beta\vartheta - \vartheta_{\rm E}^2 = 0\,, \tag{5.3}$$

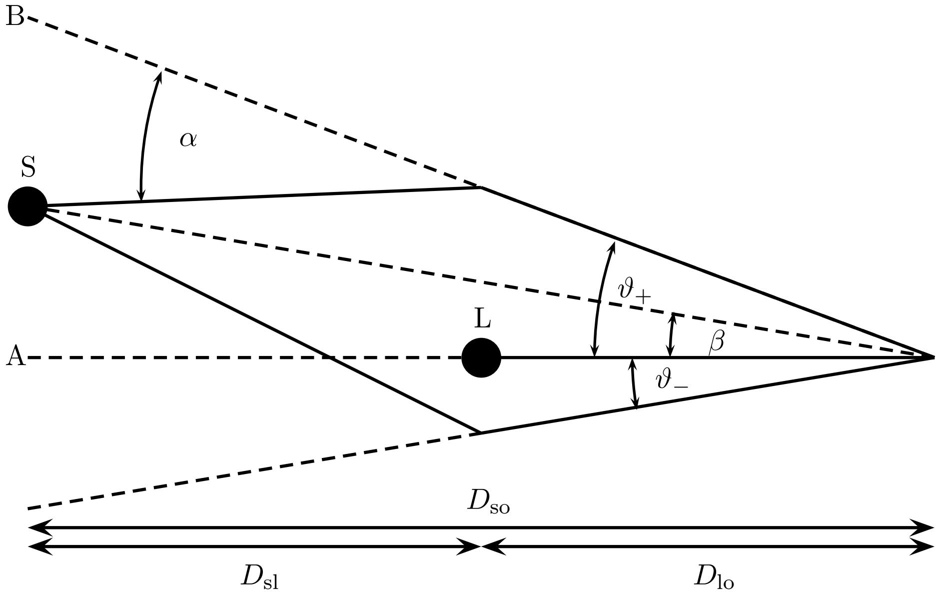


Figure 5.1: The source S that is off the optical axis OL by the angle β appears as two images on opposite sides from the optical axis OL. The two images are separated by the angles $\vartheta_\pm$ from the optical axis O.

where we introduced the Einstein angle

$$\vartheta_{\rm E} = \left(2R_S\,\frac{D_{\rm ls}}{D_{\rm os}D_{\rm ol}}\right)^{1/2}\,. \tag{5.4}$$

The two images of the source are deflected by the angles

$$\vartheta_\pm = \frac{1}{2}[\beta \pm (\beta^2 + 4\vartheta_{\rm E}^2)^{1/2}] \tag{5.5}$$

from the line-of-sight to the lens. If we do not know the lens location, measuring the separation of two lenses images, $\vartheta_+ + \vartheta_-$, provides only an upper bound on the lens mass. If observer, lens and source are aligned, then symmetry implies that $\vartheta_+ = \vartheta_- = \vartheta_E$, i.e. the image becomes a circle with radius $\vartheta_{\rm E}$. Deviations from this perfectly symmetric situations break the circle into arcs as shown in an image of the galaxy cluster Abell 2218 in Fig. 5.2.

For a numerical estimate of the Einstein angle in case of a stellar object in our own galaxy, we set $M = M_\odot$ and $D_{\rm ls}/D_{\rm os} \approx 1/2$ and obtain

$$\vartheta_{\rm E} = 0.64'' \times 10^{-4}\left(\frac{M}{M_\odot}\,\frac{D_{\rm ol}}{10\,{\rm kpc}}\right)^{1/2}\,. \tag{5.6}$$

The numerical value of order 10^{-4} of an arc-second for the deflection led to the name “microlensing.”

Magnification Without scattering or absorption of photons, the conservation of photon number implies that the intensity along the trajectory of a light-ray stays constant, [1]

$$\frac{2}{h^3}\,f(\boldsymbol{x},\boldsymbol{p}) = \frac{{\rm d}N}{{\rm d}^3x\,{\rm d}^3p} = \frac{{\rm d}N}{{\rm d}A\,{\rm d}t\,{\rm d}\Omega\,{\rm d}E} = \frac{I}{hcp^2}\,. \tag{5.7}$$

[1] We define here intensity as connected to the energy flux $\mathcal{F}$, while often the particle flux is used.

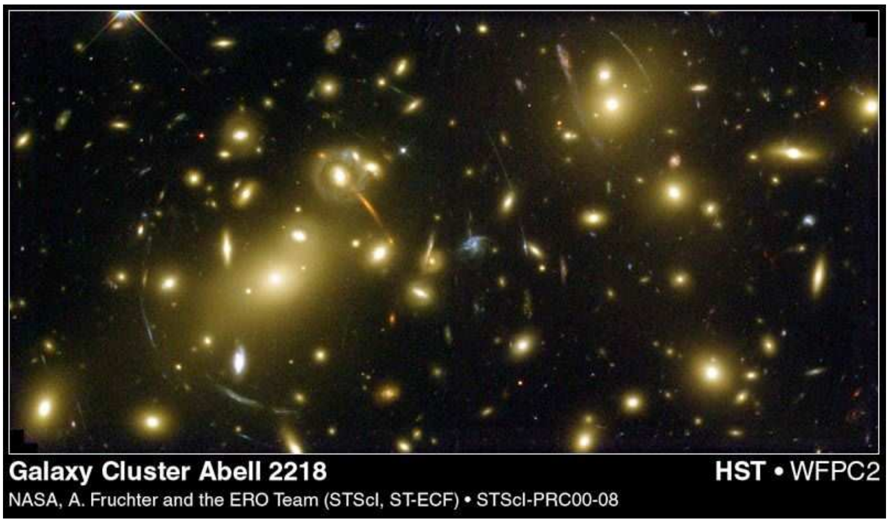


Figure 5.2: Gravitational lensing of the galaxy cluster Abell 2218.

Thus the observed intensity I equals the surface brightness B of the source.

Gravity can affect this result in two ways: First, gravity can redshift the frequency of photons, $\nu_{\rm sr} = \nu_{\rm obs}(1+z)$. This can be either the gravitational redshift as in Sec. 4.2 or a cosmological redshift due to the expansion of the universe (that will be discussed in Sec. 11.2). Thus the intensity $I_{\rm obs}$ at the observed frequency photons $\nu_{\rm obs}$ is the emitted intensity evaluated at $\nu_{\rm sr} = \nu_{\rm obs}(1+z)$ and reduced by $(1+z)^3$,

$$I_{\rm obs}(\nu_{\rm obs}) = \frac{I(\nu_{\rm sr})}{(1+z)^3}\,. \tag{5.8}$$

In both cases, this redshift depends only on the initial and the final point of the photon trajectory, but not on the actual path in-between. Thus the redshift cancels if one considers the relative magnification of a source by gravitational lensing.

Second, gravitational lensing affects the solid angle the source is seen in a detector of fixed size. As a result, the apparent brightness of a source increases proportionally to the increase of the visible solid angle, if the source cannot be resolved as a extended object. Hence we can compute the magnification of a source by calculating the ratio of the solid angle visible without and with lensing.

In Fig. 5.3, we sketch how the two lensed images are stretched: An infinitesimal small surface element $2\pi\sin\beta\mathrm{d}\beta\mathrm{d}\phi \approx 2\pi\beta\mathrm{d}\beta\mathrm{d}\phi$ of the unlensed source becomes in the lense plane $2\pi\vartheta_\pm\mathrm{d}\vartheta_\pm\mathrm{d}\phi$. Thus the images are tangentially stretched by $\vartheta_\pm/\beta$, while the radial size is changed by $\mathrm{d}\vartheta_\pm/\mathrm{d}\beta$. Thus the magnification $a_\pm$ of the source is

$$a_\pm = \left|\frac{\vartheta_\pm\mathrm{d}\vartheta_\pm}{\beta\mathrm{d}\beta}\right|\,. \tag{5.9}$$

Differentiating Eq. (5.5) gives

$$\frac{\mathrm{d}\vartheta_\pm}{\mathrm{d}\beta} = \frac{1}{2}\left[1 \pm \frac{\beta}{(\beta^2+4\vartheta_{\rm E}^2)^{1/2}}\right] = \frac{1}{2}\left[1 \pm \frac{1}{(1+4x^2)^{1/2}}\right], \tag{5.10}$$

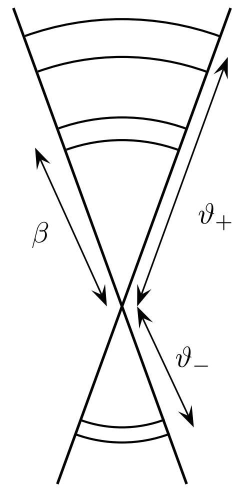


Figure 5.3: The effect of gravitational lensing on the shape of an extended source: The surface element $2\pi\beta d\beta d\phi$ of the unlensed image at position β is transformed into the two lensed images of size $2\pi\vartheta_\pm d\vartheta_\pm d\phi$ at position $\vartheta_\pm$.

where we introduced $x = \vartheta_E/\beta$. Thus the magnification of the two images becomes

$$a_\pm = \frac{1}{4}\left|\left[2 \pm (1+4x^2)^{1/2} \pm \frac{1}{(1+4x^2)^{1/2}}\right]\right|. \tag{5.11}$$

Since $y + 1/y \geq 2$, we have to choose the over-all sign of the expression in the bracket such that $a_{\rm tot} = y + 1/y \pm 2$. Then the total magnification is

$$a_{\rm tot} = a_+ + a_- = \frac{1}{2}\left[(1+4x^2)^{1/2} + \frac{1}{(1+4x^2)^{1/2}}\right]. \tag{5.12}$$

For large separation x, the magnification $a_{\rm tot}$ goes to one, while the magnification diverges for $x \to 0$ as $a_{\rm tot} \sim 1/x$: In this limit we would receive light from an infinite number of images on the Einstein circle. Physically, the approximation of a point source breaks down when x reaches the extension of the source. Since $a_{\rm tot}$ is larger than one, gravitational lensing always increases the total flux observed from a lensed source, facilitating the observation of very faint objects. As compensation, the source appears slightly dimmed to all those observers who do not see the source lensed.

Two important applications of gravitational lensing are the search for dark matter in the form of black holes or brown dwarfs in our own galaxy by microlensing and the determination of the value of the cosmological constant by weak lensing observations.

In microlensing experiments that have tried to detect dark matter in the form of MACHOs (black holes, brown dwarfs,...) one observed stars of the LMC. If a MACHO with speed $v \approx 220\,\text{km/s}$ moves through the line-of-sight of a monitored star, its light-curve is magnified temporally. If v is the perpendicular velocity of the source,

$$\beta(t) = \left[\beta_0^2 + \frac{v^2}{D_{\rm ol}^2}(t-t_0)^2\right]^{1/2} \tag{5.13}$$

The magnification $a(t)$ is symmetric around t_0 and its shape can be determined inserting typical values for $D_{\rm ol}$ and the MACHO mass.

More about lensing in Ref. [3].

6 Black holes

A black hole is a solution of Einstein's equations containing a physical singularity which in turn is covered by an event horizon. Such a horizon acts classically as a perfect unidirectional membrane which any causal influence can cross only towards the singularity. A real understanding of the physical significance of the singularities contained in the solution was obtained only in the 1960s.

Apart from the Schwarzschild solution valid for a static, uncharged black hole, we will examine also the Kerr solution for a rotating, and (very briefly) the solution for a charged black hole.

6.1 Schwarzschild black hole

Our main aim in this section to obtain a better understanding of the meaning of the hypersurface $r = 2M$ in the Schwarzschild metric. This requires to find new coordinates which are non-singular at $r = 2M$. In addition, these coordinates should facilitate the understanding of the causal structure of the spacetime. In the case of the Schwarzschild black hole, we would like to transform the two-dimensional subspace $\{t, r\}$ to new coordinates $\{T, R\}$ which are regular for $R > 0$ and conformally flat. We start this enterprise with few definitions.

Conformal flatness A conformal transformation of the metric,

$$g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x). \tag{6.1}$$

changes distances, but keeps angles invariant. Thus the causal structure of two conformally related spacetimes is identical. A spacetime is called conformally flat if it is connected by a conformal transformation to Minkowski space,

$$g_{\mu\nu}(x) = \Omega^2(x) \eta_{\mu\nu}(x) = \mathrm{e}^{2\omega(x)} \eta_{\mu\nu}(x). \tag{6.2}$$

In particular, light-rays also propagate in conformally flat spacetimes along straight lines at ± 45 degrees to the time axis. In the appendix 6.B, we show that any two-dimensional manifold is conformally flat.

Hypersurfaces We can define a three-dimensional hypersurface H by imposing a constraint, $f(x^\mu) = a$, on the spacetime coordinates x^μ. For two infinitesimally separated points in such a hypersurface, $P(x^\mu)$ and $Q(x^\mu + \mathrm{d}x^\mu)$, we can expand the constraint as

$$a = f(x^\mu + \mathrm{d}x^\mu) = f(x^\mu) + \frac{\partial f}{\partial x^\mu} \mathrm{d}x^\mu. \tag{6.3}$$

Subtracting then $a = f(x^\mu)$, we obtain

$$0 = \frac{\partial f}{\partial x^\mu} \mathrm{d}x^\mu = \nabla_\mu f \mathrm{d}x^\mu. \tag{6.4}$$

This equation defines the gradient $n_\mu = \partial_\mu f$. Because of

$$0 = n_\mu \mathrm{d}x^\mu = g_{\mu\nu} n^\mu \mathrm{d}x^\nu, \tag{6.5}$$

we see that n^μ is orthogonal to $\mathrm{d}x^\nu$ which in turn lies in H. Thus the 1-form n_μ is normal to the hypersurface H.

Horizons We define an *event horizon* as a three-dimensional hypersurface which limits a region of a spacetime which can never influence an observer. The event horizon is formed by light-rays and is therefore a null surface. Hence we require that at each point of such a surface defined by $f(x^\mu) = 0$ a null tangent vector n^μ exists that is orthogonal to two space-like tangent vectors. The normal n^μ to this surface is parallel to the gradient along the surface, $n^\mu = h\nabla^\mu f = h\partial^\mu f$, where h is an arbitrary non-zero function. From

$$0 = n_\mu n^\mu = g_{\mu\nu} n^\mu n^\nu \tag{6.6}$$

we see that the line element vanishes on the horizon, $\mathrm{d}s = 0$. Hence the (future) light-cones at each point of an event horizon are tangential to the horizon. The normal $\boldsymbol{n}$ to this surface is parallel to the gradient along the surface, $n_a = \nabla_a f = \partial_a f$. From

$$0 = n_a n^a = g^{ab} n_a n_b = g^{ab} n_a \partial_b f\,. \tag{6.7}$$

and $\mathrm{d}f = n_a \mathrm{d}x^a = 0$ we see that the normal vector is parallel to $\mathrm{d}x^a$ and thus lies inside the surface. In a stationary, axisymmetric spacetime the general equation of a surface, $f(x^\mu) = 0$, simplifies to $f(r, \vartheta) = 0$. The condition for a null surface becomes

$$0 = g^{\mu\nu}(\partial_\mu f)(\partial_\nu f) = g^{rr}(\partial_r f)^2 + g^{\vartheta\vartheta}(\partial_\vartheta f)^2. \tag{6.8}$$

Example 6.5: normal in,outside space/timelike in Minkowski space.

Eddington–Finkelstein coordinates We next try to find new coordinates which are regular at $r = 2M$ and valid in the whole range $0 < r < \infty$. Such a coordinate transformation has to be singular at $r = 2M$, otherwise we cannot hope to cancel the singularity present in the Schwarzschild coordinates. We can eliminate the troublesome factor $g_{rr} = (1 - 2M/r)^{-1}$ introducing a new radial coordinate r^* defined by

$$\mathrm{d}r^* = \frac{\mathrm{d}r}{1 - \frac{2M}{r}}. \tag{6.9}$$

Integrating (6.9) results in

$$r^*(r) = r + 2M \ln\left|\frac{r}{2M} - 1\right| + A, \tag{6.10}$$

with $A \equiv -2Ma$ as integration constant. The coordinate $r^*(r)$ is often called tortoise coordinate, because $r^*(r)$ changes only logarithmically close to the horizon. This coordinate change maps the range $r \in [2M, \infty]$ of the radial coordinate onto $r^* \in [-\infty, \infty]$. A radial null geodesics satisfies $\mathrm{d}(t \pm r^*) = 0$, and thus in- and out-going light-rays are given by

$$\tilde{u} \equiv t - r^* = t - r - 2M \ln\left|\frac{r}{2M} - 1\right| - A, \qquad \text{outgoing rays,} \tag{6.11}$$

$$\tilde{v} \equiv t + r^* = t + r + 2M \ln\left|\frac{r}{2M} - 1\right| + A, \qquad \text{ingoing rays.} \tag{6.12}$$

For $r > 2M$, Eq. (4.42) implies that $\mathrm{d}r/\mathrm{d}t > 0$ so that r increases with t. Therefore (6.11) describes outgoing light-rays, while (6.12) corresponds to ingoing light-rays for $r > 2M$.

We can extend now the Schwarzschild metric using as coordinate the "advanced time parameter $\tilde{v}$" instead of t. Forming the differential,

$$\mathrm{d}\tilde{v} = \mathrm{d}t + \mathrm{d}r + \left(\frac{r}{2M} - 1\right)^{-1} \mathrm{d}r = \mathrm{d}t + \left(1 - \frac{2M}{r}\right)^{-1} \mathrm{d}r, \tag{6.13}$$

we can eliminate $\mathrm{d}t$ from the Schwarzschild metric and find

$$\mathrm{d}s^2 = \left(1 - \frac{2M}{r}\right) \mathrm{d}\tilde{v}^2 - 2\mathrm{d}\tilde{v}\mathrm{d}r - r^2 \mathrm{d}\Omega. \tag{6.14}$$

This metric was found first by Eddington and was later rediscovered by Finkelstein. Although $g_{\tilde{v}\tilde{v}}$ vanishes at $r = 2M$, the determinant $g = r^4 \sin^2\vartheta$ is non-zero at the horizon and thus the metric is invertible. Moreover, r^* was defined by (6.10) initially only for $r > 2M$, but we can use this definition also for $r < 2M$, arriving at the same expression (6.14). Therefore, the metric using the advanced time parameter $\tilde{v}$ is regular at $2M$ and valid for all $r > 0$. We can view this metric hence as an extension of the $r > 2M$ part of the Schwarzschild solution, similar to the process of analytic continuation of complex functions. The price we have to pay for a non-zero determinant at $r = 2M$ are non-diagonal terms in the metric. As a result, the spacetime described by (6.14) is not symmetric under the exchange $t \to -t$. We will see shortly the consequences of this asymmetry.

We now study the behaviour of radial light-rays, which are determined by $\mathrm{d}s^2 = 0$ and $\mathrm{d}\phi = \mathrm{d}\vartheta = 0$. Thus radial light-rays satisfy $A\mathrm{d}\tilde{v}^2 - 2\mathrm{d}\tilde{v}\mathrm{d}r = 0$, which is trivially solved by ingoing light-rays, $\mathrm{d}\tilde{v} = 0$ and thus $\tilde{v} = \text{const}$. The solutions for $\mathrm{d}\tilde{v} \neq 0$ are given by (6.12). Additionally, the horizon $r = 2M$ which is formed by stationary light-rays satisfies $\mathrm{d}s^2 = 0$. In order to draw a spacetime diagram, it is more convenient to replace the light-like coordinate $\tilde{v}$ by a new time-like coordinate. We show in the left panel of Fig. 6.1 geodesics using as new time coordinate $\tilde{t} = \tilde{v} - r$. Then the ingoing light-rays are straight lines at 45° to the r axis. Radial light-rays which are outgoing for $r > 2M$ and ingoing for $r < 2M$ follow Eq. (6.12). A few future light-cones are indicated: they are formed by the intersection of light-rays, and they tilt towards $r = 0$ as they approach the horizon. At $r = 2M$, one light-ray forming the light-cone becomes stationary and part of the horizon, while the remaining part of the cone lies completely inside the horizon.

Let us now discuss how Fig. 6.1 would look like using the retarded Eddington–Finkelstein coordinate $\tilde{u}$. Then the outgoing radial null geodesics are straight lines at 45°. They start from the singularity, crossing smoothly $r = 2M$ and continue to spatial infinity. Such a situation, where the singularity is not covered by an event horizon is called a "white hole". The cosmic censorship hypothesis postulates that singularities formed in gravitational collapse are always covered by event horizons. This implies that the time-invariance of the Einstein equations is broken by its solutions. In particular, only the BH solution using the retarded

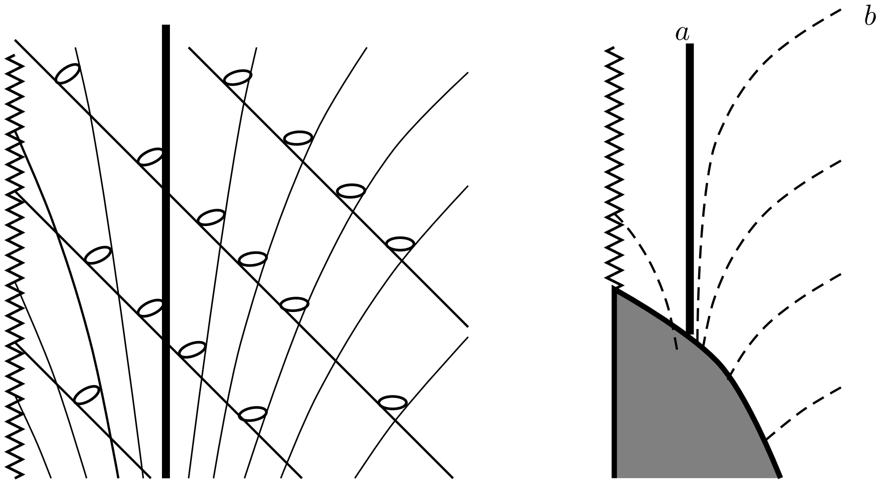


Figure 6.1: Left: The Schwarzschild spacetime using advanced Eddington–Finkelstein coordinates; the singularity is shown by a zigzag line, the horizon by a thick line and geodesics by thin lines. Right: Collapse of a star modelled by pressureless matter; dashes lines show geodesics, the thin solid line encompasses the collapsing stellar surface.

Eddington–Finkelstein coordinates should be realised by nature—otherwise we should expect causality to be violated. This behaviour may be compared to classical electrodynamics, where all solutions of the wave equation are described by the retarded Green function, while the advanced Green function seems to have no relevance.

Collapse to a BH After a star has consumed its nuclear fuel, gravity can be balanced only by the Fermi degeneracy pressure of its constituents. Increasing the total mass of the star remnant, the stellar EoS is driven towards the relativistic regime until the star becomes unstable. As a result, the collapse of its core to a BH seems to be inevitable for a sufficiently heavy star.

Let us consider a toy model for such a gravitational collapse. We describe the star by a spherically symmetric cloud of pressureless matter. While the assumption of negligible pressure is unrealistic, it implies that particles at the surface of the star follow radial geodesics in the Schwarzschild spacetime. Thus we do not have to bother about the interior solution of the star, where $T_{\mu\nu} \neq 0$ and our vacuum solution does not apply. In advanced Eddington–Finkelstein coordinates, the collapse is schematically shown in the right panel of Fig. 6.1. At the end of the collapse, a stationary Schwarzschild BH has formed. Note that in our toy model the event horizon forms before the singularity, as required by the cosmic censorship hypothesis. The horizon grows from $r = 0$ following the light-like geodesic a shown by the thin black line until it reaches its final size $R_s = 2M$. What happens if we drop a lump of matter δM on a radial geodesics into the BH? Since we do not add angular momentum to the BH, the final stage is, according to the Birkhoff's theorem, still a Schwarzschild BH. All deviations from spherical symmetry corresponding to gradient energy in the intermediate regime are being radiated away as gravitational waves. Thus in the final stage, the only change is an increase of the horizon, size $R_s \to 2(M + \delta M)$. Therefore some light-rays (e.g. b) which we expected to escape to spatial infinity will be trapped. Similarly, light-ray a, which we thought to form the horizon, will be deflected by the increased gravitational attraction towards the singularity. In essence, knowing only the spacetime up to a fixed time t, we are not able to decide which light-rays form the horizon. The event horizon of a black hole is a global property of the spacetime: It is not only independent of the observer but also influenced by the complete spacetime.

How does the stellar collapse looks like for an observer at large distances? Let us assume that the observer uses a neutrino detector and is able to measure the neutrino luminosity $L_\nu(r) = \mathrm{d}E_\nu/\mathrm{d}t = N_\nu\omega_\nu/\mathrm{d}t$ emitted by a shell of stellar material at radius r. In order to determine the luminosity $L_\nu(r)$, we have to connect the radial coordinate r of the shell and the emission time t. Using in Eq. (4.41), the leading term for $r \to 2M$ gives

$$\frac{\sqrt{r/2M}+1}{\sqrt{r/2M}-1} = \mathrm{e}^{-(t-t_0)/2M} \tag{6.15}$$

or

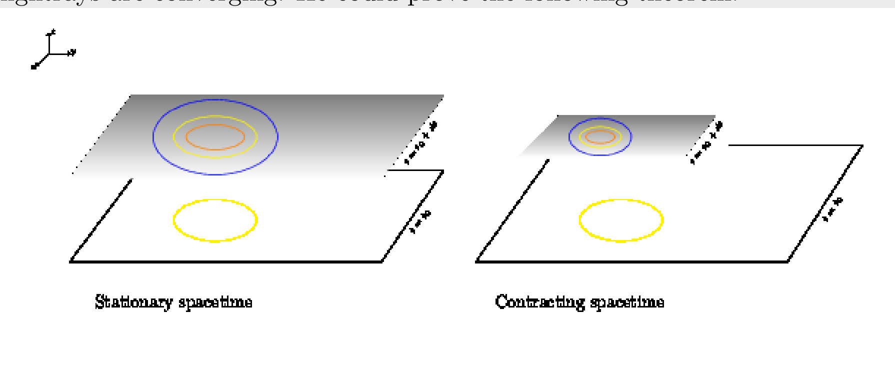
$$\frac{r}{2M} - 1 \propto \mathrm{e}^{-(t-t_0)/4M}. \tag{6.16}$$

For an observer at large distance r_0, the time difference between two pulses sent by a shell falling into a BH increases thus exponentially with the characteristic time scale $\tau = 4M$ for $r \to 2M$. As a result the energy ω_ν of an individual neutrino is also exponentially redshifted

$$\omega_\nu(r) = \omega_\nu(r_0)\mathrm{e}^{-(t-t_0)/4M}. \tag{6.17}$$

A more detailed analysis confirms the expectation that then also the luminosity decreases exponentially. Thus an observer at infinity will not see shells which slow down logarithmically as they fall towards $r \to 2M$, as suggested by Eq. (4.41). Instead the signal emitted by the shell will fade away exponentially, with the short characteristic time scale of $M = Mt_{\rm Pl}/M_{\rm Pl} \approx 10^{-5}\,$s for a stellar-size BH.

Remark 6.2: Singularity theorems: The collapse of a star within GR was first discused by Oppenheimer and Synder 1936: They concluded that "...The star thus tends to close itself off from any communication with a distant observer; only its gravitational field persists." However, the predominant attitude was that the appearance of singularities is an artefact of the assumed spherical collapse. Only after the first hints for the existence of supermassive black holes (discovery of quasars 1963), the general attitude changed.
But how to tackle the non-spherical problem? Penrose introduced the concept of a trapped surface, i.e. a closed space-like two-dimensional surface for which both the in- and out-going lightrays are converging. He could prove the following theorem:



Assume a spacetime allows for a well-posed initial value problem. Then a future incomplete light-path exists, if it contains a closed future-trapped surface, and if the matter is "normal". (For a definition of "normal" matter see remark 8.4.)

Since the theorem relies on topology, its conclusion is stable under perturbations; Moreover, it is well suited for studies in numerical relativity.

Kruskal coordinates We have been able to extend the Schwarzschild solution into two different branches; a BH solution using the advanced time parameter $\tilde{v}$ and a white hole solution using the retarded time parameter $\tilde{u}$. The analogy with the analytic continuation of complex functions leads naturally to the question of whether we can combine these two branches into one common solution. Moreover, our experience with the Rindler metric suggests that an event horizon where energies are exponentially redshifted implies the emission of a thermal spectrum. If true, our BH would not be black after all. One way to test this suggestion is to relate the vacua as defined by different observers via a Bogolyubov transformation. In order to simplify this process, we would like to find new coordinates for which the Schwarzschild spacetime is conformally flat.

An obvious attempt to proceed is to use both the advanced and the retarded time parameters. For most of our discussion, it is sufficient to concentrate on the t, r coordinates in the line element $\mathrm{d}s^2 = \mathrm{d}\bar{s}^2 + r^2\mathrm{d}\Omega$, and to neglect the angular dependence from the $r^2\mathrm{d}\Omega$ part. We start by eliminating r in favour of r^*,

$$\mathrm{d}\bar{s}^2 = \left(1 - \frac{2M}{r(r^*)}\right)(\mathrm{d}t^2 - \mathrm{d}r^{*2}), \tag{6.18}$$

where r has to be expressed through r^*. This metric is conformally flat but the definition of $r(r^*)$ on the horizon contains the ill-defined factor $\ln(2m/r - 1)$. Clearly, a new set of coordinates where this factor is exponentiated is what we are seeking.

This is achieved introducing both Eddington–Finkelstein parameters,

$$\tilde{u} = t - r^*, \qquad \tilde{v} = t + r^*, \tag{6.19}$$

for which the metric simplifies to

$$\mathrm{d}\bar{s}^2 = \left(1 - \frac{2M}{r(\tilde{u}, \tilde{v})}\right)\mathrm{d}\tilde{u}\mathrm{d}\tilde{v}. \tag{6.20}$$

From (6.10) and (6.19), it follows

$$\frac{\tilde{v} - \tilde{u}}{2} = r^*(r) = r + 2M\ln\left|\frac{r}{2M} - 1\right| - 2Ma, \tag{6.21}$$

or

$$1 - \frac{2M}{r} = \frac{2M}{r}\exp\left(\frac{\tilde{v} - \tilde{u}}{4M}\right)\exp\left(a - \frac{r}{2M}\right). \tag{6.22}$$

This allows us to eliminate the singular factor $1 - 2M/r$ in (6.20), obtaining

$$\mathrm{d}\bar{s}^2 = \frac{2M}{r}\exp\left(a - \frac{r}{2M}\right)\exp\left(-\frac{\tilde{u}}{4M}\right)\mathrm{d}\tilde{u}\exp\left(\frac{\tilde{v}}{4M}\right)\mathrm{d}\tilde{v}. \tag{6.23}$$

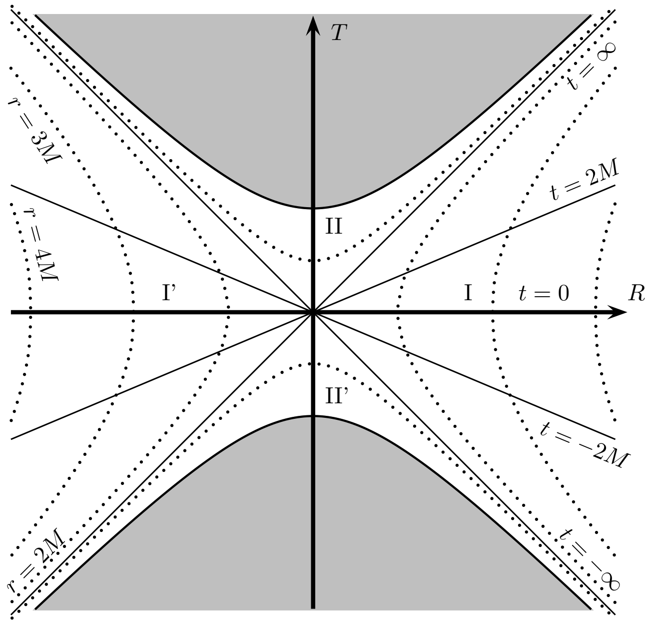


Figure 6.2: Spacetime diagram for the Kruskal coordinates T and R.

Finally, we change to Kruskal light-cone coordinates u and v defined by

$$u = -4M \exp\left(-\frac{\tilde{u}}{4M}\right) \quad \text{and} \quad v = 4M \exp\left(\frac{\tilde{v}}{4M}\right), \tag{6.24}$$

arriving at

$$\mathrm{d}s^2 = \frac{2M}{r} \exp\left(a - \frac{r}{2M}\right) \mathrm{d}u\mathrm{d}v + r^2 \mathrm{d}\Omega. \tag{6.25}$$

Kruskal diagram The coordinates $\tilde{u}, \tilde{v}$ cover only the exterior $r > 2M$ of the Schwarzschild spacetime, and thus u, v are initially only defined for $r > 2M$. Since they are regular at the Schwarzschild radius, we can extend these coordinates towards $r = 0$. In order to draw the spacetime diagram of the full Schwarzschild spacetime shown in Fig. 6.2, it is useful to go back to time- and space-like coordinates via

$$u = T - R \quad \text{and} \quad v = T + R. \tag{6.26}$$

Then the connection between the pair of coordinates $\{T, R\}$, $\{u, v\}$ and $\{t, r\}$ is given by

$$uv = T^2 - R^2 = -16M^2 \exp\left(\frac{r^*}{2M}\right) = -16M^2 \left(\frac{r}{2M} - 1\right) \exp\left(\frac{r}{2M} - a\right), \tag{6.27a}$$

$$\frac{u}{v} = \frac{T-R}{T+R} = \exp\left[-t/(2M)\right]. \tag{6.27b}$$

Lines with $r =$ const. are given by $uv = T^2 - R^2 =$ const. They are thus parabola shown as dotted lines in Fig. 6.4. Lines with $t =$ const. are determined by $u/v =$ const. and are thus given by straight (solid) lines through zero. In particular, null geodesics correspond to straight lines with angle 45° in the $R - T$ diagram. The horizon $r = 2M$ is given by to $u = 0$

or $v = 0$. Hence two separate horizons exist: a past horizon at $t = -\infty$ (for $v = 0$ and thus $T = -R$) and a future horizon at $t = +\infty$ (for $u = 0$ and thus $T = R$). Also, the singularity at $r = 0$ corresponds to two separate lines in the $R - T$ Kruskal diagram[1] and is given by

$$T = \pm\sqrt{16M^2 + R^2}. \tag{6.28}$$

The horizon lines $\{t = -\infty, r = 2M\}$ and $\{t = \infty, r = 2M\}$ divide the spacetime in four parts. The future singularity is unavoidable in part II, while in region II' all trajectories start at the past singularity. Region I corresponds to the original Schwarzschild solution outside the horizon $r > 2M$, while region I and II encompass the advanced Eddington–Finkelstein solution. The regions I' and II' represent the retarded Eddington–Finkelstein solution, where II' corresponds to a white hole. Note that I' represents a new asymptotically flat Schwarzschild exterior solution.

The presence of a past horizon $v = 0$ at $t = -\infty$ makes the complete BH solutions time-symmetric and corresponds to an eternal BH. If we model a realistic BH, that is, one that was created at finite t by a collapsing mass distribution, with Kruskal coordinates, then any effect induced by the past horizon should be considered as unphysical.

6.2 Reissner-Nordström black hole

The solution of the coupled Einstein-Maxwell equations for a point-like particle with mass M and electric charge Q was found by Reissner and Nordström. We will derive its line-element,

$$\mathrm{d}s^2 = A(r)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{A(r)} - r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\phi^2) \tag{6.29}$$

with

$$A(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \tag{6.30}$$

in section 9.2. As in the Schwarzschild case, the position of the horizon is determined by the solution of $A = 0$. However, now $A = 0$ is a quadratic equations with the solution

$$r_\pm = M \pm \sqrt{M^2 - Q^2}. \tag{6.31}$$

Thus there exists an outer and an inner horizon given by r_+ and r_-, respectively. Moreover, the charge of the black hole cannot be arbitrarly large, as $r_\pm$ becomes complex for $Q > 2\sqrt{\pi}M$. Possibly singularities are given by $A \to \infty$, i.e. at $r = 0$ and $r = r_\pm$. We will show next that the latter are coordinate singularities, determining a metric which is regular at r.

Eddington-Finkelstein coordinates Following the same approach as for a Schwarzschild black hole, we use the equation for an ingoing radial photon to introduce advanced Eddington-Finkelstein coordinates. Radial light-rays satisfy $\mathrm{d}s^2 = 0$ and $\mathrm{d}\phi = \mathrm{d}\vartheta = 0$, leading to

$$\frac{\mathrm{d}r}{\mathrm{d}t} = 1 - 2M/r + Q^2/r^2 = \pm\frac{(r - r_-)(r - r_+)}{r^2}$$

[1] Recall that we suppress two space dimension: Thus a point in the $R - T$ Kruskal diagram correspond to a sphere S^2, and a line to $\mathbb{R} \times S^2$.

where Eq. (6.36) was used in the second step. Inverting and integrating gives

$$t = r - \frac{r_-^2}{r_+ - r_-} \ln \left| \frac{r}{r_-} - 1 \right| + \frac{r_+^2}{r_+ - r_-} \ln \left| \frac{r}{r_+} - 1 \right| + \text{const.} \tag{6.32}$$

$$t = -r - + \frac{r_-^2}{r_+ - r_-} \ln \left| \frac{r}{r_-} - 1 \right| - \frac{r_+^2}{r_+ - r_-} \ln \left| \frac{r}{r_+} - 1 \right| + \text{const.} \tag{6.33}$$

where the first equation describes an outgoing and the second an ingoing light-ray. The advanced Eddington-Finkelstein coordinate uses the integration constant $p = \text{const.}$ as new coordinate. Changing then again to the time-like coordinate $t' = p - r$ gives

$$t' = t - \frac{r_-^2}{r_+ - r_-} \ln \left| \frac{r}{r_-} - 1 \right| + \frac{r_+^2}{r_+ - r_-} \ln \left| \frac{r}{r_+} - 1 \right| . \tag{6.34}$$

Differentiating gives

$$\mathrm{d}t' = \mathrm{d}p - \mathrm{d}r = \mathrm{d}t + (1/A - 1)\,\mathrm{d}r. \tag{6.35}$$

Inserting $\mathrm{d}t'$ into the metric, one finds that the spacetime is regular for all $r > 0$.

Ingoing radial light-rays satisfy

$$t' + r = \text{const.},$$

while for outgoing it holds

$$\frac{\mathrm{d}t'}{\mathrm{d}r} = \frac{2 - A}{A}.$$

Thus ingoing radial light-rays are straight lines at 45°. The gradient of the outgoing ones is determined by $(2 - A)/A$.

Next we determine the horizons of the Reissner-Nordström BH solution (for any M and Q). Coordinate singularities and horizons are given by the solution of $A = 0$,

$$r_\pm = M \pm \sqrt{M^2 - Q^2}. \tag{6.36}$$

Depending on the values of M and Q, we have to distinguish three cases: i) For $Q^2 < M^2$, both $r_\pm$ are real and two horizons exists; ii) $Q^2 = M^2$, r_- and r_+ coincide; iii) $Q^2 > M^2$, both $r_\pm$ are imaginary. No coordinate singularity exists and the metric is regular except at $r = 0$. Since the singularity is not coveed by an event horizon, this case should not be possible in the real world.

6.3 Kerr black holes

The stationary spacetime outside a rotating mass distribution can be derived by symmetry arguments similarly (but much more tortorous...) to the case of the Schwarzschild metric. It was found first accidentally by R. Kerr in 1963. The black hole solution of this spacetime is fully characterised by two quantities, the mass M and the angular momentum L of the Kerr BH. Both parameters can be manipulated, at least in a gedankenexperiment, dropping material into the BH. Examining the response of a Kerr black hole to such changes was crucial for the discovery of "black hole thermodynamics".

In Boyer–Lindquist coordinates, the metric outside of a rotating mass distribution is given by

$$\begin{aligned} \mathrm{d}s^2 = &\left(1 - \frac{2Mr}{\rho^2}\right)\mathrm{d}t^2 + \frac{4Mar\sin^2\vartheta}{\rho^2}\mathrm{d}\phi\mathrm{d}t - \frac{\rho^2}{\Delta}\mathrm{d}r^2 - \rho^2\mathrm{d}\vartheta^2 \\ &- \left(r^2 + a^2 + \frac{2Mra^2\sin^2\vartheta}{\rho^2}\right)\sin^2\vartheta\mathrm{d}\phi^2, \end{aligned} \tag{6.37}$$

with the abbreviations

$$a = L/M\,, \qquad \rho^2 = r^2 + a^2\cos^2\vartheta, \qquad \Delta = r^2 - 2Mr + a^2. \tag{6.38}$$

The metric is time-independent and axially symmetric. Hence the two Killing vectors are, as in the Schwarzschild case, $\boldsymbol{\xi} = (1, 0, 0, 0)$ and $\boldsymbol{\eta} = (0, 0, 0, 1)$, where we again order coordinates as $\{t, r, \vartheta, \phi\}$.

The presence of the mixed term $g_{t\phi}$ means that the metric is stationary, but not static—as one expects for a star or BH rotating with constant rotation velocity. Finally, the metric is asymptotically flat and the weak-field limit shows that L is the angular momentum of the rotating black hole.

Its main properties are

- The metric is asymptotically flat.
- Potential singularities at $\rho = 0$ and $\Delta = 0$.
- The weak-field limit shows that L is the angular momentum of the rotating black hole.
- The presence of the mixed term $g_{t\phi}$ means that infalling particles (and thus space-time) is dragged around the rotating black hole.

Orbits in the equatorial plane $\vartheta = \pi/2$ could be derived in the same way as for the Schwarzschild case, for $\vartheta \neq \pi/2$ the discussion becomes much more involved.

Singularity First we examine the potential singularities at $\rho = 0$ and $\Delta = 0$. The calculation of the scalar invariants formed from the Riemann tensor shows that only $\rho = 0$ is a physical singularity, while $\Delta = 0$ corresponds to a coordinate singularity. The physical singularity at $\rho^2 = 0 = r^2 + a^2\cos\vartheta^2$ corresponds to $r = 0$ and $\vartheta = \pi/2$. Thus the value $r = 0$ is surprisingly not compatible with all ϑ values. o understand this point, we consider the $M \to 0$ limit of the Kerr metric (6.37) keeping $a = L/M$ fixed,

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \frac{\rho^2}{r^2 + a^2}\mathrm{d}r^2 - \rho^2\mathrm{d}\vartheta^2 - (r^2 + a^2)\sin^2\vartheta\mathrm{d}\phi^2. \tag{6.39}$$

The comparison with the Minkowski metric shows that

$$\begin{aligned} x &= \sqrt{r^2 + a^2}\sin\vartheta\cos\phi, \qquad & z = r\cos\vartheta, \\ y &= \sqrt{r^2 + a^2}\sin\vartheta\sin\phi, \end{aligned} \tag{6.40}$$

Hence the singularity at $r = 0$ and $\vartheta = \pi/2$ corresponds to a ring of radius a in the equatorial plane $z = 0$ of the Kerr black hole.

Horizons We have defined an event horizon as a three-dimensional hypersurface, $f(x^\mu) = 0$, that is null. In a stationary, axisymmetric spacetime the general equation of a surface, $f(x^\mu) = 0$, simplifies to $f(r, \vartheta) = 0$. The condition for a null surface becomes

$$0 = g^{\mu\nu}(\partial_\mu f)(\partial_\nu f) = g^{rr}(\partial_r f)^2 + g^{\vartheta\vartheta}(\partial_\vartheta f)^2. \tag{6.41}$$

In the case of the surface defined by the coordinate singularity $\Delta = r^2 - 2Mr + a^2 = 0$ that depends only on r,

$$r_\pm = M \pm \sqrt{M^2 - a^2}, \tag{6.42}$$

the condition defining a horizons becomes simply $g^{rr} = 0$ or $g_{rr} = 1/g^{rr} = \infty$. Hence, r_- and r_+ define an inner and outer horizon around a Kerr black hole.

The surface A of the outer horizon follows from inserting r_+ together with $\mathrm{d}r = \mathrm{d}t = 0$ into the metric,

$$\mathrm{d}s^2 = \rho_+^2 \mathrm{d}\vartheta^2 + \left(r_+^2 + a^2 + \frac{2Mr_+a^2\sin^2\vartheta}{\rho_+^2}\right)\sin^2\vartheta \mathrm{d}\phi^2, \tag{6.43}$$

Using $r_\pm^2 + a^2 = 2Mr_\pm$, we obtain

$$\mathrm{d}s^2 = \rho_+^2 \mathrm{d}\vartheta^2 + \left(\frac{2Mr_+}{\rho_+}\right)^2 \sin^2\vartheta \mathrm{d}\phi^2. \tag{6.44}$$

Hence the metric determinant g_2 restricted to the angular variables is given by $\sqrt{g_2} = \sqrt{g_{\vartheta\vartheta}g_{\phi\phi}} = 2Mr_+\sin\vartheta$. As expected for an axialsymmetric metric, the two-dimensional surface described by the metric (6.44) is not the one of a sphere S^2. Integrating gives the area A of the horizon as

$$A = \int_0^{2\pi} \mathrm{d}\phi \int_0^\pi \mathrm{d}\vartheta \sqrt{g_2} = 8\pi Mr_+ = 8\pi M(M + \sqrt{M^2 - a^2}). \tag{6.45}$$

Note that the area depends on the angular momentum of the black hole that can in turn be manipulated by dropping material into the hole. The horizon area A for fixed mass M becomes maximal for a non-rotating black hole, $A = 16\pi M^2$, and decreases to $A = 8\pi M^2$ for a maximally rotating one with $a = M$. For $a > M$, the metric component $g^{rr} = \Delta$ has no real zero and thus no event horizon exists.

(For an interpretation see the space-time diagram 6.4 that uses coordinates of the advanced Eddington-Finkelstein type.)

Ergosphere and dragging of inertial frames The Kerr metric is a special case of a metric with $g_{t\phi} \neq 0$. As result, both massive and massless particles with zero angular momentum alling into a Kerr black hole will acquire a non-zero angular rotation velocity $\omega = \mathrm{d}\phi/\mathrm{d}t$ as seen by an observer from infinity.

We consider a light-ray with $\mathrm{d}\vartheta = \mathrm{d}r = 0$. Then the line element becomes

$$g_{tt}\mathrm{d}t^2 + 2g_{t\phi}\mathrm{d}t\mathrm{d}\phi + g_{\phi\phi}\mathrm{d}\phi^2 = 0. \tag{6.46}$$

Dividing by $g_{\phi\phi}\mathrm{d}t^2$, we obtain a quadratic equation for the angular rotation velocity $\omega = \mathrm{d}\phi/\mathrm{d}t$,

$$\omega^2 + 2\,\frac{g_{t\phi}}{g_{\phi\phi}}\,\omega + \frac{g_{tt}}{g_{\phi\phi}} = 0 \tag{6.47}$$

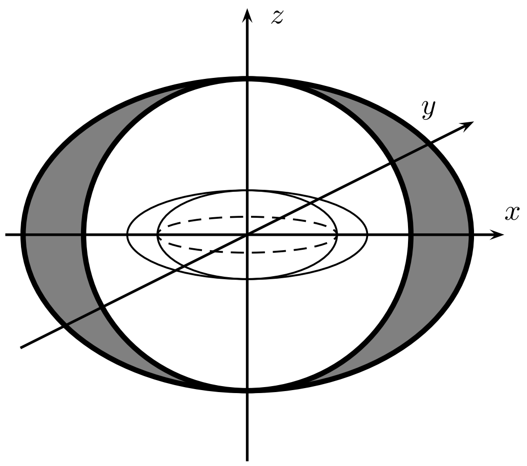


Figure 6.3: Structure of a Kerr black hole: The ergoregion (grey area) is bounded by the outer ergosurface $r_+ = M + \sqrt{M^2 - a^2 \cos^2 \vartheta}$ and the outer event horizon $r_h = M + \sqrt{M^2 - a^2}$, followed by the inner event horizon $r_h = M - \sqrt{M^2 - a^2}$, the inner ergosurface $r_- = M - \sqrt{M^2 - a^2 \cos^2 \vartheta}$ and the ring singularity $\{x^2 + y^2 = a^2, z = 0\}$.

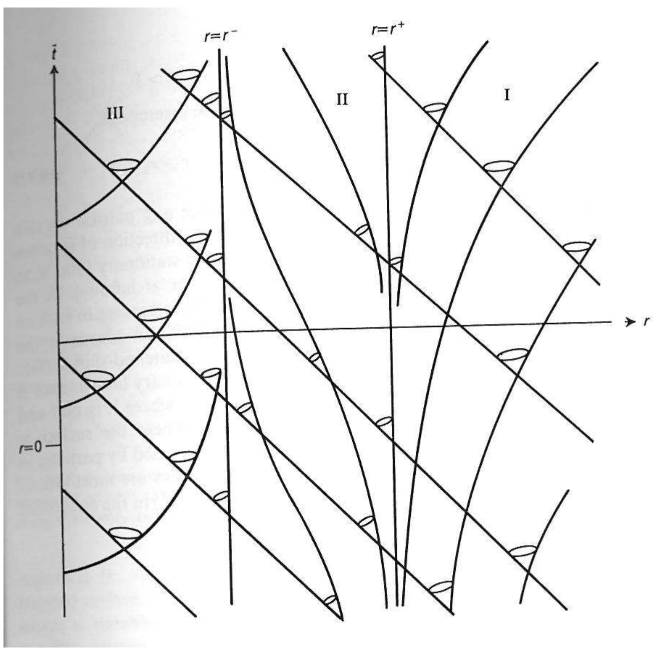


Figure 6.4: Space-time diagram in advanced Eddington-Finkelstein coordinates for a Kerr black hole with $a < M$. Between the two horizons $r_- < r < r_+$, light cones are oriented towards r_-, particles have to cross r_-. Inside the inner horizon, geodesics are possible that do not reach $r = 0$ in finite time. The behavior for $r \to 0$ (and $\vartheta \neq \pi/2$) suggests that one can extend the space-time to $r < 0$.

with the two solutions

$$\omega_{1/2} = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}}. \tag{6.48}$$

There are two interesting special cases of this equation. First, on the surface $g_{tt} = 0$, the two possible solutions of $\omega = \mathrm{d}\phi/\mathrm{d}t$ for light-rays satisfy[2]

$$\omega_1 = 0 \qquad \text{and} \qquad \omega_2 = -2\,\frac{g_{t\phi}}{g_{\phi\phi}}. \tag{6.49}$$

Hence, the rotating black hole drags spacetime at $g_{tt} = 0$ so strongly that even a photon can only co-rotate. Similarly, this condition specifies a surface inside which no stationary observers are possible. The normalisation condition $\boldsymbol{u} \cdot \boldsymbol{u} = 1$ is inconsistent with $u^a = (u^t, 0, 0, 0)$ and $g_{tt} < 0$: however strong your rocket engines are, your space-ship will not be able to hover at the same point (r, ϑ, ϕ) inside the region with $g_{tt} < 0$. Therefore one calls a surface with $g_{tt} = 0$ a *stationary limit surface*. Solving

$$g_{tt} = 1 - \frac{2Mr}{\rho^2} = 0, \tag{6.50}$$

we find the position of the two stationary limit surfaces at

$$r_{1/2} = M \pm \sqrt{M^2 - a^2 \cos^2\vartheta}. \tag{6.51}$$

The *ergosphere* is the space bounded by the outer stationary limit surface and the outer horizon.

The other interesting special case of Eq. (6.48) occurs when the allowed range of values, $\omega_1 \leq \omega \leq \omega_2$, shrinks to a single value, i.e. when

$$\omega^2 = \frac{g_{tt}}{g_{\phi\phi}} = \left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2. \tag{6.52}$$

This happens at the outer horizon r_+ and defines the rotation velocity ω_H of the black hole. In the case of a Kerr black hole, we find

$$\omega_H = \frac{a}{2Mr_+}. \tag{6.53}$$

Thus the rotation velocity of the black hole corresponds to the rotation velocity of the light-rays forming its horizon, as seen by an observer at spatial infinity.

Example 6.6: Rotation velocity of horizon—We derive Eq. (6.53) evaluating first $g_{tt}/g_{\phi\phi}$ on the outer horizon using Eq. (6.44),

$$\omega_H^2 = \frac{g_{tt}}{g_{\phi\phi}} = \frac{\rho_+^2 - 2Mr_+}{2Mr_+^2 \sin^2\vartheta}.$$

Inserting then $\rho_+^2 = r_+^2 + a^2\cos^2\vartheta$ and $r_+^2 + a^2 = 2Mr_+$, we obtain the desired result,

$$\omega_H^2 = \frac{a^2(1 - \cos^2\vartheta)}{2Mr_+^2 \sin^2\vartheta} = \frac{a^2}{2Mr_+^2}.$$

[2]Note that $\omega_1 < \omega_2$, because of $g_{\phi\phi} < 0$. Hence photons (and thus also spacetime) is corotating, as expected.

Next we consider the RHS. Because of $g_{\phi t} = g_{t\phi}$, it is $g_{t\phi} = 2Mar\sin^2\vartheta$ and thus

$$\omega_H^2 = \left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 = \left(\frac{a}{2Mr_+}\right)^2.$$

Extension of the Kerr metric The behavior of geodesics for $r \to 0$ (and $\vartheta \neq \pi/2$) suggests that one can extend the space-time to $r < 0$. For $r \to -\infty$, the extension becomes asymptotically flat, i.e. there exists a second Minkowski space that is connected to ours via the Kerr black hole. Since for negative r, Δ is always positive, $\Delta = r^2 - 2Mr + a^2 > 0$, the singularity is not protected by an event horizon in the "other" Minkowksi space. Moreover, there exist closed time-like curves: Consider a curve depending only on ϕ in the equatorial plane, the line-element for small, negative r is

$$\mathrm{d}s^2 = \left(r^2 + a^2 + \frac{2Ma^2}{r}\right)\mathrm{d}\phi^2 \sim \frac{2Ma^2}{r}\mathrm{d}\phi^2 < 0 \tag{6.54}$$

time-like.

The cosmic censorship hypothesis postulates that singularities formed in gravitational collapse are always covered by event horizons. Thus we are in the "$r > 0$" Minkowski space of all Kerr black holes – and the $r < 0$ is simply a mathematical artefact of a highly symmetrical manifold, not showing up in real physical situations.

6.4 Black hole thermodynamics

Penrose process and the area theorem The total energy of a Kerr BH consists of its rest energy and its rotational energy. These two quantities control the size of the event horizon and therefore it is important to understand how they change dropping matter into the BH.

The energy of any particle moving on a geodesics is conserved, $E = \boldsymbol{p} \cdot \boldsymbol{\xi}$. Inside the ergosphere, the Killing vector $\boldsymbol{\xi}$ is space-like and the quantity E is thus the component of a spatial momentum which can have both signs. This led Penrose to entertain the following gedankenexperiment: Suppose the spacecraft A starts at infinity and falls into the ergosphere. There it splits into two parts: B is dropped into the BH, while C escapes to infinity. In the splitting process, four-momentum has to be conserved, $\boldsymbol{p}_A = \boldsymbol{p}_B + \boldsymbol{p}_C$. We can now choose a time-like geodesics for B falling into the BH such that $E_B < 0$. Then $E_C > E_A$ and the escaping part C of the spacecraft has at infinity a higher energy than initially.

The Penrose process decreases both the mass and the angular momentum of the BH by an amount equal to that of the space craft B falling into the BH. Now we want to show that the changes are correlated in such a way that the area of the BH increases. Let us first define a new Killing vector,

$$\boldsymbol{K} = \boldsymbol{\xi} + \omega_H \boldsymbol{\eta}.$$

This Killing vector is null on the horizon and time-like outside. It corresponds to the four-velocity with the maximal possible rotation velocity. Now we use $E_B = \boldsymbol{p}_B \cdot \boldsymbol{\xi}$ and $L_B = -\boldsymbol{p}_B \cdot \boldsymbol{\eta}$ and

$$\boldsymbol{p}_B \cdot \boldsymbol{K} = \boldsymbol{p}_B \cdot (\boldsymbol{\xi} + \omega_H \boldsymbol{\eta}) = E_B - \omega_H L_B > 0, \tag{6.55}$$

to obtain the bound $L_B < E_B/\omega_H$. Since $E_B < 0$, the added angular momentum is negative, $L_B < 0$.

The mass and the angular momentum of the BH change by $\delta M = E_B$ and $\delta L = L_B$, when particle B drops into the BH. Thus

$$\delta M > \omega_H \delta L = \frac{a\delta L}{r_+^2 + a^2}. \tag{6.56}$$

Now we define the irreducible mass of BH as the mass of that Schwarzschild BH whose event horizon has the same area,

$$M_{\mathrm{irr}}^2 = \frac{1}{2}(M^2 + \sqrt{M^4 - L^2}) \tag{6.57}$$

or

$$M^2 = M_{\mathrm{irr}}^2 + \left(\frac{L}{2M_{\mathrm{irr}}}\right)^2. \tag{6.58}$$

Thus we can interpret the total mass as the Pythagorean sum of the irreducible mass and a contribution related to the rotational energy. Differentiating the relation (6.57) results in

$$\delta M_{\mathrm{irr}} = \frac{a}{4M_{\mathrm{irr}}\sqrt{M^2 - a^2}} \left(\omega_H^{-1}\delta M - \delta L\right). \tag{6.59}$$

Our bound implies now $\delta M_{\mathrm{irr}} > 0$ or $\delta A > 0$. Thus the surface of a Kerr BH can only increase, even when its mass decreases.

Bekenstein entropy We have shown that classically the horizon of a black hole can only increase with time. The only other quantity in physics with the same property is the entropy, $\mathrm{d}S \geq 0$. This suggests a connection between the horizon area and its entropy. To derive this relation, we apply the first law of thermodynamics $\mathrm{d}U = T\mathrm{d}S - P\mathrm{d}V + \ldots$ to a Kerr black hole. Its internal energy U is given by $U = M$ and thus

$$\mathrm{d}U = \mathrm{d}M = T\mathrm{d}S - \boldsymbol{\omega}\mathrm{d}\boldsymbol{L}, \tag{6.60}$$

where $\boldsymbol{\omega}\mathrm{d}\boldsymbol{L}$ denotes the mechanical work done on a rotating macroscopic body.

Our experience with the thermodynamics of non-gravitating systems suggests that the entropy is an extensive quantity and thus proportional to the volume, $S \propto V$. We now offer an argument that shows that the entropy S of a black hole is proportional to its area A. We introduce the "rationalised area" $\alpha = A/4\pi = 2Mr_+$, cf. (6.45), or

$$\alpha = 2M^2 + 2\sqrt{M^4 - L^2}. \tag{6.61}$$

The parameters describing a Kerr black hole are its mass M and its angular momentum L and thus $\alpha = \alpha(M, L)$. We form the differential $\mathrm{d}\alpha$ and find after some algebra (problem 25.**??**)

$$\frac{\sqrt{M^2 - a^2}}{2\alpha}\,\mathrm{d}\alpha = \mathrm{d}M + \frac{\boldsymbol{a}}{\alpha}\mathrm{d}\boldsymbol{L}. \tag{6.62}$$

Using now Eq. (6.45) and (6.53), we can rewrite the RHS as

$$\frac{\sqrt{M^2 - a^2}}{2\alpha}\,\mathrm{d}\alpha = \mathrm{d}M + \boldsymbol{\omega}_H\mathrm{d}L. \tag{6.63}$$

Thus the first law of black hole thermodynamics predicts the correct angular velocity ω_H of a Kerr black hole. Including the term $\Phi\mathrm{d}q$ representing the work done by adding the charge dq to a black hole, the area law of a charged black hole together with the first law of BH thermodynamics reproduces the correct surface potential Φ of a charged black hole.

The factor in front of dα is positive, as its interpretation as temperature requires. This lead Bekenstein to suggest the identification

$$T\mathrm{d}S = \frac{\sqrt{M^2 - a^2}}{2\alpha}\,\mathrm{d}\alpha. \tag{6.64}$$

If this identification is not just of formal nature, then a black hole should emit thermal radiation. This rather bold hypothesis was confirmed by Hawking 1974 who showed that a black hole in vacuum emits black-body radiation ("Hawking radiation") with the temperature suggested by Bekenstein,

$$T = \frac{2\sqrt{M^2 - a^2}}{A}. \tag{6.65}$$

Specialising to the case of a Schwarzschild BH, it is $T = 1/(8\pi M)$. Integrating then $\mathrm{d}M = T\mathrm{d}S = \mathrm{d}S/(8\pi M)$ results[3] in

$$S = \frac{A}{4} = \frac{A}{4L_{\mathrm{Pl}}^2} = \frac{kc^3}{4\hbar G}\,A. \tag{6.66}$$

Thus the entropy of a black hole is not extensive but is proportional to its surface. It is large, because its basic unit of entropy, $4L_{\mathrm{Pl}}^2$, is so tiny. The presence of $\hbar$ in the last formula, where we have inserted the natural constants, signals that the black hole entropy is a quantum property.

The heat capacity C_V of a Schwarzschild black hole follows with $U = M = 1/(8\pi T)$ from the definition

$$C_V = \frac{\partial U}{\partial T} = -\frac{1}{8\pi T^2} < 0\,. \tag{6.67}$$

As it is typical for self-gravitating systems, its heat capacity is negative. Thus a black hole surrounded by a cooler medium emits radiation, heats up the environment and becomes hotter.

6.5 *** Quantum black holes ***

6.5.1 Rindler spacetime and the Unruh effect

Rindler spacetime Recall from exercise 2.3 that the trajectory of an accelerated observer (suppressing the transverse coordinates y and z) is given by

$$t(\tau) = \frac{1}{a}\sinh(a\tau) \quad \text{and} \quad x(\tau) = \frac{1}{a}\cosh(a\tau). \tag{6.68}$$

It describes one branch of the hyperbola $x^2 - t^2 = a^{-2}$. Introducing light-cone coordinates,

$$u = t - x \quad \text{and} \quad v = t + x, \tag{6.69}$$

it follows

$$u(\tau) = -\frac{1}{a}\exp(-a\tau). \tag{6.70}$$

[3] We set the integration constant to zero, such that the third law of thermodynamics is valid for black holes.

Our aim is to determine how the uniformly accelerated observer experiences Minkowski space. As a first step, we try to find a frame $\{\xi, \chi\}$ comoving with the observer. In this frame, the observer is at rest, $\chi(\tau) = 0$, and the coordinate time ξ agrees with the proper time, $\xi = \tau$. Introducing comoving light-cone coordinates,

$$\tilde{u} = \xi - \chi \quad \text{and} \quad \tilde{v} = \xi + \chi, \tag{6.71}$$

these conditions become

$$\tilde{u}(\tau) = \tilde{v}(\tau) = \tau. \tag{6.72}$$

Moreover, we can choose the comoving coordinates such that the metric is conformally flat,

$$\mathrm{d}s^2 = \Omega^2(\xi, \chi)(\mathrm{d}\xi^2 - \mathrm{d}\chi^2) = \Omega^2(\tilde{u}, \tilde{v})\mathrm{d}\tilde{u}\mathrm{d}\tilde{v}. \tag{6.73}$$

Next we have to relate the comoving coordinates $\{\tilde{u}, \tilde{v}\}$ to Minkowski coordinates $\{t, x\}$. Since $\mathrm{d}\tilde{u}^2$ and $\mathrm{d}\tilde{v}^2$ are missing in the line element, the functions $u(\tilde{u}, \tilde{v})$ and $v(\tilde{u}, \tilde{v})$ can depend only on one of their two arguments. We can set therefore $u(\tilde{u})$ and $v(\tilde{v})$. Expressing $\dot{u}$ as

$$\frac{\mathrm{d}u}{\mathrm{d}\tau} = \frac{\mathrm{d}u}{\mathrm{d}\tilde{u}} \frac{\mathrm{d}\tilde{u}}{\mathrm{d}\tau}, \tag{6.74}$$

inserting $\dot{u} = -au$ and $\dot{\tilde{u}} = 1$ we arrive at

$$-au = \frac{\mathrm{d}u}{\mathrm{d}\tilde{u}}. \tag{6.75}$$

Separating variables and integrating we end up with $u = C_1 \mathrm{e}^{-a\tilde{u}}$. In the same way, we find $v = C_2 \mathrm{e}^{a\tilde{v}}$. Since the line element has to agree along the trajectory with the proper-time, $\mathrm{d}s^2 = \mathrm{d}\tau^2 = \mathrm{d}u\mathrm{d}v$, the two integration constants C_1 and C_2 have to satisfy the constraint $-a^2 C_1 C_2 = 1$. Choosing $C_1 = -C_2$, the desired relation between the two sets of coordinates becomes

$$u = -\frac{1}{a}\,\mathrm{e}^{-a\tilde{u}} \quad \text{and} \quad v = \frac{1}{a}\,\mathrm{e}^{a\tilde{v}}, \tag{6.76}$$

or using Cartesian coordinates,

$$t = \frac{1}{a}\,\mathrm{e}^{a\chi} \sinh(a\xi) \quad \text{and} \quad x = \frac{1}{a}\,\mathrm{e}^{a\chi} \cosh(a\xi). \tag{6.77}$$

The spacetime described by the coordinates defining the comoving frame of the accelerated observer,

$$\mathrm{d}s^2 = \mathrm{e}^{2a\chi}(\mathrm{d}\xi^2 - \mathrm{d}\chi^2), \tag{6.78}$$

is called Rindler spacetime. It is locally equivalent to Minkowski space but differs globally. If we vary the Rindler coordinates over their full range, $\xi \in \mathbb{R}$ and $\chi \in \mathbb{R}$, then we cover only the one quarter of Minkowski space with $x > |t|$. Thus for an accelerated observer an event horizon exist: Evaluating on a hypersurface of constant comoving time, $\xi =$ const., the physical distance from $\chi = -\infty$ to the observer placed at $\chi = 0$ gives

$$d = \int_{-\infty}^{0} \mathrm{d}\chi \sqrt{|g_{\chi\chi}|} = \frac{1}{a}. \tag{6.79}$$

This corresponds to the coordinate distance between the observer and the horizon in Minkowski coordinates.

definition:: The particle horizon is the maximal distance from which we can receive signals, while the event horizon defines the maximal distance to which we can send signals.

Exponential redshift Later we will discuss gravitational particle production as the effect of a non-trivial Bogolyubov transformation between different vacua. Before we apply this formalism, we will examine the basis of this physical phenomenon in a classical picture. As a starter, we want to derive the formula for the relativistic Doppler effect. Consider an observer who is moving with constant velocity v relative to the Cartesian inertial system $x^\mu = (t, x)$ where we neglect the two transverse dimensions. We can parameterise the trajectory of the observer as

$$x^\mu(\tau) = (t(\tau), x(\tau)) = (\tau\gamma, \tau\gamma v), \tag{6.80}$$

where γ denotes its Lorentz factor. A monochromatic wave of a scalar, massless field $\phi(k) \propto \exp[-\mathrm{i}\omega(t-x)]$ will be seen by the moving observer as

$$\phi(\tau) \equiv \phi(x^\mu(\tau)) \propto \exp\left[-\mathrm{i}\omega\tau\left(\gamma - \gamma v\right)\right] = \exp\left[-\mathrm{i}\omega\tau\sqrt{\frac{1-v}{1+v}}\right]. \tag{6.81}$$

Thus this simple calculation reproduces the usual Doppler formula, where the frequency ω of the scalar wave is shifted as

$$\omega' = \sqrt{\frac{1-v}{1+v}}\,\omega. \tag{6.82}$$

Next we apply the same method to the case of an accelerated observer. Then $t(\tau) = a^{-1}\sinh(a\tau)$ and $x(\tau) = a^{-1}\cosh(a\tau)$. Inserting this trajectory again into a monochromatic wave with $\phi(k) \propto \exp(-\mathrm{i}\omega(t-x)$ now gives

$$\phi(\tau) \propto \exp\left[-\frac{\mathrm{i}\omega}{a}\left[\sinh(a\tau) - \cosh(a\tau)\right]\right] = \exp\left[\frac{\mathrm{i}\omega}{a}\exp(-a\tau)\right] \equiv \mathrm{e}^{-\mathrm{i}\vartheta}. \tag{6.83}$$

Thus an accelerated observer does not see a monochromatic wave, but a superposition of plane waves with varying frequencies. Defining the instantaneous frequency by

$$\omega(\tau) = \frac{\mathrm{d}\vartheta}{\mathrm{d}\tau} = \omega\,\exp(-a\tau), \tag{6.84}$$

we see that the phase measured by the accelerated observer is exponentially redshifted. As next step, we want to determine the power spectrum $P(\nu) = |\phi(\nu)|^2$ measured by the observer, for which we have to calculate the Fourier transform $\phi(\nu)$.

Remark 6.3: Determine the Fourier transform of the wave $\phi(\tau)$.
Substituting $y = \exp(-a\tau)$ in

$$\phi(\nu) = \int_{-\infty}^{\infty} \mathrm{d}\tau \phi(\tau)\mathrm{e}^{\mathrm{i}\nu\tau} = \int_{-\infty}^{\infty} \mathrm{d}\tau \exp\left(\frac{\mathrm{i}\omega}{a}\exp(-a\tau)\right)\mathrm{e}^{\mathrm{i}\nu\tau} \tag{6.85}$$

gives

$$\phi(\nu) = \frac{1}{a}\int_0^\infty \mathrm{d}y\; y^{-\mathrm{i}\nu/a-1}\,\mathrm{e}^{\mathrm{i}(\omega/a)y}. \tag{6.86}$$

On the other hand, we can rewrite Euler's integral representation of the Gamma function as

$$\int_0^\infty \mathrm{d}t\, t^{z-1}\mathrm{e}^{-bt} = b^{-z}\,\Gamma(z) = \exp(-z\ln b)\,\Gamma(z) \tag{6.87}$$

for $\Re(z) > 0$ and $\Re(b) > 0$. Comparing these two expressions, we see that they agree setting $z = -\mathrm{i}\nu/a + \varepsilon$ and $b = -\mathrm{i}\omega/a + \varepsilon$. Here we added an infinitesimal positive real quantity $\varepsilon > 0$ to ensure the convergence of the integral. In order to determine the correct phase of b^{-z}, we have rewritten this factor as $\exp(-z\ln b)$ and have used

$$\ln b = \lim_{\varepsilon\to 0}\ln\left(-\frac{\mathrm{i}\omega}{a}+\varepsilon\right) = \ln\left|\frac{\omega}{a}\right| - \frac{\mathrm{i}\pi}{2}\mathrm{sign}(\omega/a). \tag{6.88}$$

Thus the Fourier transform $\phi(\nu)$ is given by

$$\phi(\nu) = \frac{1}{a}\left(\frac{\omega}{a}\right)^{\mathrm{i}\nu/a}\Gamma(-\mathrm{i}\nu/a)\mathrm{e}^{\pi\nu/(2a)}. \tag{6.89}$$

The Fourier transform $\phi(\nu)$ contains negative frequencies,

$$\phi(-\nu) = \phi(\nu)\mathrm{e}^{-\pi\nu/a} = \frac{1}{a}\left(\frac{\omega}{a}\right)^{\mathrm{i}\nu/a}\Gamma(-\mathrm{i}\nu/a)\mathrm{e}^{-\pi\nu/(2a)}. \tag{6.90}$$

Using the reflection formula of the Gamma function for imaginary arguments,

$$\Gamma(\mathrm{i}x)\Gamma(-\mathrm{i}x) = \frac{\pi}{x\sinh(\pi x)}, \tag{6.91}$$

we find the power spectrum at negative frequencies as

$$P(-\nu) = \frac{\pi}{a^2}\,\frac{\mathrm{e}^{-\pi\nu/a}}{(\nu/a)\sinh(\pi\nu/a)} = \frac{\beta}{\nu}\,\frac{1}{\mathrm{e}^{\beta\nu}-1} \tag{6.92}$$

with $\beta = 2\pi/a$. Remarkably, the dependence on the frequency ω of the scalar wave—still present in the Fourier transform $\phi(\nu)$—has dropped from the negative frequency part of the power spectrum $P(-\nu)$ which corresponds to a thermal Planck law with temperature $T = 1/\beta = a/(2\pi)$.

The occurrence of negative frequencies is the classical analogue for the mixing of positive and negative frequencies in the Bogolyubov method. Therefore we expect that on the quantum level a uniformly accelerated detector will measure a thermal Planck spectrum with temperature $T = 1/\beta = a/(2\pi)$. This phenomenon is called Unruh effect and $T = a/(2\pi)$ the Unruh temperature.

6.5.2 Hawking radiation from the Unruh effect

Hawking radiation Hawking could show 1974 that a black hole in vacuum emits black-body radiation ("Hawking radiation") with temperature

$$T = \frac{2\sqrt{M^2-a^2}}{A} \tag{6.93}$$

and thus

$$S = \frac{kc^3}{4\hbar G}A = \frac{A}{4L_{\mathrm{Pl}}^2}\,. \tag{6.94}$$

A black hole surrounded by a cooler medium emits radiation and heats up the environment. The entropy of a black hole is large, because its basic unit of entropy, $4L_{\mathrm{Pl}}^2$, is so tiny.

We can understand this result considering an observer in the Schwarzschild metric. The force required by of rocket to stay on a stationary orbit was calculated in exercise 6.1. The derived acceleration of a stationary observer,

$$a \equiv (-\boldsymbol{a}\cdot\boldsymbol{a})^{1/2} = \left(1-\frac{2M}{r}\right)^{-1/2}\frac{M}{r^2} = \left(1-\frac{R_s}{r}\right)^{-1/2}\frac{R_s/2}{r^2}, \tag{6.95}$$

diverges approaching the horizon, $r \to R_s = 2M$. The acceleration a close to the horizon, i.e. for $r_1 - R_s \ll R_s$, is thus much larger than the curvature $\propto 1/R_s$. We can use therefore the approximation of an accelerated observer in a flat space, who sees according to the Unruh effect a thermal spectrum with temperature $T = a_1/2\pi$ at r_1. Assume now that the observer moves from r_1 to $r_2 > r_1$. Then the spectrum is redshifted by V_1/V_2 with $V_i = \sqrt{1 - R_s/r_i}$. For $r_2 \to \infty$, it is $V_2 \to 1$ and thus $T_2 \to V_1 T_1$. Approaching also the horizon, the temperature becomes

$$T = \lim_{r_1 \to R_s} V_1 T_1 = \lim_{r_1 \to R_s} \sqrt{1 - R_s/r_1} \frac{1}{2\pi} \frac{R_s/2}{r_1^2 \sqrt{1 - R_s/r_1}} = \frac{1}{4\pi R_s} = \frac{1}{8\pi M}. \tag{6.96}$$

6.5.3 Information paradox–resolved?

Fine vs. coarse-grained entropy Entropy is a quantity of fundamental importance. It is therefore no surprise that it comes in a series of different flavours. Boltzmann introduced the entropy of a classical system with phase-space density $f(q^j, p_j, t)$ as

$$S = -\int \mathrm{d}q^j \mathrm{d}p_j \, f(q^j, p_j, t) \ln[f(q^j, p_j, t)]. \tag{6.97}$$

Liouville's theorem states that the phase-space density $f(q^j, p_j, t)$ of a Hamiltonian system stays constant along a trajectory in phase space and thus its entropy stays constant too. Analogously, in quantum mechanics the von Neumann entropy is defined as

$$S_N = -\mathrm{tr}[\rho \ln(\rho)] \tag{6.98}$$

with ρ as the density operator of the system. Now it is the unitary time evolution of a quantum system which gurantees that S_N remains constant. Both entropies are called fine-grained entropy, since they are based on our (partial) knowledge of the *microstates* of the system.

In contrast, the thermodynamic entropy S_{td} increases with time, $\mathrm{d}S_{\mathrm{td}}/\mathrm{d}t \geq 0$. What is the reason for this different time behaviour? A thermodynamic system is typically characterised by the value of only a few *macroscopic* quantities like its total energy, volume, pressure, $A_i = \{U, V, P, \ldots\}$, while the microscopic variables of the system are unknown. As result, the thermodynamical entropy S_{td} is obtained by calculating first the fine-grained entropy for all microstates compatible with the values of the macroscopic quantities A_i. Then the set of allowed value of $S(A_i)$ is maximised,

$$S_{\mathrm{td}} = \max_{A_i = A_{i,0}} \{S_{\mathrm{N}}(A_i)\}. \tag{6.99}$$

The thermodynamic entropy is therefore also called the coarse-grained entropy. From the definition, it is obvious that $S_{\mathrm{td}} \geq S_{\mathrm{N}}$. Moreover. one can show that it implies $\mathrm{d}S_{\mathrm{td}}/\mathrm{d}t \geq 0$. Since the Bekenstein entropy of a BH increases, it has to be its thermodynamic or coarse-grained entropy. In this view, the BH entropy is enormous because an extremely large set of different initial configurations leads to the same BH.

Entanglement entropy We can visualize the Hawking process as follows: A virtual particle-antiparticle pair is created close to the horizon. One particle of the pair crosses the horizon and adds negative energy to the BH, while the other escapes with positive energy to infinity.

Since the vacuum has zero charge and spin, the state of the escaping particle is entangled with the one inside the horizon. We can therefore view the entropy of the Hawking radiation as entanglement entropy.

The von Neumann entropy (6.98) specifies our knowledge of the system. In particular, for a pure state $\psi \in \mathscr{H}$, it is zero. However, if we divide the total system in two sub-systems A and B, and can measure only variables connect to, say, B, then the density matrix of the subsystem B becomes

$$\rho_B = -\mathrm{tr}_A[\rho \ln(\rho)]. \tag{6.100}$$

Its entropy becomes non-zero. Moreover, $S_A = S_B$. Applied to BH evaporation, we divide the universe in two subsystems, the BH and the outside world, where we have access only to the latter.

Page curve Hawking's claim from 1976 that BH evaporation adds a new source of uncertainty to physics can be phrased as follows: Imagine a BH formed from pure states (e.g. by the collision of gravitational plane waves). After the BH evaporated, the remaining thermal radiation is in a mixed state, described by a thermal density matrix. This is in conflict with unitary time evolution, and seems thus to require some fundamental new aspects in quantum gravity with respect to other quantum field theories.

In the modern formulation, the paradox arises as follows: As the BH evaporates, its area and thus its Bekenstein entropy decrease. At the same time, more and more thermal radiation is emitted and thus the entropy of the Hawking radiation increases. At same point, called the Page time t_P, the two entropies become equal, $S_B = S_{\mathrm{rad}}$. If the entropy of the Hawking radiation can be identified with the entanglement entropy between pairs of quanta emitted to infinity and confined in the BH. i.e. with degrees of freedom inside the BH, the paradox arises at $t \geq t_P$: The Bekenstein entropy is the coarse-gained entropy of the von Neumann entropy of the internal BH degrees of freedom. Therefore the latter has to be smaller, $S_N \leq S_B$. In turn, it has to be also smaller than the entanglement entropy of the Hawking radiation, or $S_B \geq S_{\mathrm{rad}}$. But after the Page time, there are not enough internal degrees of freedom in the BH with which the radiation can be entangled with.

The only solution is that the entropy of the Hawking radiation starting from t_P decreases, such that $S_{\mathrm{rad}} \leq S_B$. Numerically, one finds that the Page time is roughly half of the evaporation time of the BH, $t_P \simeq 0.5 t_{\mathrm{ev}}$. For a macroscopic BH, effects of quantum gravity should play no role at this time, and the semiclassical calculation of Hawking should hold. Both argumentations seems to be valid, but are contradicting each other—this is the paradox.

Modern developments The fact that the Bekenstein entropy is proportional to the surface of the BH suggests that the microstates characterising its entropy are also there localised. This led to the idea of holography: the information about the state of a gravitating system is contained in its surface. This somewhat vague proposal put forward originally by t'Hooft was made more concrete by the AdS/CFT correspondence conjectured by Maldacena: While its original version was connecting anti-de Sitter space within string/M-theory to a conformal field theory on its boundary, today the conjecture is believed to hold much more general. If the conjecture is true, then we can describe the BH physics equivalently by a conformal field theory. The latter respects unitary time evolution. Thus Hawking had to be wrong.

While the AdS/CFT correspondence suggests that information is conserved, it has been unclear how and why the entropy is decreasing after the Page time. Only in the last years, there has been progress in addressing this question directly using the path-integral approach to quantum gravity, i.e. avoiding the AdS/CFT correspondence. While these calculations are restricted to toy models, they can reproduce the Page curve. Moreover, they suggest that the problem can be understood semi-classically. For a review article introducing the basic ideas see Ref. [2]

6.A Appendix: General stationary axisymmetric metric

6.B Appendix: Conformal flatness for $d = 2$

A conformal transformation of the metric,

$$g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x). \tag{6.101}$$

changes distances, but keeps angles invariant. Thus the causal structure of two conformally related spacetimes is identical. A spacetime is called conformally flat if it is connected by a conformal transformation to Minkowski space,

$$g_{\mu\nu}(x) = \Omega^2(x)\eta_{\mu\nu}(x) = \mathrm{e}^{2\omega(x)}\eta_{\mu\nu}(x). \tag{6.102}$$

In particular, light-rays also propagate in conformally flat spacetimes along straight lines at ± 45 degrees to the time axis.

We want to show that any two-dimensional (pseudo-) Riemannian manifold is conformally flat. For this, we have to show that a transformation from

$$\mathrm{d}s^2 = g_{11}(\mathrm{d}x^1)^2 + 2g_{12}\mathrm{d}x^1\mathrm{d}x^2 + g_{12}(\mathrm{d}x^2)^2 \tag{6.103}$$

to

$$\mathrm{d}s^2 = (\mathrm{d}x^1)^2 \pm (\mathrm{d}x^2)^2 \tag{6.104}$$

exists. Let us assume that the transformations are given by

$$\tilde{x}^1 = F(x^1, x^2) \quad \text{and} \quad \tilde{x}^2 = G(x^1, x^2).$$

Using the usual transformation law for a tensor of rank two under an arbitrary coordinate transformation,

$$\tilde{g}^{\mu\nu}(\tilde{x}) = \frac{\partial \tilde{x}^\mu}{\partial x^\alpha}\frac{\partial \tilde{x}^\nu}{\partial x^\beta}\, g^{\alpha\beta}(x), \tag{6.105}$$

it follows

$$0 = \tilde{g}^{12} = (\partial_\alpha F)(\partial_\beta G)g^{\alpha\beta} = g^{11}\partial_1 F\partial_1 G + g^{12}(\partial_1 F\partial_2 G + \partial_2 F\partial_1 G) + g^{11}\partial_2 F\partial_2 G \tag{6.106}$$

and

$$0 = \tilde{g}^{11} \mp \tilde{g}^{22} = [(\partial_\alpha F)(\partial_\beta F) \mp (\partial_\alpha G)(\partial_\beta G)]g^{\alpha\beta} = \tag{6.107a}$$

$$= g^{11}\left[(\partial_1 F)^2 \mp (\partial_1 G)^2\right] + 2g^{12}(\partial_1 F\partial_2 F \mp \partial_1 G\partial_2 G) + g^{22}\left[(\partial_2 F)^2 \mp (\partial_2 G)^2\right], \tag{6.107b}$$

where the first (plus) minus corresponds to a (pseudo-) Riemannian manifold. Let us pick out the minus sign for definiteness. Inserting then the ansatz

$$\partial_1 F = \kappa(g^{21}\partial_1 G + g^{22}\partial_2 G) \tag{6.108a}$$

$$\partial_2 F = -\kappa(g^{11}\partial_1 G + g^{12}\partial_2 G) \tag{6.108b}$$

into the first condition, we see that it is satisfied identically for an arbitrary function κ. Inserting it next in the second condition, we obtain

$$\left[\kappa^2 \left(g^{11}g^{22} - (g^{12})^2\right) - 1\right] \left[g^{11}(\partial_1 G)^2 + 2g^{21}\partial_1 G\partial_2 G + g^{22}(\partial_2 G)^2\right] = 0 \tag{6.109}$$

We recognise $g^{11}g^{22} - (g^{12})^2$ as $\det(g^{ij}) = 1/\det(g_{ij}) = 1/g$. Thus the expression vanishes choosing $\kappa^2 = g$.

Further reading https://arxiv.org/pdf/0706.0622

Problems

6.1 Force of a hovering rocket. A stationary observer hovers on a radial orbit around a Schwarzschild black-hole. a.) Argue that

$$f^\alpha = m\left(\ddot{x}^\alpha + \Gamma^\alpha{}_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma\right)$$

is the correct generalisation of Newton's second law to curved spacetime. b.) Usw $\Gamma^r{}_{tt} = (1 - 2M/r)(M/r^2)$ to find the radial force required to stay on a stationary orbit. c.) The result from b.) is the radial force in the coordinate basis. Relate it to the force measured by the observer in its Cartesian inertial frame.

7 Classical field theory

In this chapter, we consider classical fields in Minkowski space. We derive the connection between symmetries and conservation laws, and discuss the scalar and massless vector field as examples. The latter is characterised by a local gauge symmetry, which has much in common with a gravitational wave in Minkowski space.

7.1 Lagrange formalism for fields

A relativistic field associates to each spacetime point x^μ a set of values. The space of field values at each point can be characterized by its transformation properties under Lorentz transformations (a scalar ϕ, vector A^μ, tensor $g^{\mu\nu}$, or spinor ψ_a field) and internal symmetry groups which are (typically) Lie groups like U(1), SU(n),… Thus we have to generalize Hamilton's principle to a collection of fields $\phi_a(x^\mu), a = 1, \ldots, k$, where the index a includes both Lorentz and group indices. To ensure Lorentz invariance, we consider a scalar Lagrange density $\mathscr{L}$ that may, analogously to $L(q, \dot{q})$, depend on the fields and its first derivatives $\partial_\mu \phi_a$. There is no explicit time-dependence, since "everything" should be explained by the fields and their interactions. The Lagrangian $L(\phi_a, \partial_\mu \phi_a)$ is obtained by integrating $\mathscr{L}$ over a given space volume V.

The action S is thus the four-dimensional integral over the Lagrange density $\mathscr{L}$,

$$S[\phi_a] = \int_a^b \mathrm{d}t\, L(\phi_a, \partial_\mu \phi_a) = \int_\Omega \mathrm{d}^4 x\, \mathscr{L}(\phi_a, \partial_\mu \phi_a)\,, \tag{7.1}$$

where $\Omega = V \times [t_a : t_b]$. We require again that the variations $\delta\phi_a = \varepsilon\tilde{\phi}_a$ vanish on the boundary, $\delta\phi_a|_{\partial\Omega} = 0$, and that ε is independent of x^μ. If the Lorentz scalar $\mathscr{L}$ is in addition a local function, i.e. it is a function of the fields and their gradients at the same spacetime point x^μ, we will obtain automatically Lorentz-invariant equations of motion.

A variation $\varepsilon\phi_a \equiv \delta\phi_a$ of the fields leads to a variation of the action,

$$\delta S = \int_\Omega \mathrm{d}^4 x \left(\frac{\partial \mathscr{L}}{\partial \phi^a}\, \delta\phi^a + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi^a)}\, \delta(\partial_\mu \phi^a) \right), \tag{7.2}$$

where we have to sum over fields ($a = 1, \ldots, k$) and the Lorentz index $\mu = 0, \ldots, 3$. We eliminate again the variation of the field gradients $\partial_\mu \phi^a$ by a partial integration using Gauß' theorem,

$$\delta S = \int_\Omega \mathrm{d}^4 x \left[\frac{\partial \mathscr{L}}{\partial \phi^a} - \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi^a)} \right) \right] \delta\phi^a = 0\,. \tag{7.3}$$

The boundary term vanishes, since we require that the variation is zero on the boundary $\partial\Omega$. Thus the Lagrange equations for the fields ϕ^a are

$$\boxed{\frac{\partial \mathscr{L}}{\partial \phi^a} - \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi^a)} \right) = 0\,.} \tag{7.4}$$

If the Lagrange density $\mathscr{L}$ is changed by a four–dimensional divergence, the same equations of motion result.

7.2 Noether's theorem and conservation laws

Conservation laws Let j^μ be a conserved vector field in Minkowski space,

$$\partial_\mu j^\mu = 0\,. \tag{7.5}$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \mathrm{d}^3x\, j^0 = -\int_{\partial V} \mathrm{d}\boldsymbol{S}\cdot\boldsymbol{j} \tag{7.6}$$

and

$$Q = \int_V \mathrm{d}^3x\, j^0 \tag{7.7}$$

is a globally conserved quantity, if there is no outgoing flux $\boldsymbol{j}$ through the boundary ∂V. To show that Q is a Lorentz invariant quantity, we have to rewrite Eq. (7.7) as a tensor equation.

Consider

$$Q(t=0) = \int d^4x\, j^\mu(x)\partial_\mu\vartheta(\boldsymbol{n}\cdot\boldsymbol{x}) \tag{7.8}$$

with ϑ the step function and $\boldsymbol{n}$ a unit vector in time direction, $\boldsymbol{n}\cdot\boldsymbol{x} = x^0 = t$. Then

$$Q(t=0) = \int d^4x\, j^0(x)\partial_0\vartheta(x^0) = \int d^4x\, j^0(x)\delta(x^0) = \int d^3x\, j^0(x) \tag{7.9}$$

and hence Eqs. (7.7) and (7.8) are equivalent. Since one of them is a tensor equation, Q is Lorentz invariant.

Example 7.1: Show explictely that observers in different frames x and x' measure the same charge.
The charge measured by the observer O' is

$$Q' = \int d^4x\, j'^\mu(x')\,\frac{\partial}{\partial x'^\mu}\,\vartheta(\boldsymbol{n}\cdot\boldsymbol{x}') = \int d^4x\, j^\mu(x)\partial_\mu\vartheta(\boldsymbol{n}\cdot\boldsymbol{x}')\,. \tag{7.10}$$

Since

$$n_\alpha x'^\alpha = n_\alpha\,\frac{\partial {x'}^\alpha}{\partial x^\beta}\,x^\beta = n'_\alpha x^\alpha \tag{7.11}$$

it follows

$$Q - Q' = \int d^4x\, j^\mu(x)\partial_\mu[\vartheta(\boldsymbol{n}\cdot\boldsymbol{x}) - \vartheta(\boldsymbol{n}'\cdot\boldsymbol{x})]\,. \tag{7.12}$$

The current is conserved, $\partial_\mu j^\mu = 0$, and we can thus move ∂_μ to the left,

$$Q - Q' = \int d^4x\,\partial_\mu[j^\mu(x)(\vartheta(\boldsymbol{n}\cdot\boldsymbol{x}) - \vartheta(\boldsymbol{n}'\cdot\boldsymbol{x}))]\,. \tag{7.13}$$

Now we can integrate by parts and use that all surface terms vanish, because for large $|\boldsymbol{x}|$ the current vanishes, while for large $|t|$ the theta functions are identical and cancel. Thus a globally, Lorentz invariant conserved charge Q exists.

In the same way, we can construct in Minkowski space globally conserved quantities Q for conserved tensors: If for instance $\partial_\mu T^{\mu\nu} = 0$, then

$$P^\nu = \int \mathrm{d}^3 x \, T^{0\nu} \tag{7.14}$$

is a globally conserved vector, and similarly for higher-rank tensors.

Symmetries and Noether's theorem Noether's theorem gives a formal connection between global, continuous symmetries of a physical system and the resulting conservation laws. Such symmetries can be divided into space-time and internal symmeties. We derive this theorem in two steps, considering in the first one only internal symmetries.

We assume that our collection of fields ϕ_a has a continuous symmetry group. Thus we can consider an infinitesimal change $\delta\phi_a$ that keeps $\mathscr{L}(\phi_a, \partial_\mu\phi_a)$ invariant,

$$0 = \delta\mathscr{L} = \frac{\delta\mathscr{L}}{\delta\phi_a} \, \delta_0\phi_a + \frac{\delta\mathscr{L}}{\delta\partial_\mu\phi_a} \, \delta_0\partial_\mu\phi_a \, . \tag{7.15}$$

Here, we used the notation δ_0 to stress that we exclude variations due to the change of spacetime point. Now we exchange $\delta\partial_\mu$ against $\partial_\mu\delta$ in the second term and use then the Lagrange equations, $\delta\mathscr{L}/\delta\phi_a = \partial_\mu(\delta\mathscr{L}/\delta\partial_\mu\phi_a)$, in the first term. Then we can combine the two terms using the Leibniz rule,

$$0 = \delta\mathscr{L} = \partial_\mu \left(\frac{\delta\mathscr{L}}{\delta\partial_\mu\phi_a} \right) \delta_0\phi_a + \frac{\delta\mathscr{L}}{\delta\partial_\mu\phi_a} \, \partial_\mu\delta_0\phi_a = \partial_\mu \left(\frac{\delta\mathscr{L}}{\delta\partial_\mu\phi_a} \, \delta_0\phi_a \right) . \tag{7.16}$$

Hence the invariance of $\mathscr{L}$ under the change $\delta_0\phi_a$ implies the existence of a conserved current, $\partial_\mu j^\mu = 0$, with

$$j^\mu = \frac{\delta\mathscr{L}}{\delta\partial_\mu\phi_a} \, \delta_0\phi_a \, . \tag{7.17}$$

If the transformation $\delta_0\phi_a$ leads to change in $\mathscr{L}$ that is a total four-divergence, $\delta_0\mathscr{L} = \partial_\mu K^\mu$, and boundary terms can be dropped, then the equation of motion are still invariant. The conserved current is changed to $j^\mu = \delta\mathscr{L}/\delta\partial_\mu\phi_a \, \delta_0\phi_a - K^\mu$.

In the second step, we consider in addition a variation of the coordinates, $x'_\mu = x_\mu + \delta x_\mu$. Such a variation implies a change of the fields

$$\phi'_a(x'_\mu) = \phi_a(x_\mu) + \delta\phi_a(x_\mu) \tag{7.18}$$

and thus also of the Lagrange density. Note that we compare now the field at different points. In order to be able recycle our old result, we split the total variation $\delta\phi_a(x_\mu)$ as follows

$$\delta\phi_a(x_\mu) = \phi'_a(x'_\mu) - \phi_a(x_\mu) = \phi'_a(x_\mu + \delta x^\mu) - \phi_a(x_\mu) \tag{7.19}$$

$$= \phi'_a(x_\mu) + \delta x^\mu \partial_\mu \phi'_a(x_\mu) - \phi_a(x_\mu) = \delta_0\phi_a(x_\mu) + \delta x^\mu \partial_\mu \phi'_a(x_\mu) \tag{7.20}$$

$$= \delta_0\phi_a(x_\mu) + \delta x^\mu \partial_\mu \phi_a(x_\mu). \tag{7.21}$$

Here we made in the second line first a Taylor expansion, and introduced then the local variation $\delta_0\phi_a(x_\mu) = \phi'_a(x_\mu) - \phi_a(x_\mu)$ which we calculated previously. Since δx^μ is already a linear term, we could replace in the third line $\phi'_a(x_\mu) \simeq \phi_a(x_\mu)$, neglecting thereby only a quadratic term.

We consider now the variation of the action S implied by the coordinate change $\tilde{x}_\mu = x_\mu + \delta x_\mu$. Such a variation implies not only a variation of $\mathscr{L}$ but also of the integration measure d^4x,

$$\delta S = \int_\Omega \left[\mathrm{d}^4x(\delta\mathscr{L}) + (\delta\mathrm{d}^4x)\mathscr{L}\right] . \tag{7.22}$$

The two integration measures d^4x and $\mathrm{d}^4\tilde{x}$ are connected by the Jacobian, i.e. the determinant of the transformation matrix

$$a^\mu_\nu = \frac{\partial\tilde{x}^\mu}{\partial x^\nu} . \tag{7.23}$$

Using again that the variation is infinitesimal, we find

$$J = \left|\frac{\partial\tilde{x}^\mu}{\partial x^\nu}\right| = \begin{pmatrix} 1+\frac{\partial\delta x^0}{\partial x^0} & \frac{\partial\delta x^0}{\partial x^1} & \dots \\ \frac{\partial\delta x^1}{\partial x^0} & 1+\frac{\partial\delta x^1}{\partial x^1} & \\ \dots & & \dots \end{pmatrix} = 1 + \frac{\partial\delta x^\mu}{\partial x^\mu}. \tag{7.24}$$

Inserting first this result and using then Eq. (7.21) applied to $\mathscr{L}$ gives

$$\delta S = \int_\Omega \mathrm{d}^4x \left[\delta\mathscr{L} + \mathscr{L}\frac{\partial\delta x^\mu}{\partial x^\mu}\right] = \int_\Omega \mathrm{d}^4x \left[\delta_0\mathscr{L} + \frac{\partial\mathscr{L}}{\partial x_\mu}\delta x_\mu + \mathscr{L}\frac{\partial\delta x^\mu}{\partial x^\mu}\right] . \tag{7.25}$$

We combine the last two terms using the Leibniz rule, and insert the known variation $\delta_0\mathscr{L}$ at the same point from Eq. (7.16), obtaining

$$\delta S = \int_\Omega \mathrm{d}^4x \frac{\partial}{\partial x_\mu}\left[\frac{\partial\mathscr{L}}{\partial(\partial^\mu\phi_a)}\,\delta_0\phi_a + \mathscr{L}\delta x_\mu\right] . \tag{7.26}$$

If the system is invariant under these transformations, the variation of the action is zero, $\delta S = 0$, and the square bracket represents a conserved current j^μ. As last step, we change from the local variation δ_0 to the full variation δ using Eq. (7.21), obtaining as final expression for the Noether current

$$j_\mu = \frac{\partial\mathscr{L}}{\partial(\partial^\mu\phi_a)}\,\delta\phi_a - \left[\frac{\partial\mathscr{L}}{\partial(\partial^\mu\phi_a)}\frac{\partial\phi_a}{\partial x^\nu} - \eta_{\mu\nu}\mathscr{L}\right]\delta x^\nu . \tag{7.27}$$

Translations Invariance under translations $x'_\mu = x_\mu + \varepsilon_\mu$ means $\phi'_a(x') = \phi_a(x)$ or $\delta\phi_a = 0$. Hence we obtain a conserved tensor

$$\Theta_{\mu\nu} = \frac{\partial\mathscr{L}}{\partial(\partial^\mu\phi_a)}\frac{\partial\phi_a}{\partial x^\nu} - \eta_{\mu\nu}\mathscr{L} \tag{7.28}$$

called the *energy-momentum stress tensor* or in short the stress tensor. We will see in the next chapter that this tenor sources gravity—being thus of crucial interest for us. If the stress tensor is derived via the Noether procedure (7.28), it is called *canonical*. In general, the canonical stress tensor is not symmetric, $\Theta_{\mu\nu} \neq \Theta_{\nu\mu}$, as it should be as source of gravity in Einstein's theory. Note however that the Noether procedure does not uniquely specificy the stress tensor, because we can add any tensor $\partial_\lambda f^{\lambda\mu\nu}$ which is a four-divergence and antisymmetric in μ and λ: Such a term drops out of the conservation law because of $\partial_\mu\partial_\lambda f^{\lambda\mu\nu} = 0$ and of the global charge because it is a four-divergence. This freedom allows us to obtain always a symmetric stress tensor. We will learn later a different method, leading directly to a symmetric energy-momentum tensor $T_{\mu\nu}$ (called the *dynamical* energy-momentum tensor).

Remark 7.1: The invention of the three-dimensional stress tensor σ_{ij} goes back to Pascal and Euler. Recall that σ_{ij} is determined via $\mathrm{d}F_i = \sigma_{ij}\mathrm{d}A_j$ as the response of a material to the force F_i on its surface element A_j. This implies that we can view the stress tensor also as an (anisotropic) pressure tensor, $P_{ij} = \mathrm{d}F_i/\mathrm{d}A_j = \sigma_{ij}$. Moreover, it follows with $f_i = \mathrm{d}F_i/\mathrm{d}V$ for the force density $f_j = \partial_i\sigma_{ij}$ as equilibrium condition (or equation of motion) of the system.
The relativistic stress tensor $T_{\mu\nu}$ was introduced by Minkowski in 1908 for electrodynamics, combining Maxwell's stress tensor (in vaccuum)

$$\sigma_{ij} = E_iE_j + B_iB_j - \frac{1}{2}(\boldsymbol{E}^2 - \boldsymbol{B}^2)\delta_{ij}$$

with the energy density $\rho = (\boldsymbol{E}^2 + \boldsymbol{B}^2)/2$, the Poynting vector (or energy flux) $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{B}$, and the momentum density $\boldsymbol{\pi}$

$$T^{\mu\nu} = \begin{pmatrix} \rho & \boldsymbol{S} \\ \boldsymbol{\pi} & \sigma_{ij} \end{pmatrix}.$$

In a relativistic theory, the energy flux equals the momentum density. Then $T^{0i} = T^{i0}$, what is sufficient condition for the symmetry of the full tensor.

Integrating we obtain four conserved Noether charges,

$$p^\nu = \int \mathrm{d}^3x\, \Theta^{0\nu}\,. \tag{7.29}$$

From the example, we know that Θ^{00} corresponds to the energy density ρ. Therefore p^0 is the energy, and thus p^μ the four-momentum of the field. This is in line with the fact that translations are generated by the four-momentum operator.

Lorentz transformations Lorentz transformation, i.e. rotations and boosts, lead to a linear change of coordinates,

$$\tilde{x}^\mu = x^\mu + \delta\omega^{\mu\nu}x_\nu\,. \tag{7.30}$$

They preserve the norm of vectors, implying that

$$\begin{aligned} x^\mu x_\mu = \tilde{x}^\mu\tilde{x}_\mu &= (x^\mu + \delta\omega^{\mu\sigma}x_\sigma)\,(x_\mu + \delta\omega_{\mu\tau}x^\tau) & (7.31)\\ &= x^\mu x_\mu + \delta\omega^{\mu\sigma}x_\sigma x_\mu + \delta\omega_{\mu\tau}x^\mu x^\tau + \mathcal{O}(\omega^2) & (7.32)\\ &= x^\mu x_\mu + (\delta\omega^{\mu\nu} + \delta\omega^{\nu\mu})x_\mu x_\nu\,. & (7.33)\end{aligned}$$

Thus the matrix parameterising Lorentz transformations is antisymmetric,

$$\omega^{\mu\nu} = -\omega^{\nu\mu}, \tag{7.34}$$

and has six independent elements. For an infinitesimal transformation, the transformed fields $\tilde{\phi}_a(\tilde{x})$ depend linearly on[1] $\delta\omega^{\mu\nu}$ and $\phi_a(x)$,

$$\tilde{\phi}_a(\tilde{x}) = \phi_a(x) + \frac{1}{2}\delta\omega_{\mu\nu}(I^{\mu\nu})_{ab}\phi_b(x)\,. \tag{7.35}$$

The symmetric part of $(I^{\mu\nu})_{ab}$ does not contribute, because of the antisymmetry of the $\delta\omega^{\mu\nu}$. Hence we can choose also the $(I^{\mu\nu})_{ab}$ as antisymmetric and thus there exists six generators

[1] We add a factor 1/2, because in the summation two terms contribute for each transformation parameter.

$(I^{\mu\nu})_{ab}$ corresponding to the three boosts and the three rotations. The explicit form of the generators I_{ab} (the "matrix representation of the Lorentz group" for spin s) depends on the spin of the considered field, as the known different transformation properties of scalar ($s = 0$), spinor ($s = 1/2$) and vector ($s = 1$) fields under rotations show.

We evaluate now the Noether current (7.27), inserting first the definition of the stress tensor,

$$j_\mu = \frac{\partial \mathscr{L}}{\partial(\partial^\mu \phi_a)}\, \delta\phi_a - \Theta_{\mu\nu}\delta x^\nu . \tag{7.36}$$

Next we use $\delta x^\mu = \delta\omega^{\mu\nu} x_\nu$ and $\delta\phi_a = \frac{1}{2}\delta\omega^{\mu\nu}(I^{\mu\nu})_{ab}\phi_b(x)$ as well as the antisymmetry of $\delta\omega^{\mu\nu}$, to obtain

$$j_\mu = \frac{\partial \mathscr{L}}{\partial(\partial^\mu \phi_a)}\, \frac{1}{2}\delta\omega^{\nu\lambda}(I_{\nu\lambda})_{ab}\phi_b(x) - \underbrace{\Theta_{\mu\nu}\delta\omega^{\nu\lambda}x_\lambda}_{\frac{1}{2}\delta\omega^{\nu\lambda}(\Theta_{\mu\nu}x_\lambda - \Theta_{\mu\lambda}x_\nu)} = \frac{1}{2}\delta\omega^{\nu\lambda} M_{\mu\nu\lambda} \tag{7.37}$$

with the definition

$$M_{\mu\nu\lambda} = \Theta_{\mu\lambda}x_\nu - \Theta_{\mu\nu}x_\lambda + \frac{\partial \mathscr{L}}{\partial(\partial^\mu \phi_a)}\, (I_{\nu\lambda})_{ab}\phi_b\, . \tag{7.38}$$

This tensor of rank three is antisymmetric in the index pair $\nu\lambda$ and conserved with respect to the index μ. In order to understand its meaning, let us consider first a scalar field $\phi(x)$. Then $\tilde{\phi}(\tilde{x}) = \phi(x)$, the last term is thus absent, and the conservation law becomes

$$0 = \partial_\mu M^{\mu\nu\lambda} = \delta^\nu_\mu\, \Theta^{\mu\lambda} - \delta^\lambda_\mu \Theta^{\mu\nu} = \Theta^{\nu\lambda} - \Theta^{\lambda\nu}\, . \tag{7.39}$$

Hence for a scalar field, the canonical stress tensor is symmetric, $\Theta^{\nu\lambda} = \Theta^{\lambda\nu}$, and agrees with the dynamical stress tensor, $\Theta^{\mu\nu} = T^{\mu\nu}$. The corresponding Noether charges are

$$M^{\nu\mu} = \int \mathrm{d}^3x\, M^{0\nu\mu} = \int \mathrm{d}^3x\, x^\nu\Theta^{0\mu} - x^\mu\Theta^{0\nu} \equiv L^{\mu\nu}\, . \tag{7.40}$$

Recalling Eq. (7.29), we see that these charges agree with the relativistic *orbital* angular momentum tensor $L^{\mu\nu}$. Since $L^{\mu\nu}$ is antisymmetric, Eq. (7.40) defines six conserved quantities, one for each of the generators of the Lorentz group. Choosing spatial indices, L^{ij} agrees with the non-relativistic orbital angular momentum, while the conservation of L^{i0} leads to the relativistic version of the constant center-of-mass motion.

For a field with non-zero spin, the last term in Eq. (7.38) does not vanish. It represents therefore the intrinsic or *spin angular momentum* density $S^{\mu\nu}$ of the field. In this case, only the total angular momentum $M^{\mu\nu}$ is conserved, not however the orbital and spin angular momentum individually. Moreover, the canonical stress tensor derived using Noether's theorem is in general not symmetric.

7.3 Perfect fluid

In cosmology, the various contributions to the energy content of the universe can be modelled as fluids, averaging over sufficiently large scales such that $N \gg 1$ particles (photons, dark matter particles, ..., galaxies) are contained in a "fluid element". In almost all cases, viscosity is negligible and the state of such an *ideal* or *perfect fluid* is fully parametrised by its energy density ρ and pressure P.

We construct the stress tensor of a perfect fluid considering first the simplest case of pressureless matter, traditionally called dust. Consider now how the energy density ρ of dust transforms. An observer moving relative to the rest frame of dust measures $\rho' = \gamma \mathrm{d}m/(\gamma^{-1}\mathrm{d}V) = \gamma^2\rho$. Hence the energy density should be the 00 component of the stress tensor tensor $T^{\alpha\beta}$, with $T^{00} = \rho$ in the rest frame. In order to find the expression valid in any frame we can use the *tensor method:* We express $T^{\alpha\beta}$ as a linear combination of all relevant tensors, which are in our case the four-velocity u^α plus the invariant tensors of Minkowski space, i.e. the metric tensor and the Levi-Civita symbol. Additionally, we impose the constraint that $T^{\alpha\beta}$ is symmetric, leading to

$$T^{\alpha\beta} = A\rho u^\alpha u^\beta + B\rho\eta^{\alpha\beta}\,. \tag{7.41}$$

In the rest-frame, $u^\alpha = (1, \mathbf{0})$, the condition $T^{00} = \rho$ leads $A - B = 1$, while $T^{11} = 0$ implies $B = 0$. Thus the stress tensor of dust is

$$T^{\alpha\beta} = \rho u^\alpha u^\beta\,. \tag{7.42}$$

Writing $u^\alpha = (\gamma, \gamma\boldsymbol{v})$, we can identify $T^{00} = \gamma^2\rho$ with the energy density, $T^{0i} = \gamma^2\rho v^i$ with the energy/momentum density flux in direction i, and $T^{ij} = \gamma^2\rho v^i v^j$ with the flow of the momentum density component i through the area with normal direction j.

Let us now check the consequences of $\partial_\alpha T^{\alpha\beta} = 0$, assuming for simplicity the non-relativistic limit. We look first at the $\beta = 0$ component,

$$\partial_t\rho + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}) = 0\,. \tag{7.43}$$

This corresponds to the mass continuity equation and, because of $E = m$ for dust, at the same time to energy conservation. Next we consider the $\beta = 1, 2, 3 = i$ components,

$$\partial_t(\rho u^j) + \partial_i(u^i u^j\rho) = 0 \tag{7.44}$$

or

$$\boldsymbol{u}\partial_t\rho + \rho\partial_t\boldsymbol{u} + \boldsymbol{u}\boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}) + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\rho\boldsymbol{u} = 0\,. \tag{7.45}$$

Taking the continuity equation into account, we obtain the Euler equation for a force-free fluid without viscosity,

$$\rho\partial_t\boldsymbol{u} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\rho\boldsymbol{u} = 0\,. \tag{7.46}$$

Hence, as anounced, the condition $\partial_\mu T^{\mu\nu} = f^\nu$ gives the equations of motion for a continuous medium.

In fluid mechanics, the derivative $\partial_t + \boldsymbol{u}\cdot\boldsymbol{\nabla}$ is often called convective derivative,

$$\frac{\mathcal{D}}{\mathcal{D}t} \equiv \partial_t + \boldsymbol{u}\cdot\boldsymbol{\nabla}. \tag{7.47}$$

Returning to four-vector notation and using $u^\alpha \simeq (1, \boldsymbol{v})$, we can rewrite the convective derivative applied to the vector V^α as

$$\frac{\mathcal{D}}{\mathcal{D}t}V^\alpha = \partial_t V^\alpha + \boldsymbol{u}\cdot\boldsymbol{\nabla}V^\alpha = u^\beta\partial_\beta V^\alpha. \tag{7.48}$$

Thus the convective derivative agrees with "our" directional (covariant) derivative (3.58).

Finally we include the effect of pressure. We know that the pressure tensor coincides with the σ_{ij} part of the stress tensor. Moreover, for a perfect fluid in its rest-frame, the pressure is isotropic $P_{ij} = P\delta_{ij}$. This corresponds to $P_{ij} = -P\eta_{ij}$ and adds $-P$ to T^{00}. Compensating for this gives

$$T^{\alpha\beta} = (\rho + P)u^\alpha u^\beta - P\eta^{\alpha\beta} . \tag{7.49}$$

7.4 Free scalar field

Real field The Klein-Gordon equation is a relativistic wave equation describing a scalar field. We first consider a real, free field ϕ. Similar as the free Schrödinger equation,

$$\mathrm{i}\partial_t\psi = \frac{p^2}{2m}\psi = \frac{\Delta}{2m}\psi , \tag{7.50}$$

can be "derived" using the replacements

$$E \to \mathrm{i}\partial_t \qquad \boldsymbol{p} \to -\mathrm{i}\boldsymbol{\nabla}_x \tag{7.51}$$

from the non-relativistic energy-momentum relation $E = p^2/(2m)$, we obtain from the relativistic $E^2 = m^2 + \boldsymbol{p}^2$

$$(\Box + m^2)\phi = 0 \qquad \text{with} \quad \Box = \eta_{\mu\nu}\partial^\mu\partial^\nu . \tag{7.52}$$

Translation invariance implys that we can choose the solutions as eigenstates of the momentum operator, $\hat{p}\phi = p\phi$. These states are plane waves with positive and negative energies $\pm\sqrt{\boldsymbol{p}^2 + m^2}$. Interpreting the Klein–Gordon equation as a relativistic wave equation for a single particle cannot therefore be fully satisfactory, since the energy of its solutions is not bounded from below.

How do we guess the correct Lagrange density $\mathscr{L}$? The correspondence $\dot{q} \leftrightarrow \partial_\mu\phi$ means that the kinetic field energy is quadratic in the field derivatives. In contrast, the mass term m^2 should be part of the potential energy, $V(\phi) \propto m^2$. The relativistic energy-momentum relation $E^2 = m^2 + \boldsymbol{p}^2$ suggests that $V(\phi)$ is also quadratic, with the same numerical coefficient as the kinetic energy. Therefore we try as Lagrange density

$$\mathscr{L} = \frac{1}{2}\eta_{\mu\nu}(\partial^\mu\phi)(\partial^\nu\phi) - V(\phi) = \frac{1}{2}\eta_{\mu\nu}(\partial^\mu\phi)(\partial^\nu\phi) - \frac{1}{2}m^2\phi^2 \equiv \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2, \tag{7.53}$$

where the factor 1/2 is convention: The kinetic energy of a *canonically normalised* real field carries the prefactor 1/2. With

$$\frac{\partial}{\partial(\partial_\alpha\phi)}\left(\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right) = \eta^{\mu\nu}\left(\delta^\alpha_\mu\partial_\nu\phi + \delta^\alpha_\nu\partial_\mu\phi\right) = \eta^{\alpha\nu}\partial_\nu\phi + \eta^{\mu\alpha}\partial_\mu\phi = 2\partial^\alpha\phi, \tag{7.54}$$

the Lagrange equation becomes

$$\frac{\partial\mathscr{L}}{\partial\phi} - \partial_\alpha\left(\frac{\partial\mathscr{L}}{\partial(\partial_\alpha\phi)}\right) = -m^2\phi - \partial_\alpha\partial^\alpha\phi = 0. \tag{7.55}$$

Thus the Lagrange density (7.53) leads to the Klein-Gordon equation. We can check if we have correctly chosen the signs by calculating the stress tensor,

$$T^{\mu\nu} = \frac{\partial \mathscr{L}}{\partial \phi_{,\mu}} \phi^{,\nu} - \eta^{\mu\nu} \mathscr{L} = \phi^{,\mu} \phi^{,\nu} - \eta^{\mu\nu} \mathscr{L} \,. \tag{7.56}$$

that is already symmetric. The corresponding 00 component is

$$T^{00} = \phi_{,0}\phi_{,0} - \mathscr{L} = \frac{1}{2}\left[(\partial_t \phi)^2 + (\nabla\phi)^2 + m^2\phi^2\right] > 0 \tag{7.57}$$

positiv definite. Thus the energy density of a scalar field is, in contrast to the energy of the single-particle solution, bounded from below.

Complex field and internal symmetries If two field exist with the same mass m, one might wish to combine the two real fields into one complex field,

$$\phi = \frac{1}{\sqrt{2}}\left(\phi_1 + \mathrm{i}\phi_2\right) \,. \tag{7.58}$$

Then one can interprete ϕ and $\phi^\dagger$ as a particle and its antiparticle, which are Hermetian conjugated fields.

The resulting Lagrangian density is just the sum,

$$\mathscr{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi \tag{7.59}$$

The presence of two fields sharing some quantum numbers (here the mass) opens up the possibility of internal symmetries. The Lagrangian (7.59) is invariant under global phase transformations, $\phi \to \mathrm{e}^{\mathrm{i}\vartheta}\phi$ and $\phi^\dagger \to \mathrm{e}^{-\mathrm{i}\vartheta}\phi^\dagger$. With $\delta\phi = \mathrm{i}\phi$ and $\delta\phi^\dagger = -\mathrm{i}\phi^\dagger$, the conserved current follows as

$$j^\mu = \mathrm{i}\left[\phi^\dagger \partial^\mu \phi - (\partial^\mu \phi^\dagger)\phi\right] . \tag{7.60}$$

The conserved charge $Q = \int \mathrm{d}^3x\, j^0$ can be also negative and thus we cannot interpret j^0 as the probability density to observe a ϕ particle. Instead, we should associate Q with a conserved additive quantum number as, for example, the electric charge.

Next we calculate the stress tensor,

$$T^{00} = 2\partial_t\phi^\dagger\partial_t\phi - \mathscr{L} = |\partial_t\phi|^2 + |\nabla\phi|^2 + m^2|\phi|^2 > 0 \,. \tag{7.61}$$

We consider now plane-wave solutions to the Klein-Gordon equation,

$$\phi = N\mathrm{e}^{-\mathrm{i}kx} . \tag{7.62}$$

If we insert $\partial_\mu\phi = \mathrm{i}k_\mu\phi$ into $\mathscr{L}$, we find $\mathscr{L} = 0$ and thus

$$T^{00} = 2|N|^2 k^0 k^0 \,. \tag{7.63}$$

Relativistic one-particle states are usually normalised as $N^{-2} = 2\omega V$. Thence the energy density $T^{00} = \omega/V$ agrees with the expectation for one particle with energy ω per volume V.

The other components are necessarily

$$T^{\mu\nu} = 2|N|^2 k^\mu k^\nu \,. \tag{7.64}$$

Since $T^{\mu\nu}$ is symmetric, we can find a frame in which $T^{\mu\nu}$ is diagonal with $T \propto \mathrm{diag}(\omega, v_x k_x, v_y k_y, v_z k_z)/V$). This agrees with the contribution of a single particle to the energy density and pressure of an ideal fluid. This holds also for other free fields, and thus we can model elementary fields as ideal fluids, distinguished only by their equation of state (E.o.S.), $w = P/\rho$.

7.5 Charge conservation and local gauge invariance

We have seen that the free charged scalar field is invariant under *global* phase transformations $\exp[\mathrm{i}q\Lambda] \in \mathrm{U}(1)$, implying a conserved current via Noether's theorem. We now ask if we can promote this global U(1) symmetry to a local one,

$$\phi(x) \to \phi'(x) = U(x)\phi(x) = \exp[\mathrm{i}q\Lambda(x)]\phi(x), \tag{7.65}$$

by making the phase U spacetime dependent. Since the partial derivatives in the Lagrangian (7.59) will lead to an additional term $\propto \partial_\mu U(x)$, its invariance is destroyed except we add an additional term as compensation to the free Lagrangian.

We proceed now similar to the case of gravity, where we modified the partial derivative such that we obtained the desired transformation properties. Thus we replace the normal derivative by a new covariant derivative D_μ,

$$\partial_\mu \to D_\mu = \partial_\mu + \mathrm{i}gA_\mu, \tag{7.66}$$

and require that D_μ transforms as the matter field ϕ,

$$D_\mu\phi(x) \to D'_\mu\phi'(x) = U(x)D_\mu\phi(x). \tag{7.67}$$

In the general case, ϕ may be a vector of N complex scalar fields, $\phi = \{\phi_1, \dots, \phi_n\}$, which are mixed by the gauge transformations. Then A_μ and U are matrices in field space, $A_\mu = (A_{ij})_\mu$. We will suppress these indices—the only important point to remember is that matrices do not necessairly commute.

We now determine the transformation properties of D_μ and A_μ demanding that (7.66) and (7.67) hold. Combining both requirements gives

$$D_\mu\phi(x) \to [D_\mu\phi]' = UD_\mu\phi = UD_\mu U^{-1}U\phi = UD_\mu U^{-1}\phi', \tag{7.68}$$

and thus the covariant derivative transforms as $D'_\mu = UD_\mu U^{-1}$. Using its definition (7.66), we find

$$[D_\mu\phi]' = [\partial_\mu + \mathrm{i}gA'_\mu]U\phi = UD_\mu\phi = U[\partial_\mu + \mathrm{i}gA_\mu]\phi. \tag{7.69}$$

We compare now the second and the fourth term, after having performed the differentiation $\partial_\mu(U\phi)$. The result

$$[(\partial_\mu U) + \mathrm{i}gA'_\mu U]\phi = \mathrm{i}gUA_\mu\phi \tag{7.70}$$

should be valid for arbitrary ϕ and hence, after multiplying from the right with U^{-1}, we arrive at

$$A_\mu \to A'_\mu = UA_\mu U^{-1} + \frac{\mathrm{i}}{g}(\partial_\mu U)U^{-1} = UA_\mu U^{-1} - \frac{\mathrm{i}}{g}U\partial_\mu U^{-1}. \tag{7.71}$$

Here we also used $\partial_\mu(UU^{-1}) = 0$. In most cases, the gauge transformation U is an unitary transformation and one sets $U^{-1} = U^\dagger$. A term changing as $U(x)D_\mu(x)U^\dagger(x)$ is said to transform homogeneously, while the potential A_μ is said to transform inhomogeneously.

Our discussion up to now was generic, applying to all (non-) abelian gauge fields as the photon, the electroweak gauge bosons or the gluons. Specialing now to the case of electrodynamics, the gauge group U(1) is abelian. Thus the transformation law for the electromagnetic field simplifies using $U = \exp[\mathrm{i}q\Lambda(x)]$ and $g \to q$ to

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \frac{\mathrm{i}}{q}U(x)\partial_\mu U^\dagger(x) = A_\mu(x) - \partial_\mu\Lambda(x), \tag{7.72}$$

agreeing with the standard result.

Finally, we consider the new Lagrangian of the complex scalar field, replacing the normal with covariant derivatives,

$$\mathscr{L} = (D_\mu\phi)^\dagger D^\mu\phi - m^2\phi^\dagger\phi. \tag{7.73}$$

Inserting the definition $D_\mu = \partial_\mu + \mathrm{i}qA_\mu$ and multiplying out, we obtain

$$\mathscr{L} = \partial_\mu\phi^\dagger\partial^\mu\phi - \mathrm{i}qA_\mu\phi^\dagger\partial^\mu\phi + \mathrm{i}qA^\mu(\partial_\mu\phi^\dagger)\phi + q^2A_\mu A^\mu\phi^\dagger\phi - m^2\phi^\dagger\phi. \tag{7.74}$$

The three terms additional to the free Lagrangian of the complex scalar field ϕ describe its interactions with the photon field A_μ: The first two connect two ϕ and one A^μ fields, the last one two ϕ and two A^μ fields. In the language of Feynman diagrams, the first two terms are represented by a vertex $\phi\phi A^\mu$ connecting three particles, the last one by a vertex $\phi\phi A^\mu A^\nu$ connecting four particles.

To summarise: the invariance of complex (scalar or Dirac) fields under *global* phase transformations $\exp[\mathrm{i}q\Lambda] \in \mathrm{U}(1)$ implies a conserved current; promoting it to a *local* U(1) symmetry requires the existence of a massless U(1) gauge boson[2] coupled via gauge-invariant derivatives to these fields. What is still is missing, is the free Lagrangian of the field A_μ.

7.6 Electrodynamics as an abelian gauge theory

Field-strength tensor We have to find the free Lagrangian for the field A_μ. The Lagrangian should be gauge invariant, and quadratic in first derivative of the field A_μ (compare with the scalar field). Thus we should use covariant instead of partial derivatives. Moreover, the commutator of covariant derivatives is antisymmetric and thus the annoying inhomogeneous term in the transformation rule of A_μ may drop out.

Calculating therefore the commutator of covariant derivatives,

$$[D_\mu, D_\nu]\phi = \mathrm{i}q([\partial_\mu, A_\nu] - [\partial_\nu, A_\mu])\phi = \mathrm{i}q\phi(\partial_\mu A_\nu - \partial_\nu A_\mu) \equiv \mathrm{i}q\phi F_{\mu\nu}. \tag{7.75}$$

where we introduced the field-strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{7.76}$$

Clearly, its antisymmetry implies its invariance under gauge transformations,

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = F_{\mu\nu} - \partial_\mu\partial_\nu\Lambda + \partial_\nu\partial_\mu\Lambda = F_{\mu\nu}\,. \tag{7.77}$$

Example 7.2: Find the connection between 3- and 4-dim. formulation of electrodynamics: The first row of $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ reads with $A_\mu = (\phi, -A_k)$ and $\partial_\nu = \partial/\partial x^\mu = (\partial/\partial t, \nabla_k)$ as

$$F_{0k} = \partial_0 A_k - \partial_k A_0$$

Setting $F_{0k} = E^k$ gives

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \partial_t\boldsymbol{A},$$

what agrees with the first row of $F_{\mu\nu}$ given in Eq. (1.57). We go in the opposite direction for

[2] The absence of such massless bosons (or classically $1/r^2$ forces) in the case of baryon number means that this is a global symmetry which cannot be promoted to a local one.

$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$. In components, we have e.g.

$$B^1 = \partial_2 A^3 - \partial_3 A^2 = \partial_3 A_2 - \partial_2 A_3 = F_{32}$$

and similarly for the other components.

Now we can rewrite the Maxwell equations as

$$\partial_\alpha F^{\alpha\beta} = j^\beta \tag{7.78}$$

and

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0\,. \tag{7.79}$$

The last equation is completely antisymmetric in all three indices, and contains therefore only four independent equations. It is equivalent to

$$\partial_\alpha \tilde{F}^{\alpha\beta} = 0\,, \tag{7.80}$$

where

$$\tilde{F}^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} \tag{7.81}$$

is the dual field-strength tensor.

The components of the electromagnetic field-strength tensor $F^{\mu\nu}$ and its dual $\tilde{F}_{\alpha\beta} = \frac{1}{2}\varepsilon_{\alpha\beta\mu\nu}F^{\mu\nu}$ are given by (see also appendix to chapter 1)

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \quad \text{and} \quad \tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}.$$

In general, the electric and magnetic fields measured by an observer with four-velocity u_α are connected to the field-strength tensor as $E_\alpha = F_{\alpha\beta}u^\beta$ and $B_\alpha = \tilde{F}_{\alpha\beta}u^\beta$.

Example 7.3: compare the measured E and B with those in $F_{\alpha\beta}$ and $F^{\alpha\beta}$.

Current conservation and gauge invariance We take the divergence of Maxwells equation (7.78),

$$\partial_\nu \partial_\mu F^{\mu\nu} = \partial_\nu j^\nu\,. \tag{7.82}$$

Since $\partial_\nu\partial_\mu$ is symmetric and $F^{\mu\nu}$ antisymmetric, the summation of the two factors has to be zero,

$$\partial_\nu \partial_\mu F^{\mu\nu} = -\partial_\nu \partial_\mu F^{\nu\mu} = -\partial_\mu \partial_\nu F^{\nu\mu} = -\partial_\nu \partial_\mu F^{\mu\nu}\,. \tag{7.83}$$

Thus current conservation,

$$\partial_\nu j^\nu = 0\,, \tag{7.84}$$

follows from the antisymmetry of $F^{\mu\nu}$. The latter followed in turn from the required gauge-invariance of the Maxwell Lagrangian.

Remark 7.2: Differential forms:
A surface in $\mathbb{R}^3$ can be described at any point either by its two tangent vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ or by the normal $\boldsymbol{n}$. They are connected by a cross product, $\boldsymbol{n} = \boldsymbol{e}_1 \times \boldsymbol{e}_2$, or in index notation,

$$n^i = \varepsilon^{ijk} e_{1,j} e_{k,2} \,. \tag{7.85}$$

In four dimensions, the ε tensor defines a map between 1-3 and 2-2 tensors. Since ε is antisymmetric, the symmetric part of tensors would be lost; Hence the map is suited for antisymmetric tensors.
Antisymmetric tensors of rank n can be seen also as differential forms: Functions are forms of order $n = 0$; differential of functions are an example of order $n = 1$,

$$\mathrm{d}f = \frac{\partial f}{\partial x^i}\,\mathrm{d}x^i \tag{7.86}$$

Thus the $\mathrm{d}x^i$ form a basis, and one can write in general

$$\boldsymbol{A} = A_i\,\mathrm{d}x^i \,. \tag{7.87}$$

For $n > 1$, the basis has to be antisymmetrized,

$$\boldsymbol{F} = \frac{1}{2}\,F_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \tag{7.88}$$

with $\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = -\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu$. Looking at $\mathrm{d}f$, we can define a differentiation of a form ω with coefficients w and degree n as an operation that increases its degree by one to $n+1$,

$$\mathrm{d}\omega = dw_{\alpha,\ldots,\beta}\mathrm{d}x^{n+1} \wedge \mathrm{d}x^\alpha \wedge \ldots \wedge \mathrm{d}x^\beta \tag{7.89}$$

Thus we have $\boldsymbol{F} = \mathrm{d}\boldsymbol{A}$. Moreover, it follows $\mathrm{d}^2\omega = 0$ for all forms. Hence a gauge transformation $\boldsymbol{F}' = \mathrm{d}(\boldsymbol{A} + \mathrm{d}\chi) = \boldsymbol{F}$.

Wave equation The Maxwell equation (7.78) consists of four equations for the six components of $\boldsymbol{F}$. Thus we need either a second equation, i.e. Eq. (7.79), or we should transform Eq. (7.78) into an equation for the four components of the four-potential $\boldsymbol{A}$. In this case, Eq. (7.79) is automatically satisfied. Let us do the latter and insert the definition of $\boldsymbol{A}$,

$$\partial_\mu F^{\mu\nu} = \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = \Box A^\nu - \partial_\mu \partial^\nu A^\mu = j^\nu \,. \tag{7.90}$$

Gauge invariance allows us to choose a potential A^μ such that $\partial_\mu A^\mu = 0$. Such a choice is called fixing the gauge, and the particular case $\partial_\mu A^\mu = 0$ is denoted as the Lorenz gauge. In this gauge, the wave equation simplifies to

$$\Box A^\mu = j^\mu. \tag{7.91}$$

Inserting then a plane wave $A^\mu \propto \varepsilon^\mu \mathrm{e}^{\mathrm{i}kx}$ into the free wave equation, $\Box A^\nu = 0$, we find that k is a light-like vector, while the Lorenz gauge condition $\partial_\mu A^\mu = 0$ results in $\varepsilon^\mu k_\mu = 0$. Imposing the Lorenz gauge, we can still add to the potential A^μ any function $\partial^\mu \chi$ satisfying $\Box\chi = 0$. We can use this freedom to set $A^0 = 0$, obtaining thereby $\varepsilon^\mu k_\mu = -\boldsymbol{\varepsilon} \cdot \boldsymbol{k} = 0$. Thus the photon propagates with the speed of light, is transversely polarised and has two polarisation states as expected for a massless particle.

Let us discuss now why gauge invariance is necessary for a massless spin-1 particle. First we consider a linearly polarised photon with polarisation vectors $\varepsilon_\mu^{(r)}$ lying in the plane perpendicular to its momentum vector $\boldsymbol{k}$. If we perform a Lorentz boost on $\varepsilon_\mu^{(1)}$, we will find

$$\tilde{\varepsilon}_\mu^{(1)} = \Lambda^\nu{}_\mu \varepsilon_\nu^{(1)} = a_1\varepsilon_\mu^{(1)} + a_2\varepsilon_\mu^{(2)} + a_3 k_\mu\,, \tag{7.92}$$

where the coefficients a_i depend on the direction $\boldsymbol{\beta}$ of the boost. Thus, in general the polarisation vector will not be anymore perpendicular to $\boldsymbol{k}$. Similarly, if we perform a gauge transformation

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \partial_\mu \Lambda(x) \tag{7.93}$$

with

$$\Lambda(x) = -\mathrm{i}\lambda \exp(-\mathrm{i}kx) + \text{h.c.}\,, \tag{7.94}$$

then

$$A'_\mu(x) = (\varepsilon_\mu + \lambda k_\mu)\exp(-\mathrm{i}kx) + \text{h.c.} = \varepsilon'_\mu \exp(-\mathrm{i}kx) + \text{h.c.} \tag{7.95}$$

Choosing, for example, a photon propagating in z direction, $k^\mu = (\omega, 0, 0, \omega)$, we see that the gauge transformation does not affect the transverse components ε_1 and ε_2. Thus only the components of ε^μ transverse to $\boldsymbol{k}$ can have physical significance. On the other hand, the time-like and longitudinal components depend on the arbitrary parameter λ and are therefore unphysical. In particular, they can be set to zero by a gauge transformation. First, $\varepsilon'_\mu k'^\mu = 0$ implies (again for a photon propagating in z direction) $\varepsilon'_0 = -\varepsilon'_3$. From $\varepsilon'_3 = \varepsilon_3 + \lambda\omega$, we see that $\lambda = -\varepsilon_3/\omega$ sets $\varepsilon'_3 = -\varepsilon'_0 = 0$. Thus the transformation law (7.92) for the polarisation vector of a massless spin-1 particles requires the existence of the gauge symmetry (7.93). The gauge symmetry in turn implies that the massless spin-1 particle couples only to conserved currents.

Lagrange density The free field equation is

$$\partial_\mu F^{\mu\nu} = 0\,. \tag{7.96}$$

In order to find $\mathscr{L}$, we multiply by a variation δA_ν that vanishes on the boundary $\partial\Omega$. Then we integrate over $\Omega = V \times [t_a : t_b]$, and perform a partial integration,

$$\int_\Omega \mathrm{d}^4x\, \partial_\mu F^{\mu\nu}\, \delta A_\nu = -\int_\Omega \mathrm{d}^4x\, F^{\mu\nu}\, \delta(\partial_\mu A_\nu) = 0\,. \tag{7.97}$$

Next we note that

$$(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) = 2(\partial_\mu A_\nu - \partial_\nu A_\mu)\partial^\mu A^\nu \tag{7.98}$$

and thus

$$F^{\mu\nu}\,\delta(\partial_\mu A_\nu) = \frac{1}{2} F^{\mu\nu}\,\delta F_{\mu\nu}\,. \tag{7.99}$$

Applying the product rule, we obtain as final result

$$-\frac{1}{4}\,\delta \int_\Omega \mathrm{d}^4x\, F_{\mu\nu}F^{\mu\nu} = \delta S[A_\mu] = 0 \tag{7.100}$$

with

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu}\,. \tag{7.101}$$

Note that although we expressed $\mathscr{L}$ through the field-strength tensor $F_{\mu\nu}$, the action is a functional of A: The latter is the dynamical field variable which enters, e.g., the Lagrange equations, giving us the a second-order (wave) equation. This is in accordance with the fact that A_μ determines the interaction (7.110) with charged particles.

In order to justify the sign of the Lagrangian, we calculate next the corresponding energy density $\rho = T^{00}$. In exercise 7.2, you are asked to show that factor 1/4 leads to a canonically normalised field.

Stress tensor According to Eq. (7.28) we have

$$\Theta_\mu^{\ \nu} = \frac{\partial A_\sigma}{\partial x^\mu} \frac{\partial \mathscr{L}}{\partial(\partial A_\sigma/\partial x^\nu)} - \delta_\mu^\nu \mathscr{L} \,. \tag{7.102}$$

Since $\mathscr{L}$ depends only on the derivatives $A^\mu_{,\nu}$, we can use the following short-cut: We know already that

$$\delta\mathscr{L} = -\frac{1}{4}\,\delta(F_{\mu\nu}F^{\mu\nu}) = F^{\mu\nu}\,\delta(\partial_\nu A_\mu)\,. \tag{7.103}$$

Thus

$$\frac{\partial \mathscr{L}}{\partial(\partial A_\sigma/\partial x^\nu)} = F^{\sigma\nu} = -F^{\nu\sigma} \tag{7.104}$$

and

$$\Theta_\mu^{\ \nu} = -\frac{\partial A_\sigma}{\partial x^\mu}\,F^{\nu\sigma} + \frac{1}{4}\,\delta_\mu^\nu F_{\sigma\tau}F^{\sigma\tau}\,. \tag{7.105}$$

Raising the index μ and rearranging σ, we have

$$\Theta^{\mu\nu} = -\frac{\partial A^\sigma}{\partial x_\mu}\,F^\nu_{\ \sigma} + \frac{1}{4}\,\eta^{\mu\nu}F_{\sigma\tau}F^{\sigma\tau}\,. \tag{7.106}$$

This result in neither gauge invariant (contains A) nor symmetric. To symmetrize it, we should add

$$\frac{\partial A^\mu}{\partial x_\sigma}\,F^\nu_{\ \sigma} = \frac{\partial}{\partial x_\sigma}(A^\mu F^\nu_{\ \sigma})\,. \tag{7.107}$$

The last step is possible for a free electromagnetic field, $\partial_\sigma F^{\nu\sigma} = 0$, and shows that we are allowed to add the LHS. Then the two terms combine to F, and we get

$$\Theta^{\mu\nu} = -F^{\mu\sigma}\,F^\nu_{\ \sigma} + \frac{1}{4}\,\eta^{\mu\nu}F_{\sigma\tau}F^{\sigma\tau}\,. \tag{7.108}$$

In this form, the stress tensor is symmetric and gauge invariant. We can thus identify the expression (7.108) with the dynamical stress tensor, $\Theta^{\mu\nu} = T^{\mu\nu}$. Note that its trace is zero, $T^\mu_\mu = 0$.

Lorentz force We start by considering a charged point particle interacting with an external electromagnetic described by the vector potential $A^\mu = (\phi, \boldsymbol{A})$. As Lagrangian for the free particle we use $L = -m\mathrm{d}s$ or

$$S_0 = -\int_a^b \mathrm{d}s\, m. \tag{7.109}$$

How can the interaction term charged particle with an electromagnetic field look like? The action should be a scalar and the simplest choice is

$$S_{\text{em}} = -q \int \mathrm{d}x^{\mu}\, A_{\mu}(x) = -q \int \mathrm{d}\sigma\, \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\sigma}\, A_{\mu}(x)\,. \tag{7.110}$$

Note that this choice for S_{em} is invariant under a change of gauge,

$$A_{\mu}(x) \to A_{\mu}(x) + \partial_{\mu}\Lambda(x)\,. \tag{7.111}$$

The resulting change in the action,

$$\delta_{\Lambda} S_{\text{em}} = -q \int_1^2 \mathrm{d}\sigma\, \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\sigma}\, \frac{\partial\Lambda(x)}{\partial x^{\mu}} = -q \int_1^2 \mathrm{d}\Lambda = q[\Lambda(2) - \Lambda(1)] \tag{7.112}$$

drops out from δS for fixed endpoints, thus not affecting the resulting equation of motion. With $\mathrm{d}s = \sqrt{\mathrm{d}x^{\mu}\mathrm{d}x_{\mu}}$, the variation of the action is

$$\delta S = -\delta \int_a^b (m\mathrm{d}s + qA_{\mu}\mathrm{d}x^{\mu}) = -\int_a^b \left(m\frac{\mathrm{d}x^{\mu}\delta\mathrm{d}x_{\mu}}{\mathrm{d}s} + qA_{\mu}\mathrm{d}(\delta x^{\mu}) + q\delta A_{\mu}\mathrm{d}x^{\mu} \right)\,. \tag{7.113}$$

We use $\delta\mathrm{d} = \mathrm{d}\delta$ in the frist term and integrate then the first two terms partially,

$$\delta S = \int_a^b \left(m\mathrm{d}\left(\frac{\mathrm{d}x_{\mu}}{\mathrm{d}s}\right)\delta x^{\mu} + q\delta x^{\mu}\mathrm{d}A_{\mu} - q\delta A_{\mu}\mathrm{d}x^{\mu} \right) \tag{7.114}$$

where we have uses as "always" that the boundary terms vanish. Next we introduce $u_{\mu} = \mathrm{d}x_{\mu}/\mathrm{d}s$ and use

$$\delta A_{\mu} = \frac{\partial A_{\mu}}{\partial x^{\nu}}\delta x^{\nu}\,, \qquad \mathrm{d}A_{\mu} = \frac{\partial A_{\mu}}{\partial x^{\nu}}\mathrm{d}x^{\nu}\,. \tag{7.115}$$

Then

$$\delta S = \int_a^b \left(m\mathrm{d}u_{\mu}\delta x^{\mu} + q\frac{\partial A_{\mu}}{\partial x^{\nu}}\delta x^{\mu}\mathrm{d}x^{\nu} - q\frac{\partial A_{\mu}}{\partial x^{\nu}}\delta x^{\nu}\mathrm{d}x^{\mu} \right)\,. \tag{7.116}$$

Finally, we rewrite in the first term $\mathrm{d}u_{\alpha} = \mathrm{d}u_{\alpha}/\mathrm{d}s\ \mathrm{d}s$, in the second and third $\mathrm{d}x^{\alpha} = u^{\alpha}\mathrm{d}s$ and exchange the summation indices μ and ν in the third term. Then

$$\delta S = \int_a^b \left[m\frac{\mathrm{d}u_{\mu}}{\mathrm{d}s} - q\left(\frac{\partial A_{\nu}}{\partial x^{\mu}} - \frac{\partial A_{\mu}}{\partial x^{\nu}}\right)u^{\nu} \right]\delta x^{\mu}\mathrm{d}s = 0\,. \tag{7.117}$$

For arbitrary variations, the brackets has to be zero and we obtain as equation of motion

$$m\frac{\mathrm{d}u_{\mu}}{\mathrm{d}s} = f_{\mu} = q\left(\frac{\partial A_{\nu}}{\partial x^{\mu}} - \frac{\partial A_{\mu}}{\partial x^{\nu}}\right)u^{\nu} \equiv qF_{\mu\nu}u^{\nu}\,. \tag{7.118}$$

This is the relativistic form of the Lorentz force.

7.7 Non-abelian gauge theories

Gauge fields as connection There is a close analogy between the covariant derivative ∇_μ introduced for a spacetime containing a gravitational field and the gauge-invariant derivative D_μ required for a spacetime containing a gauge field. In the former case, the moving coordinate basis in curved spacetime, $\partial_\mu \boldsymbol{e}^\nu \neq 0$, introduces an additional term in the derivative of vector components $V^\mu = \boldsymbol{e}^\mu \cdot \boldsymbol{V}$. Analogously, a non-zero gauge field A^μ leads to a rotation of the basis vectors $\boldsymbol{e}_i$ in group space which in turn produces an additional term $\boldsymbol{\psi} \cdot (\partial_\mu \boldsymbol{e}_i)$ performing the derivative of a $\psi_i = \boldsymbol{\psi} \cdot \boldsymbol{e}_i$.

Let us rewrite our formulae such that the analogy between the covariant gauge derivative D_μ and the covariant spacetime derivative ∇_μ becomes obvious. The vector $\boldsymbol{\psi}$ of fields with components $\{\psi_1, \dots, \psi_n\}$ transforming under a representation of a gauge group can be written as

$$\boldsymbol{\psi}(x) = \psi_i(x)\boldsymbol{e}_i(x). \tag{7.119}$$

We can pick out the component ψ_j by multiplying with the corresponding basis vector $\boldsymbol{e}_j$,

$$\psi_j = \boldsymbol{\psi} \cdot \boldsymbol{e}_j(x). \tag{7.120}$$

If the coordinate basis in group space depends on x^μ, then the partial derivative of ψ_i acquires a second term,

$$\partial_\mu \psi_i = (\partial_\mu \boldsymbol{\psi}) \cdot \boldsymbol{e}_i + \boldsymbol{\psi} \cdot (\partial_\mu \boldsymbol{e}_i) \,. \tag{7.121}$$

We can argue, as in section 3.3, that $(\partial_\mu \boldsymbol{\psi}) \cdot \boldsymbol{e}_i$ is an invariant quantity, defining therefore as gauge-invariant derivative

$$D_\mu \psi_i = (\partial_\mu \boldsymbol{\psi}) \cdot \boldsymbol{e}_i = \partial_\mu \psi_i - \boldsymbol{\psi} \cdot (\partial_\mu \boldsymbol{e}_i). \tag{7.122}$$

The change $\partial_\mu \boldsymbol{e}_i$ of the basis vector in group space should be proportional to gA_μ. Setting

$$\partial_\mu \boldsymbol{e}_i = -\mathrm{i}g(A_\mu)_{ij}\boldsymbol{e}_j \tag{7.123}$$

we are back to our old notation. In the abelian case, i.e. electrodynamics, there is a single gauge field, the photon. Then the matrix becomes diagonal, containing the charge in units of e as entries, $g(A_\mu)_{ij} \to eA_\mu \mathrm{diag}(q_i, \dots, q_n)$.

Gauge loops The correspondence between the derivatives ∇_μ and D_μ suggests that we can use the gauge field A_μ to transport fields along a curve $x^\mu(\sigma)$. In empty space, we can use the partial derivative $\partial_\mu \psi(x)$ to compare fields at different points,

$$\partial_\mu \psi(x) \propto \psi(x + \mathrm{d}x^\mu) - \psi(x). \tag{7.124}$$

If there is an external gauge field present, the field ψ is additionally rotated in group space moving it from x to $x + \mathrm{d}x$,

$$\tilde{\psi}(x + \mathrm{d}x) = \psi(x + \mathrm{d}x) + \mathrm{i}gA_\mu(x)\psi(x)\mathrm{d}x^\mu \tag{7.125a}$$

$$= \psi(x) + \partial_\mu \psi(x)\mathrm{d}x^\mu + \mathrm{i}gA_\mu(x)\psi(x)\mathrm{d}x^\mu. \tag{7.125b}$$

Then the total change is

$$\tilde{\psi}(x + \mathrm{d}x) - \psi(x) = [\partial_\mu + \mathrm{i}gA_\mu(x)]\psi(x)\mathrm{d}x^\mu = D_\mu \psi(x)\mathrm{d}x^\mu. \tag{7.126}$$

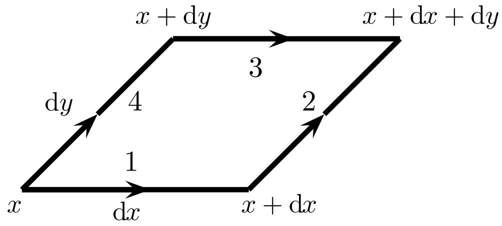


Figure 7.1: Parallelogram used to calculate the rotation of a test field ψ_i moved along a closed loop in the presence of a non-zero gauge field A^μ.

Thus we can view[3]

$$P_{\mathrm{d}x}(x) = 1 - \mathrm{i}gA_\mu(x)\mathrm{d}x^\mu \tag{7.127}$$

as an operator which allows us to transport a gauge-dependent field the infinitesimal distance from x to $x+\mathrm{d}x$.

We ask now what happens to a field $\psi_i(x)$, if we transport it along an infinitesimal parallelogram, as shown in Fig. 7.1. Calculating path 2, we find

$$\begin{aligned} P_{\mathrm{d}y}(x+\mathrm{d}x) &= 1 - \mathrm{i}gA_\nu(x+\mathrm{d}x)\mathrm{d}y^\nu \\ &= 1 - \mathrm{i}gA_\nu(x)\mathrm{d}y^\nu - \mathrm{i}g\partial_\mu A_\nu(x)\mathrm{d}x^\mu \mathrm{d}y^\nu, \end{aligned} \tag{7.128}$$

where we Taylor expanded $A_\nu(x+\mathrm{d}x)$. Combining paths 1 and 2, we arrive at

$$\begin{aligned} P_{\mathrm{d}y}(x+\mathrm{d}x)P_{\mathrm{d}x}(x) &= [1 - \mathrm{i}gA_\nu(x)\mathrm{d}y^\nu - \mathrm{i}g\partial_\mu A_\nu(x)\mathrm{d}x^\mu \mathrm{d}y^\nu][1 - \mathrm{i}gA_\mu(x)\mathrm{d}x^\mu] \\ &= 1 - \mathrm{i}gA_\mu(x)\mathrm{d}x^\mu - \mathrm{i}gA_\nu(x)\mathrm{d}y^\nu - \mathrm{i}g\partial_\mu A_\nu(x)\mathrm{d}x^\mu \mathrm{d}y^\nu \\ &- g^2 A_\nu(x)A_\mu(x)\mathrm{d}y^\nu \mathrm{d}x^\mu + \mathcal{O}(\mathrm{d}x^3). \end{aligned} \tag{7.129}$$

Instead of performing the calculation for a round trip $1\to2\to3\to4$, we evaluate next $4\to3$ which we then subtract from $1\to2$. In this way, we can reuse our result for $1\to2$ after exchanging labels, $A_\mu\mathrm{d}x^\mu \leftrightarrow A_\nu\mathrm{d}y^\nu$, obtaining

$$\begin{aligned} P_{\mathrm{d}x}(x+\mathrm{d}y)P_{\mathrm{d}y}(x) &= 1 - \mathrm{i}gA_\nu(x)\mathrm{d}y^\nu - \mathrm{i}gA_\mu(x)\mathrm{d}x^\mu - \mathrm{i}g\partial_\nu A_\mu(x)\mathrm{d}x^\mu \mathrm{d}y^\nu \\ &- g^2 A_\mu(x)A_\nu(x)\mathrm{d}x^\mu \mathrm{d}y^\nu + \mathcal{O}(\mathrm{d}x^3). \end{aligned} \tag{7.130}$$

The first three terms on the RHSs of (7.129) and (7.130) cancel in the result $P(\Box)$ for the round trip, leaving us with

$$\begin{aligned} P(\Box) \equiv P_{\mathrm{d}y}(x+dx)P_{\mathrm{d}x}(x) - P_{\mathrm{d}x}(x+dy)P_{\mathrm{d}y}(x) &= \\ - \mathrm{i}g\left\{\partial_\mu A_\nu - \partial_\nu A_\mu + \mathrm{i}g[A_\mu, A_\nu]\right\}\mathrm{d}x^\mu \mathrm{d}y^\nu&. \end{aligned} \tag{7.131}$$

If the enclosed flux is non-zero, then $P(\Box)\psi_i \neq \psi_i$ and thus the field is rotated.

Maxwell's equations inform us that the line integral of the vector potential equals the enclosed flux. The area of the parallelogram corresponds to $\mathrm{d}x^\mu \mathrm{d}y^\nu$, and the pre-factor has to be therefore the field-strength tensor. In the abelian case, i.e. electrodynamics, the

[3]Note the sign change compared to the covariant derivative: there we pull back the field from $x+\mathrm{d}x$ to x.

commutator $[A_\mu, A_\nu]$ vanishes and we are back to our definition (7.76). In the non-abelian case, we read-off

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \mathrm{i}g[A_\mu, A_\nu] \tag{7.132}$$

as the generalisation of the field-strength tensor. This is in line with the definition $F_{\mu\nu} = [D_\mu, D_\nu]/(\mathrm{i}g)$ in Eq. (7.75).

Problems

7.1 Dynamical stress tensor. Show that the definition of the dynamical stress tensor can be simplified to

$$T_{\mu\nu} = \frac{2}{\sqrt{|g|}} \frac{\delta S_\mathrm{m}}{\delta g^{\mu\nu}} = 2\frac{\partial \mathscr{L}}{\partial g^{\mu\nu}} - g_{\mu\nu}\mathscr{L}\,. \tag{7.133}$$

7.2 Maxwell Lagrangian. Show that the normalisation of the Maxwell Lagrangian corresponds to a canonically normalised field.

8 Einstein's field equation

Up to now, we have investigated the behaviour of test-particles and light-rays in a given curved spacetime determined by the metric tensor $g_{\mu\nu}$. In this chapter we discuss how the metric tensor $g_{\mu\nu}$, or the geometry of a spacetime, is connected to its matter content, and vice versa. The transition from point mechanics to a field theory means that the role of the mass m as the source of Newtonian gravity should be taken by the mass density ρ, or in the relativistic case, by the stress tensor $T_{\mu\nu}$. Thus we expect a field equation of the type $G_{\mu\nu} = \kappa T_{\mu\nu}$, where κ is proportional to Newton's constant G and $G_{\mu\nu}$ is a function of $g_{\mu\nu}$ and its derivatives.

8.1 Curvature and the Riemann tensor

We are looking for an invariant characterisation of the curvature of an manifold induced by gravity. As the discussion of normal coordinates showed, the first derivatives of the metric can be (at one point) always chosen to be zero. Hence such quantity should contain second derivatives of the metric, i.e. first derivatives of the Christoffel symbols. Curvature mainfests itself in two ways: By the rotation of a vector parallel transported and by tidal effects on nearby geodesics. Both effects can be used to define the curvature and the Riemann tensor.

Curvature as commutator of covariant derivatives We start using the first approach, employing the analogy between gauge theories and gravity. Both the gauge field A_μ and the connection $\Gamma^\mu{}_{\kappa\rho}$ transform inhomogeneously. Therefore we cannot use them to judge if a gauge or gravitational field is present. In the gauge case, we introduced therefore the field-strength $F_{\mu\nu}$. It transforms homogeneously and thus the statement $F_{\mu\nu}(x) = 0$ holds in any gauge. This suggests transforming (7.75) into a definition for a tensor measuring a non-zero curvature of spacetime by replacing D_μ with ∇_μ: The commutator of covariant derivatives applied to a tensor,

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) T^{\mu\dots}_{\nu\dots} = [\nabla_\alpha, \nabla_\beta] T^{\mu\dots}_{\nu\dots} \neq 0, \tag{8.1}$$

is obviously a tensor and contains second derivatives of the metric.

For the special case of a vector V^α we obtain with

$$\nabla_\rho V^\alpha = \partial_\rho V^\alpha + \Gamma^\alpha{}_{\beta\rho} V^\beta \tag{8.2}$$

first

$$\nabla_\sigma \nabla_\rho V^\alpha = \partial_\sigma(\partial_\rho V^\alpha + \Gamma^\alpha{}_{\beta\rho} V^\beta) + \Gamma^\alpha{}_{\kappa\sigma}(\partial_\rho V^\kappa + \Gamma^\kappa{}_{\beta\rho} V^\beta) - \Gamma^\kappa{}_{\rho\sigma}(\partial_\kappa V^\alpha + \Gamma^\alpha{}_{\beta\kappa} V^\beta). \tag{8.3}$$

The second part of the commutator follows relabelling $\sigma \leftrightarrow \rho$ as

$$\nabla_\rho \nabla_\sigma V^\alpha = \partial_\rho(\partial_\sigma V^\alpha + \Gamma^a{}_{\beta\sigma} V^\beta) + \Gamma^\alpha{}_{\kappa\rho}(\partial_\sigma V^\kappa + \Gamma^\kappa{}_{b\sigma} V^\beta) - \Gamma^\kappa{}_{\sigma\rho}(\partial_\kappa V^\alpha + \Gamma^\alpha{}_{\beta\kappa} V^\beta). \tag{8.4}$$

Now we subtract the two equations using that $\partial_\rho\partial_\sigma = \partial_\sigma\partial_\rho$ and $\Gamma^\alpha{}_{\beta\rho} = \Gamma^\alpha{}_{\rho\beta}$,

$$[\nabla_\rho, \nabla_\sigma]V^\alpha = \left[\partial_\rho\Gamma^\alpha{}_{\beta\sigma} - \partial_\sigma\Gamma^\alpha{}_{\beta\rho} + \Gamma^\alpha{}_{\kappa\rho}\Gamma^\kappa{}_{\beta\sigma} - \Gamma^\alpha{}_{\kappa\sigma}\Gamma^\kappa{}_{\beta\rho}\right] V^\beta \equiv R^\alpha{}_{\beta\rho\sigma}V^\beta. \tag{8.5}$$

The tensor $R^\alpha{}_{\beta\rho\sigma}$ is called *Riemann* or *curvature tensor*.

Remark 8.1: We can push the analogy between Yang-Mills theories and gravity further by remembering that the field-strength defined in Eq. (7.132) is a matrix. Writing out the implicit matrix indices of $F_{\mu\nu}$ in Eq. (7.75) gives

$$(F_{\mu\nu})_{ij} = \partial_\mu(A_\nu)_{ij} - \partial_\nu(A_\mu)_{ij} + \mathrm{i}g\left\{(A_\mu)_{ik}(A_\nu)_{kj} - (A_\nu)_{ik}(A_\mu)_{kj}\right\}. \tag{8.6}$$

Comparing this expression to

$$R^\alpha{}_{\beta\mu\nu} = \partial_\mu\Gamma^\alpha{}_{\beta\nu} - \partial_\nu\Gamma^\alpha{}_{\beta\mu} + \Gamma^\alpha{}_{\rho\mu}\Gamma^\rho{}_{\beta\nu} - \Gamma^\alpha{}_{\rho\nu}\Gamma^\rho{}_{\beta\mu} \tag{8.7}$$

we see that the first two indices of the Riemann tensor, α and β, correspond to the group indices ij in the field-strength tensor. This is in line with the relation of the potential $(A_\mu)_{ij}$ and the connection $\Gamma^\alpha{}_{\beta\nu}$ implied by (7.123).

*****Equation of geodesic deviation***** An alternative definition of the Riemann tensor uses how the distance of nearby geodesics changes. In the context of general relativity, this deviation is caused by the tidal forces induced by gravity.

Denote by t the tangent vector fields along a family of geodetics $x^\mu(\tau, s)$ which are parametrised by proper-time τ and their separation s. Then a second family of vector field n normal to t given by $n^\alpha = \partial x^\alpha/\partial v \delta v$ gives the separation between two neighbouring geodesics with v and $v + \delta v$. Compute the acceleration of the separation,

$$\frac{D^2 n^\alpha}{\mathrm{d}\tau} = t^\beta\nabla_\beta(t^\sigma\nabla_\sigma n^\alpha) = t^\beta\nabla_\beta(n^\sigma\nabla_\sigma t^\alpha) = (t^\beta\nabla_\beta n^\sigma)(\nabla_\sigma t^\alpha) + t^\beta n^\sigma\nabla_\beta\nabla_\sigma t^\alpha. \tag{8.8}$$

where we used first that the two vector fields satisfy $[\boldsymbol{t}, \boldsymbol{n}] = 0$ or $t^\alpha\nabla_\alpha s^\beta = s^\alpha\nabla_\alpha n^\beta$, and then the Leibiniz rule. Next we use the definition (8.5) of the Riemann tensor,

$$\frac{D^2 n^\alpha}{\mathrm{d}\tau} = (t^\beta\nabla_\beta n^\sigma)(\nabla_\sigma t^\alpha) + t^\beta n^\sigma(\nabla_\sigma\nabla_\beta t^\alpha + R^\alpha_{\rho\sigma\beta}t^\rho). \tag{8.9}$$

Then we apply in the middle term the Leibniz rule in reverse,

$$\frac{D^2 n^\alpha}{\mathrm{d}\tau} = (t^\beta\nabla_\beta n^\sigma)(\nabla_\sigma t^\alpha) + t^\beta\nabla_\sigma(n^\sigma\nabla_\beta t^\alpha) - (n^\sigma\nabla_\sigma t^\beta)\nabla_\beta t^\alpha + R^\alpha_{\rho\sigma\beta}t^\rho t^\sigma n^\beta. \tag{8.10}$$

Relabeling indices, we see that the first and third terms cancel, while the second term vanishes as t is a tangent vector. Thus the final result is

$$\frac{D^2 n^\alpha}{\mathrm{d}\tau} = R^\alpha_{\rho\sigma\beta}t^\rho t^\sigma n^\beta. \tag{8.11}$$

Specialising to a freely falling frame, we set $\boldsymbol{n} = n^a\boldsymbol{e}_a$ and $\boldsymbol{e}_0 = \boldsymbol{t}$. Thus the distance n^i of two freely falling particles changes as

$$\ddot{n}^i = R^i{}_{00j}n^j. \tag{8.12}$$

Example 8.1: Forces on a observer falling into a BH:
The freely-falling frame and the standard Schwarschild coordinates are connnected by a Lorentz transformation. For a boost η, it is

$$R'_{0101} = \Lambda^{\mu}{}_{0}\Lambda^{\nu}{}_{1}\Lambda^{\sigma}{}_{0}\Lambda^{\rho}{}_{1}R_{\mu\nu\sigma\rho} = (\underbrace{\cosh^4\eta}_{0101} - \underbrace{2\cosh^2\eta\sinh^2\eta}_{1001,0110} + \underbrace{\sinh^4\eta}_{1010})R_{0101} = R_{0101}.$$

Similarly, the other non-zero elements coincide in the two frames. Inserting the non-zero values of the Riemann tensor,

$$R_{0101} = -R_{2323} = \frac{2M}{r^3}, \qquad R_{0202} = R_{0303} = -R_{1212} = -R_{1313} = -\frac{M}{r^3} \tag{8.13}$$

into the equation of geodesic deviation (8.12), it follows

$$\ddot{n}^1 = \frac{2M}{r^3}n^1 \qquad \ddot{n}^2 = -\frac{M}{r^3}n^2 \qquad \ddot{n}^3 = -\frac{M}{r^3}n^3 \tag{8.14}$$

A volume element $\mathrm{d}m$ at the height h above the center-of-mass (in direction x^1 would be acclerated by $a = 2M/r^3h$ relative to the center-of-mass, if it could move freely. To prevent this, the force $\mathrm{d}F = a\mathrm{d}m$ has to counter-act on the mass element. The total force along the plane is

$$F = \int_0^{L/2} \mathrm{d}LL^2\,\frac{2M}{r^3}\frac{m}{L^3} = \frac{mMl}{4r^3}$$

with the volume element $\mathrm{d}LL^2$ and the density m/L^3. The resulting stresses $\sigma = -F/L^2$ follow then as

$$\sigma_{\parallel} = -\frac{mM}{4Lr^3}, \qquad \sigma_{\perp} = \frac{mM}{8Lr^3},$$

With $m = 80\,\mathrm{kg}$ and $L = 1\,\mathrm{m}$, the stresses on a human body are around

$$\sigma \sim 10^{15}\frac{\mathrm{dyn}}{\mathrm{cm}^2}\frac{M/M_{\odot}}{r/1\,\mathrm{km}}$$

(Compare with the normal pressure of Earth's atmosphere: $10^6\mathrm{dyn/cm}^2$.)

Symmetry properties The study of the symmetry properties of the Riemann tensor are simplified lowering its upper index and using normal coordinates. Inserting the definition of the Christoffel symbols, the Riemann tensor at the considered point P becomes with $\partial_\sigma g_{\alpha\beta} = 0$

$$R_{\alpha\beta\rho\sigma} = g_{\alpha\rho}R^{\rho}{}_{\beta\mu\nu} = g_{\alpha\rho}(\partial_\mu\Gamma^{\rho}{}_{\beta\nu} - \partial_\nu\Gamma^{\rho}{}_{\beta\mu}) \tag{8.15a}$$

$$= \frac{1}{2}(\partial_\sigma\partial_\beta g_{\alpha\rho} + \partial_\rho\partial_\alpha g_{\beta\sigma} - \partial_\sigma\partial_\alpha g_{\beta\rho} - \partial_\rho\partial_\beta g_{\alpha\sigma}). \tag{8.15b}$$

The tensor is antisymmetric in the indices $\rho \leftrightarrow \sigma$, antisymmetric in $\alpha \leftrightarrow \beta$ and symmetric against an exchange of the index pairs $(\alpha\beta) \leftrightarrow (\rho\sigma)$. Moreover, there exists one algebraic identity,

$$R_{\alpha\beta\rho\sigma} + R_{\alpha\sigma\beta\rho} + R_{\alpha\rho\sigma\beta} = 0, \tag{8.16}$$

which is again simplest chcked using normal coordinates. Since each pair of indices $(\alpha\beta)$ and $(\rho\sigma)$ can take six values, we can combine the antisymmetrized components of $R_{[\alpha\beta][\rho\sigma]}$ in a symmetric six-dimensional matrix. The number of independent components of this matrix is thus for $d = 4$ space-time dimensions

$$\frac{n \times (n+1)}{2} - 1 = \frac{6 \times 7}{2} - 1 = 20\,,$$

where we accounted also for the constraint (8.16). In general, the number n of independent components is in d space-time dimensions given by $n = d^2(d^2-1)/12$. The number m of field equations, $G_{\mu\nu} = \kappa T_{\mu\nu}$, is given by the number independent components of a symmetric tensor of rank in d space-time dimensions, $m = d(d+1)/2$. Thus we find

d	1	2	3	4
n	0	1	6	20
m	-	3	6	10

This implies that an one-dimensional manifold is always flat. In the appendix 6.B, we showed that a two-dimensional manifold is conformally flat. Moreover, the number of independent components of the Riemann tensor is smaller or equals the number of field equations for $d = 2$ and $d = 3$. Hence the Riemann tensor vanishes in empty space, if $d = 2, 3$. Starting from $d = 4$, already an empty space can be curved and gravitational waves exist.

The Bianchi identity is a differential constraint,

$$\nabla_\kappa R_{\alpha\beta\rho\sigma} + \nabla_\rho R_{\alpha\beta\sigma\kappa} + \nabla_\sigma R_{\alpha\beta\kappa\rho} = 0\,, \tag{8.17}$$

that is checked again simplest using normal coordinates. Then

$$\nabla_\kappa R_{\alpha\beta\rho\sigma} = \partial_\kappa R_{\alpha\beta\rho\sigma} = \frac{1}{2}\partial_\kappa\{\partial_\sigma\partial_\beta g_{\alpha\rho} + \partial_\rho\partial_\alpha g_{\beta\sigma} - \partial_\sigma\partial_\alpha g_{\beta\rho} - \partial_\rho\partial_\beta g_{\alpha\sigma}\}\,. \tag{8.18}$$

Adding then two cyclic permuatations of the first and the last two indices, all terms cancel and the Bianchi identity (8.17) follows. In the context of general relativity, this identities is a necessary consequence of the Einstein-Hilbert action which ensures the local conservation of the stress tensor.

The symmetry properties of the Riemann tensor imply that we can construct one non-zero tensor of rank two, contracting α either with the third or fourth index, $R^\rho{}_{\alpha\rho\beta} = -R^\rho{}_{\alpha\beta\rho}$. We define the Ricci tensor by

$$\boxed{R_{\alpha\beta} = R^\rho{}_{\alpha\rho\beta} = -R^\rho{}_{\alpha\beta\rho} = \partial_\rho\Gamma^\rho{}_{\alpha\beta} - \partial_\beta\Gamma^\rho{}_{\alpha\rho} + \Gamma^\rho{}_{\alpha\beta}\Gamma^\sigma{}_{\rho\sigma} - \Gamma^\sigma{}_{\beta\rho}\Gamma^\rho{}_{\alpha\sigma}.} \tag{8.19}$$

A further contraction gives the curvature scalar,

$$\boxed{R = R_{\alpha\beta}g^{\alpha\beta}.} \tag{8.20}$$

Example 8.2: Sphere S^2. Calculate the Ricci tensor R_{ij} and the scalar curvature R of the two-dimensional unit sphere S^2.
We have already determined the non-vanishing Christoffel symbols of the sphere S^2 as $\Gamma^\phi{}_{\vartheta\phi} = \Gamma^\phi{}_{\phi\vartheta} = \cot\vartheta$ and $\Gamma^\vartheta{}_{\phi\phi} = -\cos\vartheta\sin\vartheta$. We will show later that the Ricci tensor of a maximally symmetric space as a sphere satisfies $R_{ab} = Kg_{ab}$. Since the metric is diagonal, the non-diagonal elements of the Ricci tensor are zero too, $R_{\phi\vartheta} = R_{\vartheta\phi} = 0$. We calculate with

$$R_{ab} = R^c{}_{acb} = \partial_c\Gamma^c{}_{ab} - \partial_b\Gamma^c{}_{ac} + \Gamma^c{}_{ab}\Gamma^d{}_{cd} - \Gamma^d{}_{bc}\Gamma^c{}_{ad}$$

the $\vartheta\vartheta$ component, obtaining

$$R_{\vartheta\vartheta} = 0 - \partial_\vartheta(\Gamma^\phi{}_{\vartheta\phi} + \Gamma^\vartheta{}_{\vartheta\vartheta}) + 0 - \Gamma^d{}_{\vartheta c}\Gamma^c{}_{\vartheta d} = 0 - \partial_\vartheta \cot\vartheta - \Gamma^\phi{}_{\vartheta\phi}\Gamma^\phi{}_{\vartheta\phi}$$
$$= 0 - \partial_\vartheta \cot\vartheta - \cot^2\vartheta = 1\,.$$

From $R_{ab} = Kg_{ab}$, we find $R_{\vartheta\vartheta} = Kg_{\vartheta\vartheta}$ and thus $K = 1$. Hence $R_{\phi\phi} = g_{\phi\phi} = \sin^2\vartheta$. The scalar curvature is (diagonal metric with $g^{\phi\phi} = 1/\sin^2\vartheta$ and $g^{\vartheta\vartheta} = 1$)

$$R = g^{ab}R_{ab} = g^{\phi\phi}R_{\phi\phi} + g^{\vartheta\vartheta}R_{\vartheta\vartheta} = \frac{1}{\sin^2\vartheta}\,\sin^2\vartheta + 1\times 1 = 2\,.$$

Note that our definition of the Ricci tensor guaranties that the curvature of a sphere is also positive, if we consider it as subspace of a four-dimensional space-time.

8.2 Integration and differential operators

Transition to curved spacetime In special relativity, Lorentz transformations left the volume element d^4x invariant, $\mathrm{d}^4x' = \mathrm{d}t'\mathrm{d}^2x'_\perp\mathrm{d}x'_\parallel = (\gamma\mathrm{d}t)\mathrm{d}^2x_\perp(\mathrm{d}x_\parallel/\gamma) = \mathrm{d}x^4$. We allow now for arbitrary coordinate transformation for which the Jacobi determinant can deviate from one. Thus the action of a field with Lagrange density $\mathscr{L}$ becomes

$$S = \int_\Omega \mathrm{d}^4x\sqrt{|g|}\,\mathscr{L} = \int_\Omega \mathrm{d}^4x\,\mathscr{L}', \tag{8.21}$$

where g denotes the determinant of the metric tensor $g_{\mu\nu}$. Sometimes, as in the second step, we prefer to include the factor $\sqrt{|g|}$ into the definition of $\mathscr{L}$.

In the Lagrangian $\mathscr{L}_\mathrm{m}$ of the matter fields, the effects of gravity are accounted for by the replacements $\{\partial_\alpha, \eta_{\alpha\beta}\} \to \{\nabla_\alpha, g_{\alpha\beta}\}$. Note that this transition is not unique: First, there is an order ambiguity in the presence of second-order differential operators, reflecting the fact that in presence of gravity, covariant derivatives do not commute. This problem can be avoided, if one performs the transition on the level of the Lagrangian containing typically first-order derivatives. Second, in the presence of gravity new interaction terms are possible. For instance, in the case of a scalar field we can add a term $\xi R^2\phi^2$ to the usual Lagrangian. Since this term vanishes in Minkowski space, we have no way to determine the value of ξ from experiments in a flat spacetime.

Variation of the metric determinant g In order to find the equations of motion, we have to determine the variation of the metric determinant g. We consider a variation of a matrix M with elements $m_{ij}(x)$ under an infinitesimal change of the coordinates, $\delta x^a = \varepsilon x^a$,

$$\delta \ln \det M \equiv \ln\det(M + \delta M) - \ln\det(M) \tag{8.22a}$$
$$= \ln\det[M^{-1}(M + \delta M)] = \ln\det[1 + M^{-1}\delta M] = \tag{8.22b}$$
$$= \ln[1 + \mathrm{tr}(M^{-1}\delta M)] + \mathcal{O}(\varepsilon^2) = \mathrm{tr}(M^{-1}\delta M) + \mathcal{O}(\varepsilon^2). \tag{8.22c}$$

In the last step, we used $\ln(1+\varepsilon) = \varepsilon + \mathcal{O}(\varepsilon^2)$. Expressing now both the LHS and the RHS as $\delta f = \partial_\mu f \delta x^\mu$ and comparing then the coefficients of δx^μ gives

$$\partial_\mu \ln\det M = \mathrm{tr}(M^{-1}\partial_\mu M). \tag{8.23}$$

Useful formula for derivatives The replacement $\partial \to \nabla$ and the additional factor $\sqrt{|g|}$ will prevent in general the use of Gauss' law, needed to derive e.g. global conservation laws. The only exception is if we can rewrite $\nabla_\mu X^{\mu\cdots}$ as $1/\sqrt{|g|}\partial_\mu(\cdots)$.

Applying (8.23) to derivatives of g, we obtain

$$\frac{1}{2}g^{\mu\nu}\partial_\lambda g_{\mu\nu} = \frac{1}{2}\partial_\lambda \ln g = \frac{1}{\sqrt{|g|}}\partial_\lambda(\sqrt{|g|}). \tag{8.24}$$

while we find for contracted Christoffel symbols

$$\Gamma^\mu{}_{\mu\nu} = \frac{1}{2}g^{\mu\kappa}(\partial_\mu g_{\kappa\nu} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu}) = \frac{1}{2}g^{\mu\kappa}\partial_\nu g_{\mu\kappa} = \frac{1}{2}\partial_\nu \ln g = \frac{1}{\sqrt{|g|}}\partial_\nu(\sqrt{|g|}). \tag{8.25}$$

Next we consider the divergence of a vector field,

$$\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\mu{}_{\lambda\mu}V^\lambda = \partial_\mu V^\mu + \frac{1}{\sqrt{|g|}}(\partial_\mu\sqrt{|g|})V^\mu = \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}V^\mu). \tag{8.26}$$

and of an antisymmetric tensor of rank two,

$$\nabla_\mu A^{\mu\nu} = \partial_\mu A^{\mu\nu} + \Gamma^\mu{}_{\lambda\mu}A^{\lambda\nu} + \Gamma^\nu{}_{\lambda\mu}A^{\mu\lambda} = \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}A^{\mu\nu})\,. \tag{8.27}$$

In the latter case, the third term $\Gamma^\nu{}_{\lambda\mu}A^{\mu\lambda}$ vanishes because of the antisymmetry of $A^{\mu\lambda}$ so that we could combine the first two as in the vector case. This generalises to completely antisymmetric tensors of all orders. For a symmetric tensor of rank two, we find analogously

$$\nabla_\mu S^{\mu\nu} = \frac{1}{\sqrt{|g|}}\,\partial_\mu(\sqrt{|g|}S^{\mu\nu}) + \Gamma^\nu{}_{\lambda\mu}S^{\mu\lambda}. \tag{8.28}$$

Thus the covariant derivative of symmetric tensors of rank two and higher contains additional terms which prohibit the use of Gauss' theorem.

Example 8.3: Spherical coordinates 3:
Calculate for spherical coordinates $x = (r, \vartheta, \phi)$ in $\mathbb{R}^3$ the gradient, divergence, and the Laplace operator. Note that one uses normally normalized unit vectors in case of a diagonal metric: this corresponds to a rescaling of vector components $V^i \to V^i/\sqrt{g_{ii}}$ (no summation in i) or basis vectors. (Recall the analogue rescaling in the exercise "acceleration of a stationary observer in SW BH.)
We express the gradient of a scalar function f first as

$$\partial^i f\,\boldsymbol{e}_i = g^{ij}\,\frac{\partial f}{\partial x^j}\,\boldsymbol{e}_i = \frac{\partial f}{\partial r}\,\boldsymbol{e}_r + \frac{1}{r^2}\,\frac{\partial f}{\partial\vartheta}\,\boldsymbol{e}_\vartheta + \frac{1}{r^2\sin^2\vartheta}\,\frac{\partial f}{\partial\phi}\,\boldsymbol{e}_\phi$$

and rescale then the basis, $\boldsymbol{e}_i^* = \boldsymbol{e}_i/\sqrt{g_{ii}}$, or $\boldsymbol{e}_r^* = \boldsymbol{e}_r$, $\boldsymbol{e}_\vartheta^* = r\boldsymbol{e}_\vartheta$, and $\boldsymbol{e}_\phi^* = r\sin\vartheta\boldsymbol{e}_\phi$. In this new ("physical") basis, the gradient is given by

$$\partial^i f\,\boldsymbol{e}_i^* = \frac{\partial f}{\partial r}\,\boldsymbol{e}_r^* + \frac{1}{r}\,\frac{\partial f}{\partial\vartheta}\,\boldsymbol{e}_\vartheta^* + \frac{1}{r\sin\vartheta}\,\frac{\partial f}{\partial\phi}\,\boldsymbol{e}_\phi^*\,.$$

The covariant divergence of a vector field with rescaled components $X^i/\sqrt{g_{ii}}$ is with $\sqrt{g} = r^2\sin\vartheta$

given by

$$\begin{aligned}\nabla_i X^i &= \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}X^i) = \frac{1}{r^2\sin\vartheta}\left(\frac{\partial(r^2\sin\vartheta X_r)}{\partial r} + \frac{\partial(r^2\sin\vartheta X_\vartheta)}{r\partial\vartheta} + \frac{\partial(r^2\sin\vartheta X_\phi)}{r\sin\vartheta\partial\phi}\right)\\ &= \frac{1}{r^2}\frac{\partial(r^2X_r)}{\partial r} + \frac{1}{r\sin\vartheta}\frac{\partial(\sin\vartheta X_\vartheta)}{\partial\vartheta} + \frac{1}{r\sin\vartheta}\frac{\partial X_\phi}{\partial\phi}\\ &= \left(\frac{\partial}{\partial r} + \frac{2}{r}\right)X_r + \left(\frac{\partial}{\partial\vartheta} + \frac{\cot\vartheta}{r}\right)X_\phi + \frac{1}{r\sin\vartheta}\frac{\partial X_\phi}{\partial\phi}\,.\end{aligned}$$

Global conservation laws An immediate consequence of Eq. (8.26) is a covariant form of Gauß' theorem for vector fields. In particular, we can conclude from local current conservation, $\nabla_\mu j^\mu = 0$, the existence of a *globally* conserved charge. If the conserved current j^μ vanishes at infinity, then we obtain also in a general space-time

$$\int_\Omega \mathrm{d}^4x\sqrt{|g|}\,\nabla_\mu j^\mu = \int_\Omega \mathrm{d}^4x\partial_\mu(\sqrt{|g|}j^\mu) = \int_{\partial\Omega}\mathrm{d}S_\mu\sqrt{|g|}\,j^\mu = 0\,. \tag{8.29}$$

For a current which vanishes as spatial infinity but is otherwise non-zero, the volume integral over the charge density j^0 remains constant,

$$\int_\Omega \mathrm{d}^4x\sqrt{|g|}\,\nabla_\mu j^\mu = \int_{V(t_2)}\mathrm{d}^3x\sqrt{|g|}j^0 - \int_{V(t_1)}\mathrm{d}^3x\sqrt{|g|}j^0 = 0\,. \tag{8.30}$$

Thus the conservation of Noether charges of internal symmetries as the electric charge, baryon number, etc., is not affected by an expanding universe.

Next we consider the stress tensor as example for a locally conserved symmetric tensor of rank two. Now, the second term in Eq. (8.28) prevents us to convert the local conservation law into a global one. If the space-time admits however a Killing field $\boldsymbol{\xi}$, then we can form the vector field $P^\mu = T^{\mu\nu}\xi_\nu$ with

$$\nabla_\mu P^\mu = \nabla_\mu(T^{\mu\nu}\xi_\nu) = \xi_\nu\nabla_\mu T^{\mu\nu} + T^{\mu\nu}\nabla_\mu\xi_\nu = 0. \tag{8.31}$$

Here, the first term vanishes since $T^{\mu\nu}$ is conserved and the second because $T^{\mu\nu}$ is symmetric, while $\nabla_\mu\xi_\nu$ is antisymmetric. Therefore the vector field $P^\mu = T^{\mu\nu}\xi_\nu$ is also conserved, $\nabla_\mu P^\mu = 0$, and we obtain thus the conservation of the component of the energy-momentum vector in the direction of $\boldsymbol{\xi}$.

In summary, global energy conservation requires the existence of a time-like Killing vector field. Moving along such a Killing field, the metric would be invariant. Since we expect in an expanding universe a time-dependence of the metric, a time-like Killing vector field does not exist and the energy contained in a "comoving" volume changes with time.

8.3 Einstein-Hilbert action

Einstein equation in vacuum Our main guide in choosing the appropriate action for the gravitational field is simplicity. A Lagrange density has mass dimension four (or length -4) such that the action is dimensionless. In the case of gravity, we have to account for the dimensionfull coupling, Newton's constant G, and require therefore that the Lagrange density without coupling has mass dimension two. Among the possible terms we can select are

$$\begin{aligned}\mathscr{L} = \sqrt{|g|}\{\Lambda + bR + c\nabla_a\nabla_b R^{ab} + d(\nabla_a\nabla_b R^{ab})^2 + \dots \\ + f(R) + \dots\}\end{aligned} \tag{8.32}$$

Note that the terms in the first line are ordered according to the number of derivatives: $\Lambda : \partial^0$, $b : \partial^2$, $c : \partial^4$. Choosing only the first term, a constant, will not give dynamical equations. The next simplest possibility is to pick out only the second term, as it was done originally by Hilbert. The following c term will be suppressed relative to b by dimensional reasons as $\nabla_a\nabla_b/M^2 \sim E^2/M^2$. Here, E is the characteristic energy of the process considered, while we expect $1/M^2 \sim G_N$ for a theory of gravity. Alternatively, we can think of the terms in the first line as an gradient expansion. Since spacetime is stiff, the movement of most sources will lead only to small variations in the metric. Thus at low energies, the first two terms should dominate the gravitational interactions. In contrast, a term like $f(R)$ in the second line is a modification of the simple R term—the allowed size of this modification has to be constrained by experiments. We will see later that, if we do no include a constant term Λ in the gravitational action, it will pop up on the matter side. Thus we add Λ right from the start and define as the Einstein-Hilbert Lagrange density for the gravitational field

$$\boxed{\mathscr{L}_{\rm EH} = -\sqrt{|g|}(R + 2\Lambda)\,.} \tag{8.33}$$

The Lagrangian is a function of the metric, its first and second derivatives,[1] $\mathscr{L}_{\rm EH}(g_{\mu\nu}, \partial_\rho g_{\mu\nu}, \partial_\rho\partial_\sigma g_{\mu\nu})$. The resulting action

$$S_{\rm EH}[g_{\mu\nu}] = -\int_\Omega \mathrm{d}^4x\sqrt{|g|}\,\{R + 2\Lambda\} \tag{8.34}$$

is a functional of the metric tensor $g_{\mu\nu}$, and a variation of the action with respect to the metric gives the field equations for the gravitational field. We allow for variations of the metric $g_{\mu\nu}$ restricted by the condition that the variation of $g_{\mu\nu}$ and its first derivatives vanish on the boundary $\partial\Omega$. Asking that variation is zero, we obtain

$$0 = \delta S_{\rm EH} = -\delta\int_\Omega \mathrm{d}^4x\sqrt{|g|}(R + 2\Lambda) = \tag{8.35a}$$

$$= -\delta\int_\Omega \mathrm{d}^4x\sqrt{|g|}\,(g^{\mu\nu}R_{\mu\nu} + 2\Lambda) \tag{8.35b}$$

$$= -\int_\Omega \mathrm{d}^4x\left\{\sqrt{|g|}\,g^{\mu\nu}\delta R_{\mu\nu} + \sqrt{|g|}R_{\mu\nu}\delta g^{\mu\nu} + (R + 2\Lambda)\,\delta\sqrt{|g|}\right\}. \tag{8.35c}$$

Our task is to rewrite the first and third term as variations of $\delta g^{\mu\nu}$ or to show that they are equivalent to boundary terms. Let us start with the first term. Choosing inertial coordinates, the Ricci tensor at the considered point P becomes

$$R_{\mu\nu} = \partial_\rho\Gamma^\rho{}_{\mu\nu} - \partial_\nu\Gamma^\rho{}_{\mu\rho}. \tag{8.36}$$

Hence

$$g^{\mu\nu}\delta R_{\mu\nu} = g^{\mu\nu}(\partial_\rho\delta\Gamma^\rho{}_{\mu\nu} - \partial_\nu\delta\Gamma^\rho{}_{\mu\rho}) = g^{\mu\nu}\partial_\rho\delta\Gamma^\rho{}_{\mu\nu} - g^{\mu\rho}\partial_\rho\delta\Gamma^\nu{}_{\mu\nu}, \tag{8.37}$$

[1]Recall that the Lagrange equations are modified in the case of higher derivatives which is one reason why we directly vary the action in order to obtain the field equations.

where we exchanged the indices ν and ρ in the last term. Since $\partial_\rho g_{\mu\nu} = 0$ at P, we can rewrite the expression as

$$g^{\mu\nu}\delta R_{\mu\nu} = \partial_\rho(g^{\mu\nu}\delta\Gamma^\rho{}_{\mu\nu} - g^{\mu\rho}\delta\Gamma^\nu{}_{\mu\nu}) = \partial_\rho V^\rho. \tag{8.38}$$

The quantity V^ρ is a vector, since the difference of two connection coefficients transforms as a tensor, cf. with Eq. (torsion). Replacing in Eq. (8.38) the partial derivative by a covariant one promotes it therefore into a valid tensor equation,

$$g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\mu V^\mu = \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}V^\mu). \tag{8.39}$$

Thus this term corresponds to a surface term which we assume to vanish. Next we rewrite the third term using

$$\delta\sqrt{|g|} = \frac{1}{2\sqrt{|g|}}\,\delta|g| = \frac{1}{2}\sqrt{|g|}\,g^{\mu\nu}\delta g_{\mu\nu} = -\frac{1}{2}\sqrt{|g|}\,g_{\mu\nu}\delta g^{\mu\nu} \tag{8.40}$$

and obtain

$$\delta S_{\mathrm{EH}} = -\int_\Omega \mathrm{d}^4x\sqrt{|g|}\left\{R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}R - \Lambda\,g_{\mu\nu}\right\}\delta g^{\mu\nu} = 0. \tag{8.41}$$

Hence the metric fulfils in vacuum the equation

$$\boxed{-\frac{1}{\sqrt{|g|}}\frac{\delta S_{\mathrm{EH}}}{\delta g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2}\,R\,g_{\mu\nu} - \Lambda g_{\mu\nu} \equiv G_{\mu\nu} - \Lambda g_{\mu\nu} = 0,} \tag{8.42}$$

where we introduced the Einstein tensor $G_{\mu\nu}$. The constant Λ is called the cosmological constant. It has the demension of a length squared: If the cosmological constant is non-zero, empty space is curved with a curvature radius $\Lambda^{-1/2}$.

Einstein equation with matter We consider now the combined action of gravity and matter, as the sum of the Einstein-Hilbert Lagrange density $\mathscr{L}_{\mathrm{EH}}/2\kappa$ and the Lagrange density $\mathscr{L}_{\mathrm{m}}$ including all relevant matter fields,

$$\mathscr{L} = \frac{1}{2\kappa}\mathscr{L}_{\mathrm{EH}} + \mathscr{L}_{\mathrm{m}} = -\frac{1}{2\kappa}\sqrt{|g|}(R+2\Lambda) + \mathscr{L}_{\mathrm{m}}\,. \tag{8.43}$$

In $\mathscr{L}_{\mathrm{m}}$, the effects of gravity are accounted for by the replacements $\{\partial_\mu, \eta_{\mu\nu}\} \to \{\nabla_\mu, g_{\mu\nu}\}$, while we have to adjust later the constant κ such that we reproduce Newtonian dynamics in the weak-field limit. We expect that the source of the gravitational field is the energy-momentum stress tensor. More precisely, the Einstein tensor ("geometry") should be determined by the matter, $G_{\mu\nu} = \kappa T_{\mu\nu}$. Since we know already the result of the variation of S_{EH}, we conclude that the variation of S_{m} should give

$$\boxed{\frac{2}{\sqrt{|g|}}\frac{\delta S_{\mathrm{m}}}{\delta g^{\mu\nu}} = T_{\mu\nu}\,.} \tag{8.44}$$

The tensor $T_{\mu\nu}$ defined by this equation is called *dynamical energy-momentum stress stress tensor*. In order to show that this definition makes sense, we have to prove that $\nabla_\mu T^{\mu\nu} = 0$ and we have to convince ourselves that this definition reproduces the standard results we know already. We note however already that the stress tensor defined by Eq. (8.44) is by contruction symmetric. Moreover, the stresss tensor is automatically gauge invariant, if the matter Lagrangian is gauge invariant. Thus the dynamical stress tensor avoids the problems of the canonical stress tensor.

Einstein's field equation follows then as

$$\boxed{G_{\mu\nu} - \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}\,.} \tag{8.45}$$

Alternative form of the Einstein equation We can rewrite the Einstein equation such that the only geometrical term on the LHS is the Ricci tensor. Because of

$$R_\mu^{\ \mu} - \frac{1}{2} g_\mu^{\ \mu}(R + 2\Lambda) = R - 2(R + 2\Lambda) = -R - 4\Lambda = \kappa T_\mu^{\ \mu} \tag{8.46}$$

we can perform with $T \equiv T_\mu^\mu$ the replacement $R = -4\Lambda - \kappa T$ in the Einstein equation and obtain

$$R_{\mu\nu} = \kappa(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T) - g_{\mu\nu}\Lambda\,. \tag{8.47}$$

This form of the Einstein equations is often useful, when it is easier to calculate T than R. Note also that Eq. (9.69) informs us that an empty universe with $\Lambda = 0$ has a vanishing Ricci tensor, $R_{\mu\nu} = 0$. A spacetime with $R_{\mu\nu} = 0$ is callled Ricci flat.

Remark 8.2: Einstein guessed the right form of the field equations realising that $\nabla_\mu T^{\mu\nu} = 0$ implies $\nabla_\mu G^{\mu\nu} = 0$. Applying the tensor method to evaluate this constraint, we use the ansatz

$$G_{\mu\nu} = aR_{\mu\nu} + bRg_{\mu\nu}. \tag{8.48}$$

Metric compatbility ensures $\nabla_\mu g^{\mu\nu} = 0$; thus we can always add a term $\Lambda g_{\mu\nu}$ with an unconstrained coefficient Λ. To determine the ratio a/b, we consider the Bianchi identity,

$$\nabla_\kappa R_{\alpha\beta\rho\sigma} + \nabla_\rho R_{\alpha\beta\sigma\kappa} + \nabla_\sigma R_{\alpha\beta\kappa\rho} = 0\,, \tag{8.49}$$

Raising the index α and contracting with σ gives

$$\nabla_\kappa R^\alpha{}_{\beta\rho\alpha} + \nabla_\rho R^\alpha{}_{\beta\alpha\kappa} + \nabla_\alpha R^\alpha{}_{\beta\kappa\rho} = -\nabla_\kappa R_{\beta\rho} + \nabla_\rho R_{\beta\kappa} + \nabla_\alpha R^\alpha{}_{\beta\kappa\rho} = 0. \tag{8.50}$$

Next we raise the index β and contract with κ,

$$-\nabla_\beta R^\beta{}_\rho + \nabla_\rho R + \nabla_\alpha R^{\alpha\beta}{}_{\beta\rho} = -\nabla_\beta R^\beta{}_\rho + \nabla_\rho R - \nabla_\alpha R^\alpha{}_\rho = 0, \tag{8.51}$$

what gives finally $\nabla_\mu(R^{\mu\nu} - \frac{1}{2} R\, g^{\mu\nu}) = 0$.

Structure of the equations The Einstein equation is a set of ten second-order partial differential equations for the ten independent components of metric tensor. The equation is non-linear, even setting $T_{\mu\nu} = 0$ and considering only the gravitational sector. Thus we cannot superimpose solutions for, e.g., a point mass to obtain generic solutions. Moreover, attempts to derive analytical solutions are complicated by the fact that usually also the stress tensor is a non-linear function of the metric tensor.

Let us now look at the Einstein equation as an intial-value problem. At first sight, it looks like we have to prescribe the values of the metric $g_{\alpha\beta}$ and its first time-derivative $\partial_t g_{\alpha\beta}$ on a space-like hypersurface at a given time as initial values. However, the equation $\nabla_\alpha G^{\alpha\beta} = 0$ or $\nabla_\alpha T^{\alpha\beta} = 0$ imply that at least four of the Einstein equations are constraints. As a result, the time derivatives of g_{00} and g_{0i} can be eleminated from the Einstein equations. Therefore the true dynamical variables are the six space-space like components g_{ij} of the metric tensor.

Remark 8.3: Weyl tensor—The Riemann tensor has 20 independent elements, while the local matter distribution $T_{\mu\nu}$ fixes only the 10 independent elements of the Ricci tensor. Thus we should be able to decompose the Riemann tensor into a tensor determined only by the Ricci tensor, and a remainder. The latter, called the Weyl tensor, contains the non-local curvature effects of matter as well as gravitational waves.
An example for such non-local curvature effects is the Schwarzschild (exteriror) metric: For $r > R$, the space is empty, $T_{\mu\nu} = 0$, and thus Ricci flat, $R_{\mu\nu} = 0$. However, the spacetime is curved and the Riemann tensor is non-zero, with all curvature effects contained in the Weyl tensor.

8.4 Dynamical stress tensor

We start by proving that the dynamical stress tensor defined by by Eq. (8.44) is conserved. We consider the change of the matter action under variations of the metric,

$$\delta S_\mathrm{m} = \frac{1}{2}\int_\Omega \mathrm{d}^4x\sqrt{|g|}\,T_{\alpha\beta}\delta g^{\alpha\beta} = -\frac{1}{2}\int_\Omega \mathrm{d}^4x\sqrt{|g|}\,T^{\alpha\beta}\delta g_{\alpha\beta}\,. \tag{8.52}$$

We allow infinitesimal but otherwise arbitrary coordinate transformations,

$$\tilde{x}^\alpha = x^\alpha + \xi^\alpha(x^\beta)\,. \tag{8.53}$$

For the resulting change in the metric $\delta g_{\alpha\beta}$ we can use the Killing Eq. (4.11),

$$\delta g_{\alpha\beta} = \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha\,. \tag{8.54}$$

We use that $T^{\alpha\beta}$ is symmetric and that general covariance guarantees that $\delta S_m = 0$ for a coordinate transformation,

$$\delta S_\mathrm{m} = -\int_\Omega \mathrm{d}^4x\sqrt{|g|}\,T^{\alpha\beta}\nabla_\alpha\xi_\beta = 0\,. \tag{8.55}$$

Next we apply the product rule,

$$\delta S_\mathrm{m} = -\int_\Omega \mathrm{d}^4x\sqrt{|g|}\,(\nabla_\alpha T^{\alpha\beta})\xi_\beta + \int_\Omega \mathrm{d}^4x\sqrt{|g|}\,\nabla_\alpha(T^{\alpha\beta}\xi_\beta) = 0\,. \tag{8.56}$$

The second term is a four-divergence and thus a boundary term that we can neglect. The remaining first term vanishes for arbitrary $\boldsymbol{\xi}$ only, if the stress tensor is conserved,

$$\boxed{\nabla_\alpha T^{\alpha\beta} = 0\,.} \tag{8.57}$$

Hence the *local* conservation of energy-momentum is a consequence of the general covariance of the gravitational field equations, in the same way as current conservation follows from gauge invariance in electrodynamics.

We now evaluate the dynamical stress tensor for the examples of the Klein-Gordon and the photon field. Note that the replacements $\eta_{\alpha\beta} \to g_{\alpha\beta}$ requires also that we have to express summation indices as contractions with the metric tensor, i.e. we have to replace e.g. $A_\alpha B^\alpha$ by $g^{\alpha\beta} A_\alpha B_\beta$. Thus we rewrite Eq. (7.53) including a potential $V(\phi)$, that could be also a mass term, $V(\phi) = m^2\phi^2/2$, as

$$\mathscr{L} = \frac{1}{2} g^{\alpha\beta} \nabla_\alpha \phi \nabla_\beta \phi - V(\phi) \,. \tag{8.58}$$

With $\nabla_\alpha \phi = \partial_\alpha \phi$ for a scalar field, the variation of the action gives

$$\delta S_{\mathrm{KG}} = \frac{1}{2} \int_\Omega \mathrm{d}^4 x \left\{ \sqrt{|g|} \nabla_\alpha \phi \nabla_\beta \phi \, \delta g^{\alpha\beta} + [g^{\alpha\beta} \nabla_\alpha \phi \nabla_\beta \phi - 2V(\phi)] \delta \sqrt{|g|} \right\} \tag{8.59a}$$

$$= \int_\Omega \mathrm{d}^4 x \sqrt{|g|} \delta g^{\alpha\beta} \left\{ \frac{1}{2} \nabla_\alpha \phi \nabla_\beta \phi - \frac{1}{2} g_{\alpha\beta} \mathscr{L} \right\}, \tag{8.59b}$$

and thus

$$T_{\alpha\beta} = \frac{2}{\sqrt{|g|}} \frac{\delta S_{\mathrm{m}}}{\delta g^{\alpha\beta}} = \nabla_\alpha \phi \nabla_\beta \phi - g_{\alpha\beta} \mathscr{L} \,. \tag{8.60}$$

Next we consider the free electromagnetic action,

$$S_{\mathrm{em}} = -\frac{1}{4} \int_\Omega \mathrm{d}^4 x \sqrt{|g|} F_{\alpha\beta} F^{\alpha\beta} = -\frac{1}{4} \int_\Omega \mathrm{d}^4 x \sqrt{|g|} g^{\alpha\rho} g^{\beta\sigma} F_{\alpha\beta} F_{\rho\sigma} \,. \tag{8.61}$$

Noting that $F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha = \partial_\alpha A_\beta - \partial_\beta A_\alpha$, we obtain

$$\delta S_{\mathrm{em}} = -\frac{1}{4} \int_\Omega \mathrm{d}^4 x \left\{ (\delta \sqrt{|g|}) F_{\rho\sigma} F^{\rho\sigma} + \sqrt{|g|} \delta (g^{\alpha\rho} g^{\beta\sigma}) F_{\alpha\beta} F_{\rho\sigma} \right\} \tag{8.62a}$$

$$= -\frac{1}{4} \int_\Omega \mathrm{d}^4 x \sqrt{|g|} \delta g^{\alpha\beta} \left\{ -\frac{1}{2} g_{\alpha\beta} F_{\rho\sigma} F^{\rho\sigma} + 2 g^{\rho\sigma} F_{\alpha\rho} F_{\beta\sigma} \right\} . \tag{8.62b}$$

Hence the dynamical stress tensor is

$$T_{\alpha\beta} = -F_{\alpha\rho} F_\beta{}^\rho + \frac{1}{4} g_{\alpha\beta} F_{\rho\sigma} F^{\rho\sigma} \,. \tag{8.63}$$

Thus we reproduced in both cases the (symmetrised) canonical stress tensor.

Cosmological constant To understand better the meaning of the constant Λ, we ask now if one of the known stress tensors could mimick the term $g_{\alpha\beta}\Lambda$. First we consider a scalar field. The constancy of Λ requires clearly $\nabla_\alpha \phi = 0$ and thus

$$T_{\alpha\beta} = g_{\alpha\beta} V_0(\phi) \,, \tag{8.64}$$

where V_0 is the minimum of the potential $V(\phi)$. Hence a constant scalar field sitting at a non-zero minimum of its potential acts as a cosmological constant.

Next we consider a perfect fluid which E.o.S. is fixed by the two parameters density ρ and pressure P. We know already that $T^{\alpha\beta} = \mathrm{diag}\{\rho, P, P, P\}$ is valid for a perfect fluid in its rest frame. Hence a fluid with $P = -\rho$, i.e. marginally fulfilling the strong energy condition, has the same property as a cosmological constant.

Is it possible to distinguish a term like $T_{ab} = g_{ab}V_0(\phi)$ in S_m from a non-zero Λ in S_{EH}? In principle yes, since a cosmological constant fulfils $P = -\rho$ exactly and independently of all external parameters like temperature or density. The latter change with time in the universe and therefore there may be detectable differences to a fluid with $P = P(\rho, T, \ldots)$ and a scalar field with potential $V = V(\rho, T, \ldots)$, even if they mimick today very well a cosmological constant with $P = -\rho$.

Remark 8.4: Energy conditions: We can now explain what the assumption "normal matter" in Penrose's singularity theorem means. Depending on the eigenvalues of the stress tensor, one says that the stress tensor satisfies the

- weak energy condition, if $\rho \geq 0$, and $\rho + P_i > 0$, $i = \{1, 2, 3\}$;
- strong energy condition, if $\rho + \sum_i P_i \geq 0$ for $i = \{1, 2, 3\}$, and $\rho + P_i > 0$;
- dominant energy condition, if $\rho \geq 0$, and $-\rho < P_i < \rho$.

The dominant energy condition gurantees that the local speed of sound in the fluid is smaller than the speed of light. Causality implies thus that all "normal" forms of matter satisfy this condition. This condition is sufficient for the valifity of the Penrose's singularity theorems; for a proof see Wald's textbook.
On the other hand, a stress tensor dominated by a positive cosmological constant does not satisfy these assumptions. (This is not in contradiction to causality, since a cosmological constant does not support sound waves.) From a physical point of view, the repulsion induced by the negative pressure can counteract the focusing of geodesics by gravity and thus avoid the formation of singularities.

Equations of motion We show now that the Einstein equation implies that particles move along geodesics. By analogy with a pressureless fluid, $T^{\alpha\beta} = \rho u^\alpha u^\beta$, we use as ansatz[2]

$$T^{\alpha\beta}(\tilde{x}) = \frac{m}{\sqrt{|g|}} \int d\tau \, \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau} \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} \delta^{(4)}(\tilde{x} - x(\tau)) \tag{8.65}$$

for a point-particle moving along the world-line $x(\tau)$ with proper time τ. Inserting this expression into

$$\nabla_\alpha T^{\alpha\beta} = \partial_\alpha T^{\alpha\beta} + \Gamma^\alpha{}_{\sigma\alpha} T^{\sigma\beta} + \Gamma^\beta{}_{\sigma\alpha} T^{\alpha\sigma} = \frac{1}{\sqrt{|g|}} \partial_\alpha (\sqrt{|g|} T^{\alpha\beta}) + \Gamma^\beta{}_{\sigma\alpha} T^{\alpha\sigma} = 0 \tag{8.66}$$

gives

$$\int \mathrm{d}\tau \, \dot{x}^\alpha \dot{x}^\beta \frac{\partial}{\partial \tilde{x}^\alpha} \delta^{(4)}(\tilde{x} - x(\tau)) + \Gamma^\beta{}_{\sigma\alpha} \int \mathrm{d}\tau \, \dot{x}^\alpha \dot{x}^\sigma \delta^{(4)}(\bar{x} - x(\tau)) = 0 \,. \tag{8.67}$$

We can replace $\partial/\partial\tilde{x}^\alpha = -\partial/\partial x^\alpha$ acting on $\delta^{(4)}(\tilde{x} - x(\tau))$ and use moreover

$$\dot{x}^\alpha \frac{\partial}{\partial x^\alpha} \delta^{(4)}(\tilde{x} - x(\tau)) = \frac{\mathrm{d}}{\mathrm{d}\tau} \delta^{(4)}(\tilde{x} - x(\tau)) \tag{8.68}$$

to obtain

$$-\int \mathrm{d}\tau \, \dot{x}^\beta \frac{\mathrm{d}}{\mathrm{d}\tau} \delta^{(4)}(\tilde{x} - x(\tau)) + \Gamma^\beta{}_{\sigma\alpha} \int \mathrm{d}\tau \, \dot{x}^\alpha \dot{x}^\sigma \delta^{(4)}(\tilde{x} - x(\tau)) = 0 \,. \tag{8.69}$$

[2]Note the delta function is accompanied by a factor $1/\sqrt{|g|}$ such that $\int \mathrm{d}^4x \sqrt{|g|} \, f(x)\delta(x - x_0)/\sqrt{|g|} = f(x_0)$.

Integrating the first term by parts we end up with

$$\int \mathrm{d}\tau \, \left(\ddot{x}^{\beta} + \Gamma^{\beta}{}_{\sigma\alpha}\dot{x}^{\alpha}\dot{x}^{\sigma}\right) \delta^{(4)}(\tilde{x} - x(\tau)) = 0 \,. \tag{8.70}$$

The integral vanishes only, when the word-line $x^{\alpha}(\tau)$ is a geodesics. Hence the Einstein equation implies the equation of motion of a point particle, in contrast to Maxwell's theory, where the Lorentz force law has to be postulated separately.

8.5 Alternative theories

The Einstein-Hilbert action (8.33) is most likely only the low-energy limit of either the "true" action of gravity or of an unified theory of all interactions. It is therefore interesting to examine modifications of the Einstein-Hilbert action and to compare their predictions to observations.

Tensor-scalar theories The (linear) field equation for a purely scalar theory of gravity would be

$$\Box\phi = -4\pi G T^{\mu}_{\mu} \,. \tag{8.71}$$

It predicts no coupling between photons and gravitation, since $T^{\mu}_{\mu} = 0$ for the electromagnetic field. A purely vector theory for gravity fails too, since it predicts not attraction but repulsion for two masses.

However, it may well be that gravity is a mixture of scalar, vector and tensor exchange, dominated by the later. An important example for a tensor-scalar theory is the Brans-Dicke theory. Here one use $g_{\mu\nu}$ to describe gravitational interactions but assumes that the strength, G, is determined by a scalar field ϕ,

$$S = \int \mathrm{d}^4x\sqrt{|g|}\left\{-\frac{1}{2}\phi^2 R + \alpha(\partial_{\mu}\phi)^2 + \mathscr{L}_{\mathrm{m}}(g_{\mu\nu}, \psi)\right\}, \tag{8.72}$$

where ψ represents all matter fields. Rescaling the metric by

$$\tilde{g}_{\mu\nu} = g_{\mu\nu}\frac{\kappa}{\phi^2}$$

we are back to Einstein gravity, but now ϕ couples universally to all matter fields ψ.

$f(R)$ **gravity** Another important class of modified gravity models are the so-called $f(R)$ gravity models, which generalise the Einstein–Hilbert action replacing R by a general function $f(R)$. Thus the action of $f(R)$ gravity coupled to matter has the form

$$S = \int \mathrm{d}^4x\sqrt{|g|}\left\{-\frac{1}{2\tilde{\kappa}}f(R) + \mathscr{L}_{\mathrm{m}}\right\}. \tag{8.73}$$

Note that for $f(R) \neq R$, the gravitational constant $\tilde{\kappa} = 8\pi\tilde{G}$ deviates from Newton's constant G measured in a Cavendish experiment.

Extra dimensions and Kaluza-Klein theories String theory suggests that we live in a world with $d = 10$ spacetime dimensions. There are two obvious answers to this result: first, one may conclude that string theory is disproven by nature or, second, one may adjust reality. Consistency of the second approach with experimental data could be achieved, if the $d-4$ dimensions are compactified with a sufficiently small radius R, such that they are not visible in experiments sensible to wavelengths $\lambda \gg R$.

Let us check what happens to a scalar particle with mass m, if we add a fifth compact dimension y. The Klein–Gordon equation for a scalar field $\phi(x^\mu, y)$ becomes

$$(\Box_5 + m^2)\phi(x^\mu, y) = 0 \tag{8.74}$$

with the five-dimensional d'Alembert operator $\Box_5 = \Box - \partial_y^2$. The equation can be separated, $\phi(x^\mu, y) = \phi(x^\mu) f(y)$, and since the fifth dimension is compact, the spectrum of f is discrete. Assuming periodic boundary conditions, $f(x) = f(x+R)$, gives

$$\phi(x^\mu, y) = \phi(x^\mu)\cos(n\pi y/R). \tag{8.75}$$

The energy eigenvalues of these solutions are $\omega^2_{\boldsymbol{k},n} = \boldsymbol{k}^2 + m^2 + (n\pi/R)^2$. From a four-dimensional point of view, the term $(n\pi/R)^2$ appears as a mass term, $m_n^2 = m^2 + (n\pi/R)^2$. Since we usually consider states with different masses as different particles, we see the five-dimensional particle as a tower of particles with mass m_n but otherwise identical quantum numbers. Such theories are called Kaluza–Klein theories, and the tower of particles Kaluza–Klein particles. If $R \ll \lambda$, where λ is the length-scale experimentally probed, only the $n = 0$ particle is visible and physics appears to be four-dimensional.

Since string theory includes gravity, one often assumes that the radius R of the extra-dimensions is determined by the Planck length, $R = 1/M_{\rm Pl} = (8\pi G_N)^{1/2} \sim 10^{-34}$ cm. In this case it is difficult to imagine any observational consequences of the additional dimensions. Of greater interest is the possibility that some of the extra dimensions are large,

$$R_{1,\ldots,\delta} \gg R_{\delta+1,\ldots,6} = 1/M_{\rm Pl}.$$

Since the $1/r^2$ behaviour of the gravitational force is not tested below $d_* \sim$ mm scales, one can imagine that large extra dimensions exists that are only visible to gravity: Relating the $d = 4$ and $d > 4$ Newton's law $F \sim \frac{m_1 m_2}{r^{2+\delta}}$ at the intermediate scale $r = R$, we can derive the "true" value of the Planck scale in this model: Matching of Newton's law in 4 and $4+\delta$ dimensions at $r = R$ gives

$$F(r=R) = G_N \frac{m_1 m_2}{R^2} = \frac{1}{M_D^{2+\delta}} \frac{m_1 m_2}{R^{2+\delta}}. \tag{8.76}$$

This equation relates the size R of the large extra dimensions to the true fundamental scale M_D of gravity in this model,

$$G_N^{-1} = 8\pi M_{\rm Pl}^2 = R^\delta M_D^{\delta+2}, \tag{8.77}$$

while Newton's constant G_N becomes just an auxiliary quantity useful to describe physics at $r \gtrsim R$. (You may compare this to the case of weak interactions where Fermi's constant $G_F \propto g^2/m_W^2$ is determined by the weak coupling constant g and the mass m_W of the W-boson.) Thus in such a set-up, gravity is much weaker than the weak interaction because the gravitational field is diluted into a large volume.

Next we ask, if $M_D \sim$ TeV is possible, what would allow one to test such theories at accelerators as LHC. Inserting the measured value of G_N and $M_D = 1\,\mathrm{TeV}$ in Eq. (8.77) we find the required value for the size R of the large extra dimension as $10^{13}\,\mathrm{cm}$ and $0.1\,\mathrm{cm}$ for $\delta = 1$ and 2, respectively. Thus the case $\delta = 1$ is excluded by the agreement of the dynamics of the solar system with four-dimensional Newtonian physics. The cases $\delta \geq 2$ are possible, because Newton's law is experimentally tested only for scales $r \gtrsim 1\,\mathrm{mm}$.

8.A Appendix: Electrodynamics in curved spacetime

Wave equation in a curved spacetime Most of the derivations performed in section 7.6 for Minkowski spacetime can be simply recycled in a curved spacetime using the replacement $\{\partial_\alpha, \eta_{\alpha\beta}\} \to \{\nabla_\alpha, g_{\alpha\beta}\}$. Thus the Maxwell equations in curved spacetime become

$$\nabla_\alpha F^{\alpha\beta} = j^\beta \quad \text{and} \quad \nabla_\alpha \tilde{F}^{\alpha\beta} = 0\,, \tag{8.78}$$

while the definition of the field-strength tensor,

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{8.79}$$

is, due to its antisymmetry, unchanged. Inserting $F^\alpha_{\ \nu} = g^{\mu\alpha} F_{\mu\nu}$ into the inhomogenous Maxwell gives

$$g^{\mu\alpha} \nabla_\alpha \nabla_\mu A_\nu - \nabla_\alpha \nabla_\nu A^\alpha = \Box A_\nu + \nabla_\nu \nabla_\alpha A^\alpha + R_{\nu\alpha} A^\alpha = j_\nu. \tag{8.80}$$

In the second step, we introduced $g^{\alpha\beta}\nabla_\alpha\nabla_\beta \equiv \Box$ as the generalisation of the usual wave- or d'Alembert operator to a curved spacetime. Moreover, we used Eq. (13.7) to anticommute the derivatives $\nabla_\alpha \nabla_\nu$. Employing then the Lorenz gauge, $\nabla_\nu A^\nu = 0$, we obtain as wave equation

$$\Box A^\mu + R_{\mu\nu} A^\nu = j^\mu. \tag{8.81}$$

Note that the transition to curved spacetime is not unique: Performing the transition directly in the wave equation, we would miss the coupling $R_{\mu\nu} A^\nu$ of the photon field to the Riccci tensor.

*****Geometrical optics***** Studying the paths of light-rays in the Schwarzschild or Kerr metric, we have implicitely worked in the limit of geometrical optics: This is an aproximation to wave-propagation in Maxwell's theory, Eq. (8.81), which assumes that the wave-length of the photons constituting the light-ray is much smaller than any other length scale in the problem. In particular, the wave-length λ has to be smaller than the scale r of curvature, which we identify with the square root of a "typical" component of the Riemann tensor in an inertial system of interest. Moreover, the wave-lenght has to be small compared to the length L over which the wave varies. Thus we require $\lambda \ll \min\{L, r\}$.

We split the wave into $A^\mu = \Re[f^\mu e^{i\phi}]$, with a fast varying real phase ϕ and a slowly varying complex amplitude f^μ. Introducing then the parameter $\varepsilon = \lambda / \min\{L, r\}$, we can expand the amplitude as

$$f^\mu = a^\mu + i\varepsilon b^\mu + \mathcal{O}(\varepsilon^2), \tag{8.82}$$

where the constants a^μ and b^μ are independent of λ. The limit of geometrical optics corresponds to the case when the $\mathcal{O}(\varepsilon^2)$ term can be neglected. Note that the phase ϕ is of order $1/\lambda \sim \varepsilon$, and therefore we replace $\phi \to \phi/\varepsilon$. Thus our ansatz becomes

$$A^\mu \simeq \Re[a^\mu + \mathrm{i}\varepsilon b^\mu + \mathcal{O}(\varepsilon^2)]\mathrm{e}^{\mathrm{i}\phi/\varepsilon}. \tag{8.83}$$

The expansion parameter ε can be set to one at the end of the derivation.

In order to prove that this ansatz corresponds to the geometrical optics limit, we have to show that the wave-vector k_μ is a null vector, that it is tangential to the light-ray and satisfies the geodesic equation. In the local intertial frame of an observer, it is $\phi = k_\mu x^\mu$ and $k_\mu = \partial_\mu \phi$. Since the phase ϕ is a scalar, the last relation holds in general. Differentiating (8.83) once, we obtain

$$\nabla_\mu A^\nu = \left[\frac{\mathrm{i}}{\varepsilon} k_\mu a^\nu - k_\mu b^\nu + \nabla_\mu a^\nu + \mathcal{O}(\varepsilon)\right] \mathrm{e}^{\mathrm{i}\phi/\varepsilon}. \tag{8.84}$$

Differentiating a second time, it follows

$$\Box A^\mu = \left[-\frac{k^2}{\varepsilon^2} a^\mu + \frac{\mathrm{i}}{\varepsilon}\left(k^2 b^\mu + 2k^\nu \nabla_\nu a^\mu + (\nabla_\nu k^\mu) a^\nu\right) + \mathcal{O}(\varepsilon)\right] \mathrm{e}^{\mathrm{i}\phi/\varepsilon}. \tag{8.85}$$

Since $R^\mu_\nu A^\nu = \mathcal{O}(\varepsilon^0)$, the free Maxwell equation (8.81) implies that $\Box A^\mu$ vanishes at $\mathcal{O}(\varepsilon^{-2})$ and $\mathcal{O}(\varepsilon^{-1})$. Hence the $\mathcal{O}(\varepsilon^{-2})$ term implies $k^2 = 0$, while the $\mathcal{O}(\varepsilon^{-1})$ term gives a differential equation for the evolution of the amplitude a,

$$k^\nu \nabla_\nu a^\mu = -\frac{1}{2}(\nabla_\nu k^\mu) a^\nu. \tag{8.86}$$

We now show that the relation $k_\mu = \partial_\mu \phi$ together with the null condition implies the geodesice equation. Differentiating the null condition gives

$$0 = 2k^\mu \nabla_\nu k_\mu = 2k^\mu \nabla_\nu \nabla_\mu \phi = 2k^\mu \nabla_\mu \nabla_\nu \phi = 2k^\mu \nabla_\mu k_\nu, \tag{8.87}$$

where we used in the next-to-last step that two covariant derivatives commute acting on a scalar. Thus we have shown that the ansatz (8.84) specifies the vector potential in the limit of geometrical optics. Moreover, we have shown that the approximation of geometrical optics breaks down, if the condition $\lambda \ll \min\{L, r\}$ does not hold.

8.B Appendix: More differential geometry

8.B.1 Differential forms

We have seen that we can define a generalise partial integration for completely antisymmetric tensor fields. Moreover, the observation that Maxwell equations are conformally invariant, suggests that one can define such a theory without the need to introduce a metric tensor. In order to achive this, we define first differential forms.

Remark 8.5: Antisymmetric tensors of rank n can also be seen as differential forms. We already know functions as forms of order $n = 0$ and co-vectors as forms of order $n = 1$. Since differentials $\mathrm{d}f = \partial_i f \,\mathrm{d}x^i$ of functions are forms of order $n = 1$, the $\mathrm{d}x^i$ form a basis, and one can write in general $\boldsymbol{A} = A_i \,\mathrm{d}x^i$. For $n > 1$, the basis has to be antisymmetrised. Hence, a two-form as the field-strength tensor is given by

$$\boldsymbol{F} = \frac{1}{2}\, F_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \tag{8.88}$$

with $\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = -\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu$. Looking at $\mathrm{d}f$ suggestions to define the differentiation of a form ω with coefficients w and degree n as an operation that increases its degree by one to $n+1$,

$$\mathrm{d}\omega = \frac{1}{n!}\,(\partial_\beta w_{\alpha_1\ldots,\alpha_n})\mathrm{d}x^\beta \wedge \mathrm{d}x^{\alpha_1} \wedge \ldots \wedge \mathrm{d}x^{\alpha_n}. \tag{8.89}$$

Thus we have $\boldsymbol{F} = \mathrm{d}\boldsymbol{A}$. Moreover, it follows $\mathrm{d}^2\omega = 0$ for all (smooth) forms. Hence we can write an abelian gauge transformation as $\boldsymbol{F}' = \mathrm{d}(\boldsymbol{A} - \mathrm{d}\Lambda) = \boldsymbol{F}$.

Next, we want to rewrite the Maxwell equations using differential forms.
integration, Stokes theorem using differential forms

8.B.2 Cartan's structure equation and anholonomic coordinates

The derivation of our explicit expressions for the connection and the Riemann tensor were all performed using as coordinates the tangential vectors to the coordinate lines, $\{\boldsymbol{e}_\mu\} = \{\partial/\partial x^\mu\}$. In this appendix, we generalize these expressions to arbitrary ("anholonomic") coordinates.

8.B.3 Vielbein formalism and spin connection

Gravity can be seen as a local gauge theory of Lorentz transformation SO(1,3); this point of view allows one to derive the spin connection for fermionic fields, for an introductionsee my book or lecture notes on QFT.

Problems

8.1 Hyperbolic plane H^2 II. Calculate the Riemann tensor $R^a{}_{bcd}$ and the scalar curvature R for the Hyperbolic plane H^2.

8.2 Weyl tensor. Split the Riemann tensor into a tensor determined soley by the Ricci tensor, and a reminder. Show that the latter, called th Weyl tensor, is invariant under conformal transformations.

9 Relativistic stars

In this chapter, we will examine the Schwarzschild interior solution and use it to derive the equations of stellar structure for a relativistic, compact star. The most important example for stars where general relativistic effects are important are neutron stars. In order to obtain some manageable equations, we consider a static and spherically symmetric star. The matter inside a neutron star can be described as an ideal fluid, and we will assume an equation of state (E.o.S.) which depends on density but not on temperature, $P = P(\rho(r))$. Then the equations of stellar structure reduce to a set of two, ordinary differential equations, the continuity and the hydrostatic equilibrium equation. We start the chapter generalising the hydrodynamics of an ideal fluid to general relativity.

Be aware of a mix of sign conventions in this chapter!

9.1 General-relativistic hydrodynamics

We reserve the letter ρ to the total energy density, and write mn for the mass density of the fluid. Then the total energy density $\rho = mn + \varepsilon$ includes the internal (thermodynamic) energy density ε and the mass density mn. In addition, the fluid is characterised by the mass current $j^\alpha = mnu^\alpha$ and the pressure P, assumed to be isotropic.

The densities are all measured in an inertial frame that is momentarily comoving with a selected fluid element; this frame is attached to an observer moving freely in the gravitational field. The mass density mn satisfies the conservation law[1],

$$\nabla_\alpha j^\alpha = \nabla_\alpha (mnu^\alpha) = 0, \tag{9.1}$$

if the production and destruction of nucleons can be negelected. Using the product rule, it follows

$$m\frac{\mathrm{d}n}{\mathrm{d}\tau} + mn\nabla_\alpha u^\alpha = 0, \tag{9.2}$$

where $\mathrm{d}n/\mathrm{d}\tau = u^\alpha \nabla_\alpha n$ is the convective (or Lagrangian) derivative of the density n. For an ideal fluid, the stress tensor (7.49) becomes in curved space-time replacing $\eta^{\alpha\beta}$ with $g^{\alpha\beta}$

$$T^{\alpha\beta} = (\rho + P)u^\alpha u^\beta - P\eta^{\alpha\beta}. \tag{9.3}$$

Local energy-momentum conservation, $\nabla_\beta T^{\alpha\beta} = 0$, leads to

$$u^\alpha \left[\frac{\mathrm{d}\rho}{\mathrm{d}\tau} + (\rho + P)\nabla_\beta u^\beta\right] + (\rho + P)\frac{\mathrm{d}u^\alpha}{\mathrm{d}\tau} + (g^{\alpha\beta} - u^\alpha u^\beta)\nabla_\beta P = 0. \tag{9.4}$$

In section 7.3, we split the conservation law $\partial_\alpha T^{\alpha\beta} = 0$ valid in Minkowski space by hand into two separate conservation laws for energy and momentum. We want to perform this splitting now in a covariant way: In order to do so, we recall that the acceleration $\mathrm{d}u^\alpha/\mathrm{d}\tau$ is orthogonal

[1] More properly, it is the baryon number which is conserved.

to the velocity u^α. Thus we can divide this equation into two independent equations, one for the projection parallel to u^α and one for the projection perpendicular to u^α. While this splitting is obvious for the first two terms, we will show next that the third term contains already the apropriate projection operator.

Example 9.1: Projection operators—Recall from quantum mechanics that a set of two projection operators $\mathcal{P}_\pm$ should satisfy

$$\mathcal{P}_\pm^2 = \mathcal{P}_\pm, \qquad \mathcal{P}_\pm \mathcal{P}_\mp = 0, \quad \text{and} \quad \mathcal{P}_+ + \mathcal{P}_- = 1.$$

We want to show that

$$\mathcal{P}_\alpha^{\ \beta} = \delta_\alpha^{\ \beta} - n_\alpha n^\beta \tag{9.5}$$

is an operator which projects any vector on the three-dimensional subspace orthogonal to the unit vector n^α. First, we verify that this operator satisfies $\mathcal{P}^2 = \mathcal{P}$,

$$\mathcal{P}_\alpha^{\ \beta} \mathcal{P}_\beta^{\ \gamma} = (\delta_\alpha^{\ \beta} - n_\alpha n^\beta)(\delta_\beta^{\ \gamma} - n_\beta n^\gamma) = \delta_\alpha^{\ \gamma} - n_\alpha n^\gamma = \mathcal{P}_\alpha^{\ \gamma}. \tag{9.6}$$

Morover, it is $n^\alpha \mathcal{P}_\alpha^{\ \beta} n^\beta = 0$ for all unit vectors $\boldsymbol{n}$; Thus $\mathcal{P}$ projects indeed any vector on the subspace orthogonal to $\boldsymbol{n}$.

Raising one index in Eq. (9.5), we see that $\mathcal{P}^{\alpha\beta} = g^{\alpha\beta} - u^\alpha u^\beta$ projects the pressure gradient $\nabla_\beta P$ on the subspace orthogonal to the fluid velocity u^α. Thus we obtain two independent equations, one for the component parallel to u^α,

$$\frac{\mathrm{d}\rho}{\mathrm{d}\tau} + (\rho + P)\nabla_\alpha u^\alpha = 0 \tag{9.7}$$

and one for the components perpendicular to u^α,

$$(\rho + P)\frac{\mathcal{D}u^\alpha}{\mathrm{d}\tau} = -(g^{\alpha\beta} + u^\alpha u^\beta)\nabla_\beta P. \tag{9.8}$$

This is the general-relativistic generalisation of the Euler equation: The acceleration of a fluid element is given by the gradient of the pressure perpendicular to $\boldsymbol{u}$, as it should be for a pure force. Using mass conservation, we can rewrite the parallel equation as

$$\rho\frac{\mathrm{d}\varepsilon}{\mathrm{d}\tau} - (\varepsilon + P)m\frac{\mathrm{d}n}{\mathrm{d}\tau} = 0. \tag{9.9}$$

This equation expresses the first law of thermodynamics for an ideal fluid, i.e. the limit if a fluid when entropy production is negligible.

Hydrostatic equilibrium equation The equilibrium condition implies that the metric is static,

$$\mathrm{d}s^2 = g_{00}(\boldsymbol{x})\mathrm{d}t^2 + g_{ij}(\boldsymbol{x})\mathrm{d}x^i \mathrm{d}x^j\,. \tag{9.10}$$

We set $g_{00}(\boldsymbol{x}) = \mathrm{e}^{-2\Phi(\boldsymbol{x})}$, such that Φ becomes in the non-relativistic limit the Newtonian potential. Since the space-time is static, the fluid variables do also not depend on time. The only non-zero component of the four-velocity is u^0, and thus $u_\alpha u^\alpha = 1$ implies $u^0 = \mathrm{e}^{-\Phi(\boldsymbol{x})}$. Calculating the (spatial) acceleration of the fluid-element, we find

$$\frac{\mathcal{D}u^i}{\mathrm{d}\tau} = g^{ij}\partial_j\Phi. \tag{9.11}$$

Inserting the acceleration into the spatial part of the Euler equation (9.8) results ins

$$(\rho + P)g^{ij}\partial_j\Phi + g^{ij}\nabla_j P = 0, \tag{9.12}$$

or, in three-vector notation,

$$(mn + \varepsilon + P)\boldsymbol{\nabla}\Phi = -\boldsymbol{\nabla}P. \tag{9.13}$$

This is the relativistic version of the hydrostatic equilibrium equation, where in addition to the mass density mn the internal energy ε and the pressure P source the gravitational acceleration.

9.2 Reissner-Nordström exterior solution

The solution outside of s static star is decribed by the Schwarzschild metric we discussed in detail in chapter 4. We start by deriving this metric from the Einstein equations. Since it does not cost much more effort, we allow for a non-zero electric charge, deriving thereby the Reissner-Nordström solution for a charged black hole.

In the appendix 4.A, we have shown that the metric of a stationary, isotropic spacetime is determined by two functions,

$$\begin{aligned} \mathrm{d}s^2 &= A(r)\mathrm{d}t^2 - B(r)\mathrm{d}r^2 - r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\phi^2) &(9.14)\\ &= \mathrm{e}^{\sigma(r)}\mathrm{d}t^2 - \mathrm{e}^{\lambda(r)}\mathrm{d}r^2 - r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\phi^2)\,. &(9.15)\end{aligned}$$

The second equation makes it clear that these two functions are positive. Moreover, the second version is sometimes preferred since the combinations $A'/A = \sigma'$ and $B'/B = \lambda'$ (where prime denotes derivatives w.r.t. r) appear often. The non-zero components of the Ricci tensor in this metric are given by

$$R_{00} = \frac{A''}{2B} - \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{A'}{rB}, \tag{9.16}$$

$$R_{11} = -\frac{A''}{2A} + \frac{A'}{4A}\left(\frac{A'}{A} - \frac{B'}{B}\right) - \frac{B'}{rB}, \tag{9.17}$$

$$R_{22} = 1 - \frac{1}{B} - \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right), \tag{9.18}$$

$$R_{33} = R_{22}\sin^2\vartheta, \tag{9.19}$$

where we order coordinates as $x^\mu = (t, r, \vartheta, \phi)$.

We consider next the inhomogenuous Maxwell equation $\nabla_\mu F^{\mu\nu} = j^\nu$ for a point charge in the metric (9.14). The electric field measured by an observer with four-velocity u_α is $E_\alpha = F_{\alpha\beta}u^\beta$, i.e. it is associated with the field-strength tensor with lower indices. For a static charge at $r = 0$, the only non-zero field component is the radial electric field, $F_{01} = -F_{10} = E(r)$. Raising the indices, we have $F^{01} = g^{00}g^{11}F^{01} = E/(AB)$. In order to evaluate the Maxwell equation as

$$\nabla_\mu F^{\mu\nu} = \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}F^{\mu\nu}) = j^\nu, \tag{9.20}$$

we have to determine the determinant of the metric, obtaining

$$\sqrt{|g|} = \sqrt{AB}r^2\sin^2\vartheta. \tag{9.21}$$

For $r > 0$, the current j^μ is zero and thus

$$\partial_r\left(\sqrt{AB}r^2F^{01}\right) = \partial_r\left(\frac{r^2E}{\sqrt{AB}}\right) = 0 \tag{9.22}$$

can be integrated, with the result

$$E(r) = \frac{\sqrt{AB}Q}{4\pi r^2}. \tag{9.23}$$

Here, the integration constant k can be identified with $Q/(4\pi)$, because of $A, B \to 1$ for $r \to \infty$. The homogeneous Maxwell equation is automatically satisfied, since there exists a potential A_0 with $E(r) = -\partial_r A_0$.

We use the Einstein equation in its "reversed" form (9.69), which avoids the calculation of R. The stress tensor of the electromagnetic field is always traceless[2]

$$T_\mu{}^\mu = -F_{\mu\sigma}\,F^{\mu\sigma} + \frac{1}{4}\,\delta^{\mu\mu}F_{\sigma\tau}F^{\sigma\tau} = 0\,. \tag{9.24}$$

Thus the Einstein equation becomes [**changed sign convention**]

$$R_{\mu\nu} = -\kappa(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T) + g_{\mu\nu}\Lambda = \kappa T_{\mu\nu}\,. \tag{9.25}$$

Evaluating

$$T_{\mu\nu} = -F_{\mu\sigma}\,F_\nu{}^\sigma + \frac{1}{4}\,\eta_{\mu\nu}F_{\sigma\tau}F^{\sigma\tau} \tag{9.26}$$

using also $F_0{}^1 = g^{11}F_{01} = E/B$ and $F_1{}^0 = g^{00}F_{10} = E/A$, we find that they are given explicitely (setting $\kappa = 1$) by

$$R_{00} = -E^2/(2B), \tag{9.27}$$

$$R_{11} = E^2/(2A), \tag{9.28}$$

$$R_{22} = -E^2r^2/(2AB), \tag{9.29}$$

$$R_{33} = R_{22}\sin^2\vartheta. \tag{9.30}$$

Combining (9.27) and (9.28), it follows

$$BR_{00} + AR_{11} = 0. \tag{9.31}$$

Next we insert (9.16) and (9.17), obtaining

$$A'B + AB' = 0. \tag{9.32}$$

Then $(AB)' = 0$ or $AB =$ const. Assuming that spacetime becomes Minkowskian at large distance, it follows $A(r)B(r) = 1$. We see now that the choice of the radial coordinate is such that r corresponds to the "luminosity distance" or, in other words, such that a $1/r^2$ law is valid for the flux from a point source.

Finally, we use the R_{22} component of the Einstein equation to determine $A(r)$ and $B(r)$. We insert $AB = 1$ and (9.23) into (9.29),

$$R_{22} = -E^2r^2/(2AB) = -\frac{1}{2}E^2r^2 = -\frac{1}{2}\frac{Q^2}{(4\pi)^2r^2}. \tag{9.33}$$

[2] This is true for all fields without dimensionfull parameter in their Lagrangian, which are conformally invariant.

Next we simplify (9.18) using $AB=1$, obtaining

$$R_{22} = A - 1 + A'r \tag{9.34}$$

and thus

$$A + A'r = 1 - \frac{1}{2}\frac{Q^2}{(4\pi)^2 r^2}. \tag{9.35}$$

Using $A + A'r = (Ar)'$ and integrating, it follows

$$A(r) = 1 + \frac{k}{r} + \frac{Q^2}{2(4\pi)^2 r^2} = 1 - \frac{2GM}{r} + \frac{GQ^2}{4\pi r^2} \quad \text{and} \quad B(r) = 1/A(r), \tag{9.36}$$

where the integration constant k was fixed by requiring that we obtain the Schwarzschild metric for $Q=0$. In the last step, we changed also from $\kappa = 1$ to $\kappa = 8\pi G$.

9.3 Schwarzschild interior solution

TOV equations – standard approach The non-zero components of the Ricci tensor are again given by Eqs.(9.16-9.19). Proceeding like for the exterior solution, we could determine the two functions A and B for given energy and pressure profiles $\rho(r)$ and $P(r)$. However, pressure and energy are not independent but connected by the equation of state of the stellar matter. Thus we should determine how $\rho(r)$ and $P(r)$ evolve as function of r for a given equation of state and boundary conditions.

Modelling the stellar matter as ideal fluid, the stress tensor is

$$T^{\alpha\beta} = (\rho + P)u^\alpha u^\beta - P g^{\alpha\beta}. \tag{9.37}$$

Its trace follows as

$$T = T_\alpha^{\ \alpha} = (\rho + P)u_\alpha u^\alpha - P\delta_\alpha^\alpha = (\rho + P) - 4P = \rho - 3P. \tag{9.38}$$

The star is static, and thus $u_\alpha = (u_t, 0) = ((g^{tt})^{-1/2}, 0) = (1/\sqrt{A}, 0)$. Hence we obtain

$$T_{00} = A(\rho + P) - PA \tag{9.39}$$
$$T_{11} = -BP \tag{9.40}$$
$$T_{22} = -r^2 P \tag{9.41}$$
$$T_{33} = T_{22}\sin^2\vartheta, \tag{9.42}$$

and

$$R_{00} = \frac{1}{2}\kappa(\rho + 3P)A \tag{9.43}$$
$$R_{11} = \frac{1}{2}\kappa(\rho - P)B \tag{9.44}$$
$$R_{22} = \frac{1}{2}\kappa(\rho - P)r^2 \tag{9.45}$$
$$R_{33} = \sin^2\vartheta r^2 R_{22} \tag{9.46}$$

The two angular Einstein equations are not independent. We can use either three independent Einstein equations, or two plus the conservation of the stress tensor. We use the latter first, since we expect it to be directly connected to the hydrostatic equilibrium condition. We differentiate the stres tensor, using in the second term $\nabla_\alpha g^{\alpha\beta} = 0$ und in the first one Eq. (8.28), obtaining

$$\nabla_\alpha T^{\alpha\beta} = \frac{1}{\sqrt{|g|}}\partial_\alpha \left[\sqrt{|g|}(\rho+P)u^\alpha u^\beta\right] + (\rho+P)\Gamma^\beta{}_{\sigma\alpha}u^\alpha u^\sigma - g^{\alpha\beta}\partial_\alpha P. \tag{9.47}$$

The first term vanishes because only u^t is non-zero, and the star is static. For the same reason, the second term becomes $(\rho+P)\Gamma^\beta{}_{00}/A$. Using the definition of the Christoffel symbols, it is

$$\Gamma^\beta{}_{00} = -\frac{1}{2}g^{\alpha\beta}\partial_\alpha g_{00} = -\frac{1}{2}g^{\alpha\beta}\partial_\alpha A. \tag{9.48}$$

Combining everything, it follows

$$\frac{(\rho+P)}{2A}g^{\alpha\beta}\partial_\alpha A + g^{\alpha\beta}\partial_\alpha P = 0 \tag{9.49}$$

or

$$\frac{(\rho+P)}{2A}\partial_\alpha A + \partial_\alpha P = 0. \tag{9.50}$$

The function A depends only on r, so the 0 (static pressure) and $2,3$ (no pressure gradients in tangential direction because of spherical symmetry) components are trivial. For the r component, it follows

$$(\rho+P)\frac{A'}{A} + P' = 0. \tag{9.51}$$

In addition to this constraint on A'/A, we require an equation for B. The linear combination

$$\frac{R_{00}}{A} + \frac{R_{11}}{B} + \frac{2R_{22}}{r^2} = \frac{1}{r^2} - \frac{2}{Br^2} + \frac{2B'}{B^2 r} = 16\pi\rho \tag{9.52}$$

depend only on ρ and B. Expressed by λ, it is

$$-\mathrm{e}^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) + \frac{1}{r^2} = 8\pi\rho \tag{9.53}$$

We note next that the metric (9.14) has to become equal to the Schwarzschild solution at $r \geq R$. This is ensured by setting

$$B(r) = \mathrm{e}^\lambda = \left(1 - \frac{2m(r)}{r}\right)^{-1} \tag{9.54}$$

and $m(R) = M$. Solving Eq. (9.66) for $m(r)$ and differentiating the result gives

$$2\frac{\mathrm{d}m}{\mathrm{d}r} = 1 - \mathrm{e}^{-\lambda} + r\lambda'\mathrm{e}^{-\lambda}. \tag{9.55}$$

Thus

$$\frac{2}{r^2}\frac{\mathrm{d}m}{\mathrm{d}r} = \mathrm{e}^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right)\frac{1}{r^2} = 8\pi\rho \tag{9.56}$$

or in integrated form

$$m(r) = 4\pi\int_0^r \mathrm{d}r'\, r'^2\rho(r'). \tag{9.57}$$

Finally we note that in the R_{22} equation A and A' enter only via their ratio. Using the constraint to eleminate A'/A and Eq. (9.66) for B, we evaluate first the three ingredients

$$1 - \frac{1}{B} = 1 - \left(1 - \frac{2m(r)}{r}\right), \tag{9.58}$$

$$\frac{r}{2B}\frac{B'}{B} = 4\pi\rho r^2 - \frac{M(r)}{r}, \tag{9.59}$$

and

$$\frac{r}{2B}\frac{A'}{A} = \left(1 - \frac{2m(r)}{r}\right)\left(-\frac{rP'}{P+\rho}\right). \tag{9.60}$$

Combining everything we arrive at

$$1 - \left(1 - \frac{2m(r)}{r}\right)\left(1 - \frac{rP'}{P+\rho}\right) + 4\pi\rho r^2 - \frac{M(r)}{r} = 4\pi(\rho + P)r^2. \tag{9.61}$$

Solving for P' and factoring $1/r^2$ and $(1 - 2m(r)/r)^{-1/2}$ out, we obtain the Tolman-Oppenheimer-Volkov (TOV) equation for the pressure gradient of a relativistic star,

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{\rho + P}{r^2}\left[m(r) + 4\pi r^3 P\right]\left(1 - \frac{2m(r)}{r}\right)^{-1}. \tag{9.62}$$

For a given E.o.S., the continuity equation (9.64) and the TOV equation can be solved numerically using as boundary value $M(0) = 0$. Choosing in addition an arbitrary value for the core density, ρ_c, determines via the E.o.S. also the core pressure P_c. Integrating then outside until the pressure is zero defines the stellar radius R and the stellar mass M via $P(R) = 0$ and $M = M(R)$.

In the non-relativistic limit (small velocities), pressure is negligible. Moreover, the $2m(r)/r$ correction is negligible. Then the TOV equation reduces to

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{G\rho(r)m(r)}{r^2} \tag{9.63}$$

where we added also G. Note that all correction terms increase the pressure gradient relative to the Newtonian case.

The continuity equation looks identical to the non-relativstic case, except that ρ has the meaning of the total energy density. Recalling however that the proper volume is $\mathrm{d}V = 4\pi r^2\sqrt{g_{rr}}\mathrm{d}r$, the integrated energy density is

$$\widetilde{m}(r) = 4\pi\int_0^r \mathrm{d}r'\, r'^2 \frac{\rho(r')}{(1 - 2M(r')/r)^{1/2}}. \tag{9.64}$$

The difference $E = \widetilde{m}(R) - m(R)$ is the gravitational binding energy of the star, i.e. the amount of energy required to disperse the stellar material to $r \to \infty$.

TOV equations using hydrodynamics We can short-cut the rather lengthy derivation of the TOV equations using the results from section 9.1 on hydrodynamics. In this apprach, we use the Einstein equations in the standard form $G_\mu^{\ \nu} = \kappa T_\mu^{\ \nu}$. We evaluate the components of the Einstein tensor for the metric of a stationary, isotropic spacetime, setting

$$\mathrm{d}s^2 = \mathrm{e}^{\sigma(r)}\mathrm{d}t^2 - \mathrm{e}^{\lambda(r)}\mathrm{d}r^2 - r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\phi^2), \tag{9.65}$$

From the derivation of the hydrostatic equilibrium equation, we know already that we can identify $\sigma = 2\Phi$. The non-relativistic stellar structure equations suggest that the second function λ is connected with the relativistic generalisation of the enclosed mass. Moreover, we use that the metric (9.14) has to become equal to the Schwarzschild solution at $r \geq R$. This is ensured by setting

$$B(r) = \mathrm{e}^{\lambda} = \left(1 - \frac{2m(r)}{r}\right)^{-1} \tag{9.66}$$

and $m(R) = M$. Evaluating then the Einstein tensor using, e.g., the python program from the webpage, we find

$$G_0^{\ 0} = -\frac{2G}{r^2}\partial_r m(r) \tag{9.67}$$

$$G_r^{\ r} = -\frac{2}{r} f \partial_r \Phi(r); \tag{9.68}$$

the other components of $G_\mu^{\ \nu}$ are not needed. Inserting these expressions into the Einstein equations and solving for $\partial_r m(r)$ and $\partial_r \Phi(r)$ gives

$$\partial_r m(r) = -4\pi r^2 T_0^{\ 0} = -4\pi r^2 (mn + \varepsilon) \tag{9.69}$$

$$\partial_r \Phi(r) = -\frac{G}{r^2 B}\left(m + 4\pi r^3 T_r^{\ r}\right) = -\frac{G}{r^2 B}\left(m + 4\pi r^3 P\right). \tag{9.70}$$

In the last step, we inserted the relevant components of the stress tensor, $T_0^0 = \rho = mn + \varepsilon$ and $T_r^r = P$.
meaning: gravitational acceleration, mass

Next we use the hydrostatic equilibrium equation (9.13) applied to a spherically symmetric system,

$$(mn + \varepsilon + P)\frac{\mathrm{d}\Phi}{\mathrm{d}r} = -\frac{\mathrm{d}P}{\mathrm{d}r}, \tag{9.71}$$

and insert the gravitational acceleration, obtaining the TOV equation

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{\rho + P}{r^2}\left[m(r) + 4\pi r^3 P\right]\left(1 - \frac{2m(r)}{r}\right)^{-1}. \tag{9.72}$$

9.4 Simple solutions

Incompressible star The simplest possible model for a star is to assume a constant density ρ, which is the idealisation of an ultra-stiff E.o.S. The enclosed mass $m(r)$ of such an incompressible star with mass M and radius R is given by

$$m(r) = \frac{4\pi}{3}\rho r^3 = (r/R)^3 M. \tag{9.73}$$

Inserting $m(r)$ into the TOV equation, the equation can be separated,

$$\int_{P_0}^{P} \frac{\mathrm{d}P}{(\rho + P)(\rho + 3P)} = -\frac{4\pi G}{3}\int_0^r \mathrm{d}r' \frac{r'}{1 - 8\pi G \rho r'^2/3}. \tag{9.74}$$

The integration results with $P_c \equiv P(r = 0)$ in

$$\frac{\rho + 3P}{\rho + P} = \frac{\rho + 3P_c}{\rho + P_c}\left(1 - \frac{8\pi G}{3}\rho r^2\right)^{1/2}. \tag{9.75}$$

At the stellar surface, the pressure is zero and thus the LHS equals one. Thus we obtain an equation for the stellar radius R as function of the central pressure P_c,

$$R^2 = \frac{3}{8\pi G}\left[1 - \left(\frac{\rho + P_c}{\rho + 3P_c}\right)^2\right]. \tag{9.76}$$

Evaluateing on the other hand Eq. (9.75) at $r = 0$, we can express the central pressure as

$$P_c = \rho\frac{1 - \sqrt{1 - R_S/R}}{3\sqrt{1 - R_S/R} - 1}\,\rho, \tag{9.77}$$

where we introduced the Schwarzschild radius $R_S = 2GM$. As expected does the central pressure increase as R decreases. More surpridingly, the pressure diverges for a finite value of R, since the denominator vanishes for

$$R_{\rm cr} = \frac{9}{8}R_S = \frac{9M}{4}. \tag{9.78}$$

Thus there exits a minimal radius $R_{\rm cr}$ for an incompressible star with a given M. Using $m(r)$, we can rephrase this as an upper limit on M for a given density,

$$M \leq 5.7M_\odot\left(\frac{XXX}{\rho_0}\right)^{1/2}, \tag{9.79}$$

where we inserted a value for ρ_0 typical for the density of nuclear matter.

Buchdahl's theorem states that this mass limits holds for any E.o.S. Thus within general relativity, any object violating the condition $R/M > 9/4$ has to collapse to a black hole. This implies also an upper limit on the gravitational redshift (4.26):

$$1 + z \equiv \frac{\omega_\infty}{\omega(R)} > \sqrt{1 - \frac{2M}{R_{\rm cr}}} = \frac{1}{3}. \tag{9.80}$$

Thus any observed redshift $z > 4/3$ has to be of cosmological origin.

Relativistic polytropes A relativistic star satisfying the E.o.S.

$$P = K\rho^{(n+1)/n} = K\rho^\gamma \quad \text{and} \quad \varepsilon = n\rho. \tag{9.81}$$

at each r with the same adiabatic exponent $\gamma = (n + 1)/n$ is called a relativistic polytrope. In the relativistic case, the continuty and TOV equation cannot be combined into a single second order differential equation. Instead, they

10 Linearized gravity and gravitational waves

In any relativistic theory of gravity, the effects of an accelerated point mass on the surrounding spacetime can propagate maximally with the speed of light. Thus one expects that, in close analogy to electromagnetic waves, gravitational waves exist. Such waves correspond to ripples in spacetime which lead to local stresses and transport energy. Although gravitational waves were already predicted by Einstein in 1916, their existence was questioned (also by Einstein himself) until the 1950s: Since locally the effects of gravity can be eliminated, it was doubted that they cause any measurable effects. Similarly, the non-existence of a stress tensor for the gravitational field raised the question how, e.g., the momentum and energy flux of gravitational waves can be properly defined. Only in 1957, at the now famous "Chapell Hill Conference", this controversy was decided: First, Pirani presented a formalism how coordinate independent effects of a gravitational wave could be deduced. Second, Feynman suggested the following simple gedankenexperiment: A gravitational wave passing a rod with sticky beads would move the beads along the rod; friction would then produce heat, implying that the gravitational wave had done work. Soon after that the first gravitational wave detectors were developed, but only in 2015 the first detection was accomplished.

10.1 Linearized gravity

In electrodynamics, the photon is uncharged and the Maxwell equations are thus linear. In contrast, gravitational fields carry energy, are thus self-interacting and in turn the Einstein equations are non-linear. In order to derive a wave equation, we have therefore to linearize the field equations as first step.

10.1.1 Metric perturbations as a tensor field

We are looking for small perturbations $h_{\mu\nu}$ around the Minkowski[1] metric $\eta_{\mu\nu}$,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad \text{with} \qquad |h_{\mu\nu}| \ll 1. \tag{10.1}$$

These perturbations may be caused either by the propagation of gravitational waves or by the gravitational potential of a star. In the first case, current experiments show that we should not hope for h larger than $\mathcal{O}(h) \sim 10^{-22}$. Keeping only terms linear in h is therefore an excellent approximation. Choosing in the second case as application the final phase of the spiral-in of a neutron star binary system, deviations from Newtonian limit can become large. Hence one needs a systematic "post-Newtonian" expansion or even a numerical analysis to describe properly such cases.

[1] The same analysis could be performed for small perturbations around an arbitrary metric $g^{(0)}_{\mu\nu}$, adding however considerable technical complexity.

$\Box A^\alpha = j^\alpha$	wave equation	$\Box \bar{h}^{ab} = T^{ab}$
$\partial_\alpha A^\alpha = 0$	covariant gauge condition	$\partial_a \bar{h}^{\alpha\beta} = 0$
transverse	polarization	transverse, traceless
$A^\alpha(x) = \int \mathrm{d}^3x' \frac{J^\alpha(t_r,\boldsymbol{x}')}{\lvert\boldsymbol{x}-\boldsymbol{x}'\rvert}$	solution	$\bar{h}^{\alpha\beta}(x) = \int \mathrm{d}^3x' \frac{T^{\alpha\beta}(t_r,\boldsymbol{x}')}{\lvert\boldsymbol{x}-\boldsymbol{x}'\rvert}$
$L_{\mathrm{em}} = -\frac{2}{3}\ddot{d}_a\ddot{d}^a$	energy loss	$L_{\mathrm{gr}} = -\frac{G}{5}\dddot{I}_{ab}\dddot{I}^{ab}$

Table 10.1: Comparison of basic formulas for electromagnetic and gravitational radiation.

We choose a Cartesian coordinate system x^μ and ask ourselves which transformations are compatible with the splitting (10.1) of the metric. If we consider global Lorentz transformations $\Lambda^\nu{}_\mu$, then $\tilde{x}^\nu = \Lambda^\nu{}_\mu x^\mu$, and the metric tensor transforms as

$$\tilde{g}_{\alpha\beta} = \Lambda^\rho{}_\alpha\Lambda^\sigma{}_\beta g_{\rho\sigma} = \Lambda^\rho{}_\alpha\Lambda^\sigma{}_\beta(\eta_{\rho\sigma} + h_{\rho\sigma}) = \eta_{\alpha\beta} + \Lambda^\rho{}_\alpha\Lambda^\sigma{}_\beta h_{\rho\sigma} = \tilde{\eta}_{\alpha\beta} + \Lambda^\rho{}_\alpha\Lambda^\sigma{}_\beta h_{\rho\sigma}. \tag{10.2}$$

Since $\tilde{h}_{\alpha\beta} = \Lambda^\rho{}_\alpha\Lambda^\sigma{}_\beta h_{\rho\sigma}$, we see that global Lorentz transformations respect the splitting (10.1). Thus $h_{\mu\nu}$ transforms as a rank-2 tensor under global Lorentz transformations. We can view therefore the perturbation $h_{\mu\nu}$ as a symmetric rank-2 tensor field defined on Minkowski space that satisfies as wave equation the linearized Einstein equation, similar as the photon field fulfills a wave equation derived from Maxwell's equations.

The splitting (10.1) is however clearly not invariant under general coordinate transformations, as they allow, for example, the finite rescaling $g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$. We restrict therefore ourselves to infinitesimal coordinate transformations,

$$\tilde{x}^\mu = x^\mu + \xi^\mu(x^\nu) \tag{10.3}$$

with $|\xi^\mu| \ll 1$. Then the Killing equation (4.7) simplifies to

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu, \tag{10.4}$$

because the term $\xi^\rho\partial_\rho h_{\mu\nu}$ is quadratic in the small quantities $h_{\mu\nu}$ and ξ_μ and can be neglected. Recall that the $\xi^\rho\partial_\rho h_{\mu\nu}$ term appeared, because we compared the metric tensor at different points. In its absence, it is more fruitful to view Eq. (10.4) not as a coordinate but as a gauge transformation analogous to Eq. (7.95). In this interpretation, we stay in Minkowski space and the fields $\tilde{h}_{\mu\nu}$ and $h_{\mu\nu}$ describe the same physics, since the gravitational field equations do not fix uniquely $h_{\mu\nu}$ for a given source.

Comparison with electromagnetism In Table 10.1, basic properties of electromagnetic and gravitational waves are compared. The procedure to determine the physical polarisation states, their wave equation and solutions is in both cases very similar.

10.1.2 Linearized Einstein equation in vacuum

From $\partial_\mu\eta_{\nu\rho} = 0$ and the definition

$$\Gamma^\mu{}_{\nu\lambda} = \frac{1}{2}g^{\mu\kappa}(\partial_\nu g_{\kappa\lambda} + \partial_\lambda g_{\nu\kappa} - \partial_\kappa g_{\nu\lambda}) \tag{10.5}$$

we find for the change of the connection linear in $h_{\mu\nu}$

$$\delta\Gamma^{\mu}{}_{\nu\lambda} = \frac{1}{2}\eta^{\mu\kappa}(\partial_\nu h_{\kappa\lambda} + \partial_\lambda h_{\nu\kappa} - \partial_\kappa h_{\nu\lambda}) = \frac{1}{2}(\partial_\nu h^\mu_\lambda + \partial_\lambda h^\mu_\nu - \partial^\mu h_{\nu\lambda}). \tag{10.6}$$

Note that we used $\eta^{\mu\nu}$ to lower indices which is appropriate in the linear approximation. Remembering the definition of the Riemann tensor,

$$R^{\mu}{}_{\nu\lambda\kappa} = \partial_\lambda\Gamma^{\mu}{}_{\nu\kappa} - \partial_\kappa\Gamma^{\mu}{}_{\nu\lambda} + \Gamma^{\mu}{}_{\rho\lambda}\Gamma^{\rho}{}_{\nu\kappa} - \Gamma^{\mu}{}_{\rho\kappa}\Gamma^{\rho}{}_{\nu\lambda}, \tag{10.7}$$

we see that we can neglect the terms quadratic in the connection terms. Thus we find for the change

$$\delta R^{\mu}{}_{\nu\lambda\kappa} = \partial_\lambda\delta\Gamma^{\mu}{}_{\nu\kappa} - \partial_\kappa\delta\Gamma^{\mu}{}_{\nu\lambda} \tag{10.8a}$$

$$= \frac{1}{2}\left\{\partial_\lambda\partial_\nu h^\mu_\kappa + \partial_\lambda\partial_\kappa h^\mu_\nu - \partial_\lambda\partial^\mu h_{\nu\kappa} - (\partial_\kappa\partial_\nu h^\mu_\lambda + \partial_\kappa\partial_\lambda h^\mu_\nu - \partial_\kappa\partial^\mu h_{\nu\lambda})\right\} \tag{10.8b}$$

$$= \frac{1}{2}\left\{\partial_\lambda\partial_\nu h^\mu_\kappa + \partial_\kappa\partial^\mu h_{\nu\lambda} - \partial_\lambda\partial^\mu h_{\nu\kappa} - \partial_\kappa\partial_\nu h^\mu_\lambda\right\}. \tag{10.8c}$$

The change in the Ricci tensor follows by contracting μ and λ,

$$\delta R^{\lambda}{}_{\nu\lambda\kappa} = \frac{1}{2}\left\{\partial_\lambda\partial_\nu h^\lambda_\kappa + \partial_\kappa\partial^\lambda h_{\nu\lambda} - \partial_\lambda\partial^\lambda h_{\nu\kappa} - \partial_\kappa\partial_\nu h^\lambda_\lambda\right\}. \tag{10.9}$$

Next we introduce $h \equiv h^\mu_\mu$, $\Box = \partial_\mu\partial^\mu$, and relabel the indices,

$$\delta R_{\mu\nu} = \frac{1}{2}\left\{\partial_\mu\partial_\rho h^\rho_\nu + \partial_\nu\partial_\rho h^\rho_\mu - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h\right\}. \tag{10.10}$$

We now rewrite all terms apart from $\Box h_{\mu\nu}$ as derivatives of the vector

$$\xi_\mu = \partial_\nu h^\nu_\mu - \frac{1}{2}\partial_\mu h, \tag{10.11}$$

obtaining

$$\delta R_{\mu\nu} = \frac{1}{2}\left\{-\Box h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu\right\}. \tag{10.12}$$

Looking back at the properties of $h_{\mu\nu}$ under gauge transformations, Eq. (10.4), we see that we can gauge away the second and third term. Thus the linearised Einstein equation in vacuum, $\delta R_{\mu\nu} = 0$, becomes simply

$$\boxed{\Box h_{\mu\nu} = 0,} \tag{10.13}$$

if the harmonic gauge[2]

$$\xi_\mu = \partial_\nu h^\nu_\mu - \frac{1}{2}\partial_\mu h = 0 \tag{10.14}$$

is chosen. Hence the familiar wave equation holds for all independent components of $h_{\mu\nu}$, and the perturbations propagate with the speed of light: Inserting plane waves $h_{\mu\nu} = \varepsilon_{\mu\nu}\exp(-\mathrm{i}kx)$ into the wave equation, one finds immediately that k is a null vector.

The characteristic property of gravity that we can introduce in each point an inertial coordinate system implies that we can set the perturbation $h_{\mu\nu}$ equal to zero in a single

[2] Alternatively, this gauge is called Hilbert, Loren(t)z, de Donder,…, gauge.

point. This ambiguity was one of the reasons that the existence of gravitational waves was doubted for long time. In Section 8.1, we introduced therefore the Riemann tensor as an umambigious signature for the non-zero curvature of space-time. The derivation of a wave equation for the Riemann tensor,

$$\Box R^{\mu}{}_{\nu\lambda\kappa} = 0, \tag{10.15}$$

by Pirani in 1956 (which follows by using (10.13) in (10.8c)), can be therefore seen as the theoretical proof for the existence of gravitational waves in Einstein gravity: Ripples in space-time, or more formally perturbations of the curvature tensor, propagate with the speed of light.

10.1.3 Linearized Einstein equation with sources

We found $2\delta R_{\mu\nu} = -\Box h_{\mu\nu}$. By contraction it follows $2\delta R = -\Box h$. Combining then both terms gives

$$\Box\left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h\right) = -2\left(\delta R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\delta R\right) = -2\kappa\delta T_{\mu\nu}. \tag{10.16}$$

Since we assumed an empty universe in zeroth order, $\delta T_{\mu\nu}$ is the complete contribution to the stress tensor. We omit therefore in the following the δ in $\delta T_{\mu\nu}$. Next we introduce as useful short-hand notation the "trace-reversed" amplitude as

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h\,. \tag{10.17}$$

The harmonic gauge condition becomes then

$$\partial^{\mu}\bar{h}_{\mu\nu} = 0 \tag{10.18}$$

and the linearised Einstein equation in the harmonic gauge follows as

$$\boxed{\Box\bar{h}_{\mu\nu} = -2\kappa T_{\mu\nu}.} \tag{10.19}$$

Because of $\bar{\bar{h}}_{\mu\nu} = h_{\mu\nu}$ and Eq. (9.69), we can rewrite this wave equation also as

$$\Box h_{\mu\nu} = -2\kappa\bar{T}_{\mu\nu} \tag{10.20}$$

with the trace-reversed stress tensor $\bar{T}_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T$.

Newtonian limit We are now in the position to fix the value of the constant κ, comparing the wave equation (10.19) with the Schwarzschild metric in the Newtonian limit. This limit corresponds to $v/c \to 0$ and thus the only non-zero element of the stress tensor becomes $T^{00} = \rho$. Moreover, the d'Alembert operator can be approximated by minus the Laplace operator, $\Box \to -\Delta$. The Schwarzschild metric in the weak-field limit is

$$\mathrm{d}s^2 = (1+2\Phi)\mathrm{d}t^2 - (1-2\Phi)\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right) \tag{10.21}$$

with $\Phi = -GM/r$ as the Newtonian gravitational potential. Comparing this metric to Eq. (10.1), we find as the static metric perturbations caused by a point mass M at the distance r,

$$h_{00} = 2\Phi, \qquad h_{ij} = 2\delta_{ij}\Phi, \qquad h_{0i} = 0\,. \tag{10.22}$$

With $h_i^j = -h_{ij}$ and thus $h = -4\Phi$, it follows

$$-\Delta\left(h_{00} - \frac{1}{2}\eta_{00}h\right) = -4\Delta\Phi = -2\kappa\rho\,. \tag{10.23}$$

Hence the linearised Einstein equation has in the Newtonian limit the same form as the Poisson equation $\Delta\Phi = 4\pi G\rho$, and the constant κ equals $\kappa = 8\pi G$.

10.1.4 Polarizations states

TT gauge We consider a plane wave $h_{\mu\nu} = \varepsilon_{\mu\nu}\exp(-\mathrm{i}kx)$. The symmetric matrix ε_{ab} is called polarization tensor. Its ten independent components are constrained both by the wave equation and the gauge condition. The harmonic gauge $\partial^\mu \bar{h}_{\mu\nu} = 0$ corresponds to four constraints and reduces thereby the number of independent polarisation states from ten to six. Even after fixing the harmonic gauge $\partial^\mu \bar{h}_{\mu\nu} = 0$, we can still perform a gauge transformation using four functions ξ_μ satisfying $\Box\xi_\mu = 0$. We can choose them such that four additional components of $h_{\mu\nu}$ vanish. In the transverse traceless (TT) gauge, we set ($i = 1, 2, 3$)

$$h_{0i} = 0, \quad \text{and} \quad h = 0. \tag{10.24}$$

The harmonic gauge condition becomes $\xi_\alpha = \partial_\beta h_\alpha^\beta = 0$ or

$$\xi_0 = \partial_\beta h_0^\beta = \partial_0 h_0^0 = -\mathrm{i}\omega\varepsilon_{00}\mathrm{e}^{-\mathrm{i}kx} = 0\,, \tag{10.25a}$$

$$\xi_a = \partial_\beta h_a^\beta = \partial_b h_a^b = \mathrm{i}k^b\varepsilon_{ab}\mathrm{e}^{-\mathrm{i}kx} = 0. \tag{10.25b}$$

Thus $\varepsilon_{00} = 0$ and the polarisation tensor is transverse, $k^a\varepsilon_{ab} = k^b\varepsilon_{ab} = 0$. If we choose the plane wave propagating in the z direction, $\boldsymbol{k} = k\boldsymbol{e}_z$, the last row and column of the polarisation tensor vanish too. Accounting for $h = 0$ and $\varepsilon_{\alpha\beta} = \varepsilon_{\beta\alpha}$, only two independent elements are left,

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \varepsilon_{11} & \varepsilon_{12} & 0 \\ 0 & \varepsilon_{12} & -\varepsilon_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{10.26}$$

Thus the two base states for a linearly polarised GW are

$$\varepsilon_{\alpha\beta}^{(+)} = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \quad \text{and} \quad \varepsilon_{\alpha\beta}^{(\times)} = \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}, \tag{10.27}$$

where we omit the zero columns and rows of the tensor.

Let us re-discuss the procedure of determing the physical polarisation states of a gravitational wave following the same approach that we used for the photon in Eqs. (7.93)-(-7.95). We consider first as gauge transformation

$$\xi^\mu(x) = -\mathrm{i}\lambda^\mu \exp(-\mathrm{i}kx)\,, \tag{10.28}$$

obtaining[3]

$$\tilde{h}^{\mu\nu} = h^{\mu\nu} + \partial^\mu\xi^\nu + \partial^\nu\xi^\mu = (\varepsilon^{\mu\nu} + \lambda^\mu k^\nu + \lambda^\nu k^\mu)\exp(-\mathrm{i}kx) = \tilde{\varepsilon}^{\mu\nu}\exp(-\mathrm{i}kx). \tag{10.29}$$

[3]Simpler to consider trace-reversed $\bar{h}^{\mu\nu}$...

Choosing again a photon propagating in z direction, $k^\mu = (\omega, 0, 0, \omega)$, it follows

$$\lambda^\mu k^\nu + \lambda^\nu k^\mu = \omega \begin{pmatrix} 2\lambda^0 & \lambda^1 & \lambda^2 & \lambda^0\lambda^3 \\ \lambda^1 & 0 & 0 & \lambda^1 \\ \lambda^2 & 0 & 0 & \lambda^2 \\ \lambda^0\lambda^3 & \lambda^1 & \lambda^2 & 2\lambda^3 \end{pmatrix}. \tag{10.30}$$

Thus the gauge transformation does not affect the non-zero components in the TT gauge, which are therefore the only physical ones. On the other hand, the arbitrariness of λ^μ allows us to set all other elements of the polarisation tensor to zero. To see this, we note that the harmonic gauge implies

$$k^\mu \varepsilon_{\mu\nu} = \frac{1}{2} k_\nu \varepsilon^\mu_{\ \mu}. \tag{10.31}$$

Then it follows for $\nu = \{1, 2\}$

$$\varepsilon_{01} + \varepsilon_{31} = \varepsilon_{02} + \varepsilon_{32} = 0, \tag{10.32}$$

while the $\nu = \{0, 3\}$ components result with $\varepsilon_{ij} = -\varepsilon^{\mathrm{i}}_{\ j}$ and $k_\mu = (\omega, 0, 0, -\omega)$ in

$$\varepsilon_{00} + \varepsilon_{30} = \frac{1}{2}(\varepsilon_{00} - \varepsilon_{11} - \varepsilon_{22} - \varepsilon_{33}) = -(\varepsilon_{03} + \varepsilon_{33}). \tag{10.33}$$

Thus we can eleminate four elements of the polarisation tensor. We choose to eleminate ε_{0i}, using first $\varepsilon_{01} = -\varepsilon_{31}$ and $\varepsilon_{02} = -\varepsilon_{32}$. Next we combine the LHS and the RHS of Eq. (10.33) using $\varepsilon_{30} = \varepsilon_{03}$, obtaining

$$\varepsilon_{03} = -\frac{1}{2}(\varepsilon_{00} + \varepsilon_{33}). \tag{10.34}$$

Finally, we use this relation to eleminate ε_{03} in the $\nu = 3$ equation,

$$\frac{1}{2}\varepsilon_{00} - \frac{1}{2}\varepsilon_{33} = \frac{1}{2}(\varepsilon_{00} - \varepsilon_{11} - \varepsilon_{22} - \varepsilon_{33}). \tag{10.35}$$

and thus $\varepsilon_{11} = -\varepsilon_{22}$. Apart from the invariant physical elements, $\tilde{\varepsilon}_{11} = \varepsilon_{11}$ and $\tilde{\varepsilon}_{12} = \varepsilon_{12}$, the remaining four elements of the polarisation tensor transform as

$$\tilde{\varepsilon}^{13} = \varepsilon^{13} + \omega\lambda^1, \qquad \tilde{\varepsilon}^{23} = \varepsilon^{23} + \omega\lambda^2, \tag{10.36}$$
$$\tilde{\varepsilon}^{33} = \varepsilon^{33} + 2\omega\lambda^3, \qquad \tilde{\varepsilon}^{00} = \varepsilon^{00} + 2\omega\lambda^0, \tag{10.37}$$

Since each of the four elements depends on a different λ^μ, they can be set to zero choosing

$$\lambda^0 = -\frac{\varepsilon_{00}}{2\omega}, \qquad \lambda^1 = -\frac{\varepsilon_{13}}{\omega}, \qquad \lambda^2 = -\frac{\varepsilon_{23}}{\omega}, \qquad \lambda^3 = -\frac{\varepsilon_{33}}{2\omega}. \tag{10.38}$$

Helicity We determine now how a metric perturbation $h_{\mu\nu}$ transforms under a rotation with the angle α. We choose the wave propagating in z direction, $\boldsymbol{k} = k\boldsymbol{e}_z$, the TT gauge, and the rotation in the xy plane. Then the general Lorentz transformation Λ becomes

$$\Lambda_\mu^{\ \nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{10.39}$$

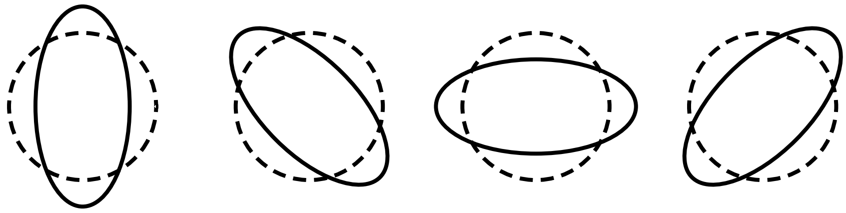


Figure 10.1: The effect of a right-handed polarised gravitational wave on a ring of transverse test particles as function of time; the dashed line shows the state without gravitational wave.

Since $\boldsymbol{k} = k\boldsymbol{e}_z$ and thus $\Lambda_\mu^{\ \nu} k_\nu = k_\mu$, the rotation affects only the polarisation tensor. We rewrite $\varepsilon'_{\mu\nu} = \Lambda_\mu^{\ \rho}\Lambda_\nu^{\ \sigma}\varepsilon_{\rho\sigma}$ in matrix notation, $\varepsilon' = \Lambda\varepsilon\Lambda^T$. It is sufficient to perform the calculation for the xy sub-matrices. The result after introducing circular polarisation states $\varepsilon_\pm = \varepsilon_{11} \pm \mathrm{i}\varepsilon_{12}$ is

$$\varepsilon'^{\mu\nu}_\pm = \exp(\mp 2\mathrm{i}\alpha)\varepsilon^{\mu\nu}_\pm \,. \tag{10.40}$$

The same calculation for a circularly polarised photon gives $\varepsilon'^{\mu}_\pm = \exp(\mp\mathrm{i}\alpha)\varepsilon^{\mu}_\pm$. Any plane wave ψ which is transformed into $\psi' = \mathrm{e}^{-\mathrm{i}h\alpha}\psi$ by a rotation of an angle α around its propagation axis is said to have helicity h. Thus if we say that a photon has spin 1 and a graviton has spin 2, we mean more precisely that electromagnetic and gravitational plane waves have helicity 1 and 2, respectively. Doing the same calculation in an arbitrary gauge, one finds that the remaining, unphysical degrees of freedom transform as helicity 1 and 0 (problem 9.6). In general, a massive tensor field of rank n contains states with helicity $h = -n, \dots, n$, containing thus $2n+1$ polarisation states. In contrast, a massless tensor field of rank n contains only the two polarisation states with maximal helicity, $h = -n$ and $h = n$.

Detection principle of gravitational waves Let us consider the effect of a gravitational wave on a free test particle that is initially at rest, $u^\alpha = (1,0,0,0)$. Then the geodesic equation simplifies to $\dot{u}^\alpha = -\Gamma^\alpha{}_{00}$. The four relevant Christoffel symbols are in the linearised approximation, cf. Eq. (10.6),

$$\Gamma^\alpha{}_{00} = \frac{1}{2}(\partial_0 h_0^\alpha + \partial_0 h_0^\alpha - \partial^\alpha h_{00}) \,. \tag{10.41}$$

We are free to choose the TT gauge in which all component of $h_{\alpha\beta}$ appearing on the RHS are zero. Hence the acceleration of the test particle is zero and its coordinate position is unaffected by the gravitational wave: the TT gauge defines a comoving coordinate system. The physical distance l between two test particles is given by integrating

$$\mathrm{d}l^2 = g_{ab}\mathrm{d}\xi^a\mathrm{d}\xi^b = (h_{ab} - \delta_{ab})\mathrm{d}\xi^a\mathrm{d}\xi^b \,, \tag{10.42}$$

where g_{ab} is the spatial part of the metric and $\mathrm{d}\xi$ the spatial coordinate distance between infinitesimal separated test particles. Hence the passage of a gravitational wave, $h_{\alpha\beta} \propto \varepsilon_{\alpha\beta}\cos(\omega t)$, results in a periodic change of the separation of freely moving test particles. Figure 10.1 shows that a gravitational wave exerts tidal forces, stretching and squashing test particles in the transverse plane. The relative size of the change, $\Delta L/L$, is given by the

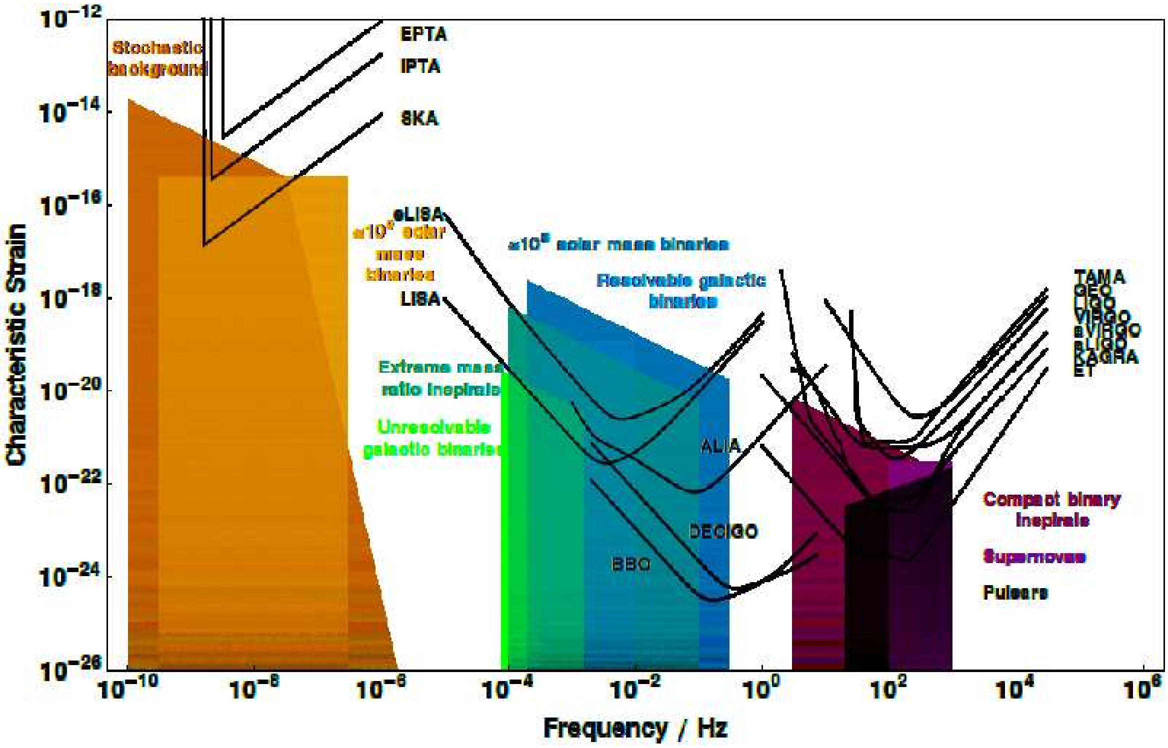


Figure 10.2: Sensitivity of present and future experiments compared to the expectations for the amplitude $h = \Delta L/L$ for various gravitational wave sources.

amplitude h of the gravitational wave. It is this tiny periodic change, $\Delta L/L \lesssim 10^{-21}\cos(\omega t)$, which gravitational wave experiments aim to detect.

There are two basic types of gravitational wave experiments. In the first, one uses the fact that the tidal forces of a passing gravitational wave excite lattice vibrations in a solid state. If the wave frequency is resonant with a lattice mode, the vibrations might be amplified to detectable levels. In the second type of experiment, the free test particles are replaced by mirrors. Between the mirrors, a laser beam is reflected multipe times, thereby increasing the effective length L and thus ΔL, before two beams at $90°$ interfere.

A collection of potential gravitational wave sources is compared to the sensitivity of present and future experiments in Fig. 10.2. As the most promising gravitational wave source the inspiral of binary systems composed of neutron stars or black holes has been suggested. In September 2015, the Advanced Laser Interferometer Gravitational-Wave Observatory (Advanced LIGO) detected such a signal for the first time [1, 4]. Since then, such merger have been observed on a regular basis: Currently, 47 compact binary mergers have been detected. Other, weaker sources in the frequency range ~ 100 Hz are supernova explosions. The coalescence of supermassive black holes during the merger of two galaxies proceeds on much longer time scales. Correspondingly, the frequency of these events is much lower, and experiments searching for them are space-based interferometers. Additionally, a stochastic background of gravitational waves might be produced during inflation and phase transitions in the early universe. This background can be detected by searching for correlated changes in the arrival times of the signals from Galactic pulsars.

10.2 Stress pseudo-tensor for gravity

Stress pseudo-tensor We consider again the splitting (10.1) of the metric, but we take into account now terms of second order in $h_{\alpha\beta}$. We rewrite the Einstein equation by bringing the Einstein tensor on the RHS and adding the linearized Einstein equation,

$$R^{(1)}_{\alpha\beta} - \frac{1}{2}\, R^{(1)}\, \eta_{\alpha\beta} = \kappa T_{\alpha\beta} + \left(-R_{\alpha\beta} + \frac{1}{2}\, R\, g_{\alpha\beta} + R^{(1)}_{\alpha\beta} - \frac{1}{2}\, R^{(1)}\, \eta_{\alpha\beta} \right) . \tag{10.43}$$

The LHS of this equation is the LHS of the usual gravitational wave equation, while the RHS now includes as source not only matter but also the gravitational field itself. It is therefore natural to define

$$R^{(1)}_{\alpha\beta} - \frac{1}{2}\, R^{(1)}\, \eta_{\alpha\beta} = \kappa\, (T_{\alpha\beta} + t_{\alpha\beta}) \tag{10.44}$$

with $t_{\alpha\beta}$ as the *stress pseudo-tensor* for gravity. If we expand all quantities,

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h^{(1)}_{\alpha\beta} + h^{(2)}_{\alpha\beta} + \mathcal{O}(h^3)\,, \qquad R_{\alpha\beta} = R^{(1)}_{\alpha\beta} + R^{(2)}_{\alpha\beta} + \mathcal{O}(h^3)\,, \tag{10.45}$$

we can set, assuming $h_{\alpha\beta} \ll 1$, $R_{\alpha\beta} - R^{(1)}_{\alpha\beta} = R^{(2)}_{\alpha\beta} + \mathcal{O}(h^3)$, etc. Hence we find as stress pseudo-tensor for the metric perturbations $h_{\alpha\beta}$ at $\mathcal{O}(h^3)$

$$t_{\alpha\beta} = -\frac{1}{\kappa} \left(R^{(2)}_{\alpha\beta} - \frac{1}{2}\, R^{(2)}\, \eta_{\alpha\beta} \right) . \tag{10.46}$$

This tensor is symmetric, quadratic in $h_{\alpha\beta}$ and conserved because of the Bianchi identity. Moreover, it transforms as a tensor in Minkowski space. This implies that one can derive global conservation laws for the energy and the angular momentum of the gravitational field, if we assume $|h_{\alpha\beta}| \to 0$ for $x \to \infty$. However, the splitting into a background metric and a perturbation used to derive $t_{\alpha\beta}$ does not hold under general coordinate transformations. Moreover, the interpretation of $t_{\alpha\beta}$ as the local stress tensor of gravity is murky, because it can be transformed at each point to zero by a suitable coordinate transformation. Therefore, no stress tensor for gravity can be defined, and $t_{\alpha\beta}$ is called the stress *pseudo*-tensor for gravity.

In the case of gravitational waves, we may expect that averaging $t_{\alpha\beta}$ over a volume large compared to the wave-length considered solves this problem. Moreover, such an averaging simplifies the calculation of $t_{\alpha\beta}$, since all terms odd in kx cancel. Nevertheless, the calculation is extremly messy. We will use therefore a short-cut via the following two digressions.

Quadratic Einstein-Hilbert action We construct the action of gravity quadratic in $h_{\mu\nu}$ from the wave equation (10.19), following the same logic as in the Maxwell case. We multiply by a variation $\delta h^{\mu\nu}$ and integrate, obtaining

$$0 = \delta S^{\text{harm}}_{\text{EH}} + \delta S^{\text{harm}}_{\text{m}} = \int \mathrm{d}^4x \sqrt{|g|}\, \left[\frac{1}{4\kappa} \delta h^{\mu\nu} \Box \bar{h}_{\mu\nu} + \frac{1}{2} \delta h^{\mu\nu} T_{\mu\nu} \right] . \tag{10.47}$$

Here, we divided by two such that we obtain the correctly normalised stress tensor of matter using Eq. (8.44). Now we can restrict ourselves again to stay in Minkowski space, setting $\sqrt{|g|} = 1$. Our aim is to massage the first term into a form similar to the kinetic energy of a scalar field: We insert first the definition of $\bar{h}_{\mu\nu}$, use then the product rule and perform finally a partial integration,

$$\delta h^{\mu\nu}\Box\bar{h}_{\mu\nu} = \delta h^{\mu\nu}\Box h_{\mu\nu} - \frac{1}{2}\delta h^{\mu\nu}\eta_{\mu\nu}\Box h = \delta h^{\mu\nu}\Box h_{\mu\nu} - \frac{1}{2}\delta h\Box h \tag{10.48a}$$

$$= \delta\left[\frac{1}{2}h^{\mu\nu}\Box h_{\mu\nu} - \frac{1}{4}h\Box h\right] = -\delta\left[\frac{1}{2}(\partial_\kappa h^{\mu\nu})^2 - \frac{1}{4}(\partial_\kappa h)^2\right]. \tag{10.48b}$$

Thus the quadratic Einstein-Hilbert action in the harmonic gauge becomes

$$S_{\rm EH}^{\rm harm} = -\frac{1}{32\pi G}\int \mathrm{d}^4x\,\left[\frac{1}{2}(\partial_\rho h^{\mu\nu})^2 - \frac{1}{4}(\partial_\rho h)^2\right]. \tag{10.49}$$

Specializing (10.49) to the TT gauge, we obtain

$$S_{\rm EH}^{\rm TT} = -\frac{1}{32\pi G}\int \mathrm{d}^4x\,\frac{1}{2}\left(\partial_\rho h_{ij}\right)^2. \tag{10.50}$$

We can express an arbitrary polarisation state as the sum over the polarisation tensors for linear polarised waves,

$$h_{\mu\nu} = \sum_{a=+,\times} h^{(a)}\varepsilon_{\mu\nu}^{(a)}. \tag{10.51}$$

Inserting this decomposition into (10.50) and using $\varepsilon_{\mu\nu}^{(a)}\varepsilon^{\mu\nu(b)} = \delta^{ab}$, the action becomes

$$S_{\rm EH}^{\rm TT} = -\frac{1}{32\pi G}\sum_a\int \mathrm{d}^4x\,\frac{1}{2}\left(\partial_\rho h^{(a)}\right)^2. \tag{10.52}$$

Thus the gravitational action in the TT gauge consists of two degrees of freedom, which we can choose as h^+ and $h^\times$. These two smplitudes determine the relative contribution of the two polarisation states. Apart from the pre-factor, the action is the same as the one of two scalar fields. This means that we can shortcut many calculations involving gravitational waves by using simply the corresponding results for scalar fields. We can understand this equivalence by recalling that the part of the action action quadratic in the fields just enforces the relativistic energy–momentum relation via a Klein–Gordon equation for each field component. The remaining content of (10.49) is just the rule how the unphysical components in $h^{\mu\nu}$ have to be eliminated. In the TT gauge, we have already applied this information, and thus the two scalar wave equations for $h^{(\pm)}$ summarise the Einstein equation at $\mathcal{O}(h^2)$.

Averaged stress tensor The stress tensor of a scalar field is in general given by

$$T_{\alpha\beta} = \frac{2}{\sqrt{|g|}}\frac{\delta S_{\rm m}}{\delta g^{\alpha\beta}} = \partial_\alpha\phi\partial_\beta\phi - g_{\alpha\beta}\mathscr{L}. \tag{10.53}$$

We consider now a free field, i.e. set now $V(\phi) = 0$, and take the average over a volume Ω large compared to the typical wavelength of the field,

$$\langle T_{\alpha\beta}\rangle = \frac{1}{\Omega}\int \mathrm{d}^4x\,T_{\alpha\beta} = \langle\partial_\alpha\phi\partial_\beta\phi\rangle - \frac{1}{2}\eta_{\alpha\beta}\langle(\partial_\rho\phi)^2\rangle. \tag{10.54}$$

Performing a partial integration of the second term, we can drop the surface term, and use then the equation of motion,

$$\langle(\partial_\rho\phi)^2\rangle = -\langle\phi\Box\phi\rangle = 0. \tag{10.55}$$

Hence $\langle T_{\alpha\beta}\rangle = \langle \partial_\alpha\phi\partial_\beta\phi\rangle$. Comparing now S_{KG} and S_{EH} in the TT gauge suggests that the averaged stress pseudo-tensor of the gravitational field is given in this gauge by

$$\langle t_{\alpha\beta}\rangle = \frac{1}{32\pi G}\langle \partial_\alpha h_{ij}\partial_\beta h^{ij}\rangle\,. \tag{10.56}$$

Bootstrap Derivation of full Einstein equation out of linear ansatz.

10.3 Emission of gravitational waves

The first, indirect, evidence for gravitational waves has been the observation of close neutron star-neutron star binaries showing that such systems loose energy, leading to a shrinkage of their orbit with time. These observations are consistent with the prediction for the energy loss by the emission of gravitational waves.

The steps in deriving this energy loss formula are similar to the corresponding derivation for the dipole emission formula of electromagnetic radiation. Step one, the derivation of the Green function for the wave equation (10.19) is exactly the same, after having fixed the gauge freedom. In the second step, we have to connect the amplitude of the field at large distances ("in the wave zone") to the source, i.e. the current j^α and the stress tensor $T^{\alpha\beta}$, respectively. Finally, we use the connection between the field and its (pseudo) stress tensor ($T^{\alpha\beta}_{\text{em}}$ or $t^{\alpha\beta}$) to derive the energy flux through a sphere around the source.

Quadrupol formula Gravitational waves in the linearized approximation fulfil the superposition principle. Hence, if the solution for a point source is known,

$$-\Box_x G(x-x') = \delta(x-x')\,, \tag{10.57}$$

the general solution can be obtained by integrating the Green function over the sources,

$$\bar{h}_{\alpha\beta}(x) = -2\kappa\int \mathrm{d}^4x' G(x-x')T_{\alpha\beta}(x')\,. \tag{10.58}$$

The Green function $G(x-x')$ is not completely specified by Eq. (10.57): We can add solutions of the homogeneous wave equation and we have to specify how the poles of $G(x-x')$ are treated. In classical physics, one chooses the retarded Green function $G^{(+)}(x-x')$ defined by

$$G^{(+)}(x-x') = -\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|}\delta[|\boldsymbol{x}-\boldsymbol{x}'|-(t-t')]\vartheta(t-t')\,, \tag{10.59}$$

picking up the contributions along the past light-cone; for a derivation see appendix 10.B.

Inserting the retarded Green function into Eq. (10.58), we can perform the time integral using the delta function and obtain

$$\bar{h}_{\alpha\beta}(x) = 4G\int \mathrm{d}^3x'\,\frac{T_{\alpha\beta}(t-|\boldsymbol{x}-\boldsymbol{x}'|,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\,. \tag{10.60}$$

The retarded time $t_r \equiv t-|\boldsymbol{x}-\boldsymbol{x}'|$ denotes the emission time t_r of a signal emitted at $\boldsymbol{x}'$ that reaches $\boldsymbol{x}$ at time t propagating with the speed of light.

We perform now a Fourier transformation from time to angular frequency,

$$\bar{h}_{\alpha\beta}(\omega,\boldsymbol{x}) = \frac{1}{\sqrt{2\pi}}\int \mathrm{d}t\,\mathrm{e}^{\mathrm{i}\omega t}\bar{h}_{\alpha\beta}(t,\boldsymbol{x}) = \frac{4G}{\sqrt{2\pi}}\int \mathrm{d}t\int \mathrm{d}^3x'\,\mathrm{e}^{\mathrm{i}\omega t}\frac{T_{\alpha\beta}(t_r,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\,. \tag{10.61}$$

Next we change from the integration variable t to t_r,

$$\bar{h}_{\alpha\beta}(\omega,\boldsymbol{x}) = \frac{4G}{\sqrt{2\pi}}\int \mathrm{d}t_r\int \mathrm{d}^3x'\,\mathrm{e}^{\mathrm{i}\omega t_r}\mathrm{e}^{\mathrm{i}\omega|\boldsymbol{x}-\boldsymbol{x}'|}\frac{T_{\alpha\beta}(t_r,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}, \tag{10.62}$$

and introduce the Fourier transformed $T_{\alpha\beta}(\omega,\boldsymbol{x}')$,

$$\bar{h}_{\alpha\beta}(\omega,\boldsymbol{x}) = 4G\int \mathrm{d}^3x'\,\mathrm{e}^{\mathrm{i}\omega|\boldsymbol{x}-\boldsymbol{x}'|}\frac{T_{\alpha\beta}(\omega,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\,. \tag{10.63}$$

We proceed using the same approximations as in electrodynamics: We restrict ourselves to slowly moving, compact sources observed in the wave zone and choose the coordinate system such that $|\boldsymbol{x}'| \ll |\boldsymbol{x}|$. Then most radiation is emitted at frequencies such that $|\boldsymbol{x}-\boldsymbol{x}'| \simeq |\boldsymbol{x}| \equiv r$ and thus

$$\bar{h}_{\alpha\beta}(\omega,\boldsymbol{x}) = 4G\,\frac{\mathrm{e}^{\mathrm{i}\omega r}}{r}\int \mathrm{d}^3x'\,T_{\alpha\beta}(\omega,\boldsymbol{x}')\,. \tag{10.64}$$

Finally, we Fourier transform first back to the observation time t, and introduce then the stress tensor evaluated at the retarded time $t_r = t - r$,

$$\bar{h}_{\alpha\beta}(t,\boldsymbol{x}) = \frac{1}{\sqrt{2\pi}}\int \mathrm{d}\omega\,\mathrm{e}^{-\mathrm{i}\omega t}\bar{h}_{\alpha\beta}(\omega,\boldsymbol{x}) \tag{10.65a}$$

$$= \frac{4G}{\sqrt{2\pi}\,r}\int \mathrm{d}\omega\,\mathrm{e}^{-\mathrm{i}\omega(t-r)}\int \mathrm{d}^3x'\,T_{\alpha\beta}(\omega,\boldsymbol{x}')\,. \tag{10.65b}$$

$$= \frac{4G}{r}\int \mathrm{d}^3x'\,T_{\alpha\beta}(t_r,\boldsymbol{x}')\,. \tag{10.65c}$$

Next we want to eleminate all elements of $T_{\alpha\beta}$ except T_{00}. We note first that $T_{00} = \rho$, $T_{0i} = \rho u_i$, and $T_{ij} = \rho u_i u_j$ implies $h_{00} = 4GM/r$, $h_{0i} \propto P_i$, while h_{ij} is proportional to the stresses. Thus h_{00} contains the constant monopole term generated by the total mass of the system. The terms h_{0i} are sourced by the total momentum P_i of the system. Hence they can be set to zero choosing the center-of-mass as the origin of the coordinate system. Thus we have only to express the elements T_{ab} via T_{00}. We use first (flat-space) energy-momentum conservation,

$$\frac{\partial}{\partial t}T^{00} + \frac{\partial}{\partial x^b}T^{0b} = 0\,, \tag{10.66a}$$

$$\frac{\partial}{\partial t}T^{a0} + \frac{\partial}{\partial x^b}T^{ab} = 0\,. \tag{10.66b}$$

Then we differentiate Eq. (10.66a) with respect to time and use Eq. (10.66b), obtaining

$$\frac{\partial^2}{\partial t^2}T^{00} = -\frac{\partial^2}{\partial x^b\partial t}T^{0b} = \frac{\partial^2}{\partial x^a\partial x^b}T^{ab}\,. \tag{10.67}$$

Multiplying with $x^a x^b$ and integrating gives then

$$\frac{\partial^2}{\partial t^2}\int \mathrm{d}^3x\;x^a x^b T^{00} = \int \mathrm{d}^3x\;x^a x^b\frac{\partial^2}{\partial x^i x^j}T^{ij} = 2\int \mathrm{d}^3x\;T^{ab}\,. \tag{10.68}$$

Here we performed two partial integerations on the RHS, dropping the surface terms as the source is compact. In the harmonic gauge, we need to calculate only the components $h^{ij}(t_r, \boldsymbol{x})$. We define as quadrupole moment of the source stress tensor

$$I^{ab}(t_r) = \int d^3x\, x^a x^b T^{00}(t_r, \boldsymbol{x})\,. \tag{10.69}$$

Then the *quadrupole formula* for the emission of gravitational waves results,

$$\boxed{\bar{h}_{ab}(t, \boldsymbol{x}) = \frac{2G}{c^6 r}\, \ddot{I}_{ab}(t_r)\,,} \tag{10.70}$$

where we added also c.

Example 10.1: Equal mass binary system on circular orbits:
We choose coordinates such that the orbits are centered at the origin in the xy-plane. Then

$$x_1 = -x_2 = R\cos\Omega t\,, \qquad\qquad y_1 = -y_2 = R\sin\Omega t. \tag{10.71}$$

The corresponding energy density is

$$T^{00} = M\delta(z)[\delta(x - R\cos\Omega t)\delta(y - R\sin\Omega t) + \delta(x + R\cos\Omega t)\delta(y + R\sin\Omega t)]. \tag{10.72}$$

The quadrupole moments follow as

$$\begin{aligned}
I_{xx} &= 2MR^2\cos^2\Omega t = MR^2(1+\cos 2\Omega t) && (10.73\text{a})\\
I_{yy} &= 2MR^2\sin^2\Omega t = MR^2(1-\cos 2\Omega t) && (10.73\text{b})\\
I_{xy} &= I_{yx} = 2MR^2\cos\Omega t\sin\Omega t = MR^2\sin 2\Omega t && (10.73\text{c})
\end{aligned}$$

and $I_{iz} = 0$. Differentiating twice gives $\ddot{I}_{ij} = -4\Omega^2 I_{ij}$ and thus the GW amplitude follows as

$$\bar{h}_{ij}(t, \boldsymbol{x}) = -\frac{8GM}{r}\,(\Omega R)^2 \begin{pmatrix} \cos 2\Omega t_r & \sin 2\Omega t_r & 0 \\ \sin 2\Omega t_r & -\cos 2\Omega t_r & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{10.74}$$

The frequency ω of the GW is twice the rotation frequency of the stars, $\omega = 2\Omega$, indicating again that the helicity of GWs is $h = 2$. Neglecting the oscillating factor, the amplitude is

$$h = \frac{8GM}{r}\,(\Omega R)^2$$

For a numerical estimate, it is convinient to eleminate $(\Omega R)^2$ using the Keplerian velocity and to replace $2GM$ by the Schwarzschild radius R_S,

$$h \simeq 2\frac{R_s^2}{ar}.$$

The amplitude is maximised for $a \simeq R_S$, i.e. during the coalesence of two black holes. Close to the merging black holes, the amplitude is $h \sim 1$ and perturbation theory is clearly not valid. At typical distance of $r = 100\,\mathrm{Mpc}$, it follows $h \simeq 10^{-22}$ for BHs with $M = 20M_\odot$.

Finally, we want to determine the physical part of the ampltiude, i.e. fixing the TT gauge. In general, the projection operator defined in the appendix 10.A has to be applied to obtain the physical states of the GW. If the coordinate system is aligned with the wave-vector of the GW, we can set the non-transverse components zo zero by hand, and subtract then half the resulting trace. In the TT gauge, we can use also that $\bar{h}_{ij}^{\rm TT} = h_{ij}^{\rm TT}$.

For an observer along the z direction, the result is already in the TT gauge, which we can express as

$$h_{\mu\nu}^{\rm tt} \propto \Re[(\varepsilon_{\mu\nu}^{(1)} - {\rm i}\varepsilon_{\mu\nu}^{(2)})\exp(2{\rm i}\Omega t_r)].$$

This corresponds to a right-handed circularly polarized wave, $\varepsilon_{\mu\nu}^{(-)} = \varepsilon_{\mu\nu}^{(1)} - {\rm i}\varepsilon_{\mu\nu}^{(2)}$.

Next consider an observer on the x axis. Transforming to the TT form, we obtain

$$\bar{h}_{ij}(t,\boldsymbol{x}) = \frac{4GM}{r}\,(\Omega R)^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\cos 2\Omega t_r & 0 \\ 0 & 0 & \cos 2\Omega t_r \end{pmatrix}.$$

This corresponds to a linearly polarized wave, $\propto \varepsilon_{\mu\nu}^{(1)}$.

Since $h^{\alpha\beta}$ is traceless, the trace of $I^{\alpha\beta}$ does not produce gravitational waves: It is connected to the scalar helicity component of a gravitational wave which is non-zero only in extension of Einstein gravity. Thus it is more convenient to replace $I^{\alpha\beta}$ by the reduced (trace-less, irreducible) quadrupole moment

$$Q^{ab} = \int d^3x\,\left[x^a x^b - \frac{1}{3}\delta^{ab}r^2\right] T^{00}(x)\,. \tag{10.75}$$

Our derivation neglected perturbations of flat space and seems therefore not applicable to a self-gravitating system. However, our final result depends only on the motion of the particles, not how it is produced. An analysis at next order in perturbation theory shows indeed that our result applies to self-gravitating systems like binary stars.

Note the following peculiarity of a gravitational wave experiment: Such an experiment measures the amplitude $h^{ab} \propto 1/r$ of a metric perturbation, while the sensitivity of all other experiments (light, neutrinos, cosmic rays, ...) is proportional to the energy flux $\propto 1/r^2$ of radiation. This difference is connected to the fact that a gravitational wave is caused by the coherent motion of the source, and can be thus observed as a coherent wave over time. In particular, one can measure the phase of h_{ab} as function of time. In contrast, light observed from an astrophysical source is a incoherent superposition of individual photons. As a result, increasing the sensitivity of a gravitational wave detector by a factor ten increases the number of potential sources by a factor 1000, in contrast to a factor $10^{3/2}$ for other detectors.

One may wonder if this behavior contradicts the fact that also the energy flux of a gravitational wave follows as $1/r^2$ law. However, the energy dissipated from a gravitational wave crossing the Earth (including our experimental set-up) is extremely tiny, while the energy density of gravitational wave with amplitude as small as $h \sim 10^{-22}$ is surprisingly large (check it e.g. with (10.80)).

Energy loss We evaluate now the pseudo stress tensor given in Eq. (10.56) for a plane-wave,

$$h_{ij} = A_{ij}\cos(kx), \tag{10.76}$$

with amplitudes A_{ij} which we choose to be real. Using $\langle\sin^2(kx)\rangle = 1/2$, we obtain

$$\langle t_{\alpha\beta}\rangle = \frac{1}{64\pi G} k_\alpha k_\beta A_{ij}A^{ij}\,. \tag{10.77}$$

The energy-flux $\mathcal{F}$, i.e. the energy crossing an unit area per unit time, in the direction $\boldsymbol{n}$ is in general $\mathcal{F} = -ct^{0i}n_i$. For a plane-wave with wave-vector k^μ, it follows

$$\mathcal{F} = -t^{0i}\hat{k}_i = \frac{1}{64\pi G} k^0 k^i \hat{k}_i A_{ij} A^{ij} = ct^{00}, \tag{10.78}$$

where we used $k^0 = -k^i\hat{k}_i$. Thus we got the reasonable result that the energy-flux is simply the energy-density t^{00} multiplied with the wave-speed c. Expressing the energy flux as a sum over linearly polarised waves,

$$h_{\mu\nu} = \sum_{a=+,\times} h^{(a)} \varepsilon^{(a)}_{\mu\nu}, \tag{10.79}$$

it follows with $a = h^{(+)}$ and $b = h^{(\times)}$

$$\mathcal{F} = \frac{\omega^2}{32\pi G}\left(a^2 + b^2\right). \tag{10.80}$$

In the case of a spherical wave emitted from the origin, we choose $\boldsymbol{n} = \boldsymbol{e}_r$. Then

$$\mathcal{F}(\boldsymbol{e}_r) = \frac{1}{32\pi G}\langle t^{0i} n_i\rangle = \frac{1}{32\pi G}\langle(\partial_t h_{ij})(\boldsymbol{n}\cdot\boldsymbol{\nabla})h^{ij}\rangle = \frac{1}{32\pi G}\langle(\partial_t h_{ij})(\partial_r h^{ij})\rangle. \tag{10.81}$$

Since the quadrupol moment is a function of retarded time, $Q_{ij} = h_{ij}(t_r, \boldsymbol{x})$ with $t_r = t - r$, we can replace the space with a time derivative, $\partial_r = -\partial_t$. Then

$$\partial_t h_{ij} = \frac{2G}{r}\dddot{Q}_{ij} \tag{10.82}$$

$$\partial_r h_{ij} = -\frac{2G}{r^2}\ddot{Q}_{ij} - \frac{2G}{r}\dddot{Q}_{ij} \simeq -\frac{2G}{r}\dddot{Q}_{ij} \tag{10.83}$$

Thus

$$\mathcal{F}(\boldsymbol{e}_r) = \frac{G}{8\pi r^2}\dddot{Q}^{\mathrm{TT}}_{ij}\dddot{Q}^{ij}_{\mathrm{TT}} \tag{10.84}$$

and

$$L_{\mathrm{gr}} = -\frac{\mathrm{d}E}{\mathrm{d}t} = -\int \mathrm{d}\Omega r^2\, \mathcal{F}(\boldsymbol{e}_r) = \frac{G}{8\pi}\int \mathrm{d}\Omega\, \dddot{Q}^{\mathrm{TT}}_{ij}\dddot{Q}^{ij}_{\mathrm{TT}}. \tag{10.85}$$

The remaining tasks, performing the TT projection and carrying out the integrals, are deferred to the appendix. Inserting the results found there in the quadrupole formula, one finds

$$L_{\mathrm{gr}} = -\frac{\mathrm{d}E}{\mathrm{d}t} = -\int \mathrm{d}\Omega r^2 \mathcal{F}(\boldsymbol{e}_r) = \frac{G}{5c^5}\dddot{Q}_{ij}\dddot{Q}^{ij}, \tag{10.86}$$

where we show explicitely the dependence on c. Finally, we can express Q_{ij} through I_{ij}, obtaining

$$\dddot{Q}_{ij}\dddot{Q}^{ij} = \left(\dddot{I}_{ij} - \frac{1}{3}\delta_{ij}\dddot{I}\right)\left(\dddot{I}^{ij} - \frac{1}{3}\delta^{ij}\dddot{I}\right) = \dddot{I}_{ij}\dddot{I}^{ij} - \frac{2}{3}\dddot{I}^2 + \frac{1}{3}\dddot{I}^2 = \dddot{I}_{ij}\dddot{I}^{ij} - \frac{1}{3}\dddot{I}^2 \tag{10.87}$$

with $I \equiv I_{ij}\delta^{ij}$.

10.4 Gravitational waves from binary systems

10.4.1 Weak field limit

The emission of gravitational radiation is negligible for all systems where Newtonian gravity is a good approximation. One of the rare examples where general relativistic effects can become important are close binary systems of compact stars. The first such example was found 1974 by Hulse and Taylor who discovered a pulsar in a binary system via the Doppler-shift of its radio pulses. The extreme precision of the periodicity of the pulsar signal makes this binary system to an ideal laboratory to test various effect of special and general relativity:

- The pulsar's orbital speed changes by a factor of four during its orbit and allows us to test the usual (special relativistic) Doppler effect.
- At the same time, the gravitational field alternately strengthens at periastron and weakens at apastron, leading to a periodic gravitational redshift of the pulse.
- The small size of the orbit leads to a precession of the Perihelion by 4.2°/yr.
- The system emits gravitational waves and looses thereby energy. As a result the orbit of the binary shrinks by 4mm/yr.

In the following, we will derive some of these predictions.

Gravitational wave emission on eccentric orbits In the first step, we derive the instantanuous energy loss of the binary system due to gravitational wave emission. Since we assume that the losses are small, we can treat the orbital parameters a and e as constant. The quadrupole moments follow for an orbit in the xy plane as

$$I_{xx} = m_1 x_1^2 + m_2 x_2^2 = \mu r^2 \cos^2\phi, \tag{10.88a}$$

$$I_{yy} = \mu r^2 \sin^2\phi, \tag{10.88b}$$

$$I_{xy} = \mu r^2 \cos\phi \sin\phi, \tag{10.88c}$$

$$I \equiv I_{xx} + I_{yy} = \mu r^2. \tag{10.88d}$$

In order to find the derivates of I_{ik}, we have to determine first $\dot{r}$ and $\dot{\phi}$. Eleminating L using Eq. (2.48) we obtain

$$\dot{\phi} = \frac{L}{\mu r^2} = \frac{[a(1-e^2)M]^{1/2}}{r^2}. \tag{10.89}$$

Differentiating then Eq. (2.46) and inserting $\dot{\phi}$, we find

$$\dot{r} = \frac{a(1-e^2)e\sin\phi\,\dot{\phi}}{(1+e\cos\phi)^2} = \left(\frac{M}{a(1-e^2)}\right)^{1/2} e\sin\phi. \tag{10.90}$$

We are now in the position to calculate, e.g.,

$$\dot{I}_{xx} = 2\mu\cos\phi\left(r\dot{r}\cos\phi - r^2\dot{\phi}\sin\phi\right) \tag{10.91}$$

as function of r and ϕ. With

$$r^2\dot{\phi} = [a(1-e^2)M]^{1/2} \equiv A \tag{10.92}$$

and

$$r\dot{r} = A\frac{e\sin\phi}{1+e\cos\phi}, \tag{10.93}$$

it follows

$$r\dot{r}\cos\phi - r^2\dot{\phi}\sin\phi = A\sin\phi\left(\frac{e\sin\phi}{1+e\cos\phi} - 1\right) = -\frac{Mr}{[a(1-e^2)]^{1/2}}\sin\phi. \tag{10.94}$$

Thus we obtain

$$\dot{I}_{xx} = -\frac{2m_1m_2r}{[a(1-e^2)M]^{1/2}}\,\cos\phi\sin\phi. \tag{10.95}$$

The calculation of the other elements of the quadrupol tensor and the higher derivatives proceeds in the same way, leading to

$$\ddot{I}_{xx} = -\frac{2m_1m_2}{a(1-e^2)}\left(\cos 2\phi + e\cos^3\phi\right), \tag{10.96a}$$

$$\dddot{I}_{xx} = \frac{2m_1m_2}{a(1-e^2)}\left(2\sin 2\phi + 3e\cos^2\phi\sin\phi\right)\dot{\phi}, \tag{10.96b}$$

$$\dot{I}_{yy} = \frac{2m_1m_2}{[a(1-e^2)M]^{1/2}}r\left(\cos\phi\sin\phi + e\sin\phi\right), \tag{10.96c}$$

$$\ddot{I}_{yy} = \frac{2m_1m_2}{a(1-e^2)}\left(\cos 2\phi + e\cos\phi + e\cos^3\phi + e^2\right), \tag{10.96d}$$

$$\dddot{I}_{yy} = -\frac{2m_1m_2}{a(1-e^2)}\left(2\sin^2\phi + e\sin\phi + 3e\cos^2\phi\sin\phi\right)\dot{\phi}, \tag{10.96e}$$

$$\dot{I}_{xy} = \frac{m_1m_2r}{[a(1-e^2)M]^{1/2}}\left(\cos^2\phi - \sin^2\phi + e\cos\phi\right) \tag{10.96f}$$

$$\ddot{I}_{xy} = -\frac{2m_1m_2}{a(1-e^2)}\left(\sin 2\phi + e\sin\phi + e\sin\phi\cos^2\phi\right) \tag{10.96g}$$

$$\dddot{I}_{xy} = -\frac{2m_1m_2}{a(1-e^2)}\left(2\cos 2\phi - e\cos\phi + 3e\cos^3\phi\right)\dot{\phi}, \tag{10.96h}$$

$$\dddot{I} = \dddot{I}_{xx} + \dddot{I}_{yy} = -\frac{2m_1m_2}{a(1-e^2)}\,e\sin\phi\dot{\phi}. \tag{10.96i}$$

Inserting these expressions into

$$L_{\mathrm{gr}} = -\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{G}{5}\dddot{Q}_{ij}\dddot{Q}^{ij} = \frac{G}{5}\left(\dddot{I}^2_{xx} + 2\dddot{I}^2_{xy} + \dddot{I}^2_{yy} - \frac{1}{3}\dddot{I}^2\right) \tag{10.97}$$

results in

$$-\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{8m_1^2m_2^2}{15a(1-e^2)^2}\,\left[12(1+e\cos\phi)^2 + e^2\sin^2\phi\right]\dot{\phi}^2. \tag{10.98}$$

for the instantanous energy loss. In order to obtain the average energy loss, we have to average this expression over one period,

$$-\left\langle\frac{\mathrm{d}E}{\mathrm{d}t}\right\rangle = -\frac{1}{T}\int_0^T \mathrm{d}t\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{1}{T}\int_0^{2\pi}\frac{\mathrm{d}\phi}{\dot{\phi}}\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{32}{5}\,\frac{m_1^2m_2^2M}{a^5}\,f(e) \tag{10.99}$$

with

$$f(e) = \frac{1+\frac{73}{24}e^2+\frac{37}{96}e^4}{(1-e^2)^{7/2}} \tag{10.100}$$

Time evolution Now we can determine how the orbital parameters change over time. The major axis a decreases with time as

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{m_1 m_2}{2E^2}\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{2a^2}{m_1 m_2}\frac{\mathrm{d}E}{\mathrm{d}t}, \tag{10.101}$$

or averaged over one period,

$$-\left\langle \frac{\mathrm{d}a}{\mathrm{d}t} \right\rangle = \frac{64}{5}\frac{m_1^2 m_2^2 M}{a^3} f(e). \tag{10.102}$$

The orbital period changes as

$$\frac{\dot{P}}{P} = \frac{3}{2}\frac{\dot{E}}{E} = -\frac{3}{2}\frac{\dot{a}}{a} = -\frac{96}{5}\frac{m_1^2 m_2^2 M}{a^4} f(e). \tag{10.103}$$

What remains to do is to work out the change of the eccentricity,

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{M}{m_1^3 m_2^3\, e}\left(L^2\frac{\mathrm{d}E}{\mathrm{d}t} + 2EL\frac{\mathrm{d}L}{\mathrm{d}t}\right). \tag{10.104}$$

Determining the loss $\dot{L}$ of angular momentum due to gravitational wave emission is more involved than the energy loss: Since $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ contains a factor r, we have to take into account terms $h \propto 1/r^2$ what requires to include term of $\mathcal{O}(h^3)$ in $\langle t_{\mu\nu}\rangle$. This calculation was performed first by Peters 1964 [6, 7], who obtained

$$-\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{2G}{5}\varepsilon^{3ij}\ddot{Q}_i^k \dddot{Q}_{jk}, \tag{10.105}$$

where it is assumed that the orbit is in the xy plane. Alternatively, we can use the back-reaction force as shown in the next box. Either way, the instantaneous loss of angular momentum follows as

$$-\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{G}{5}\left[\ddot{I}_{xy}(\dddot{I}_{yy} - \dddot{I}_{xx}) + \dddot{I}_{xy}(\ddot{I}_{xx} - \ddot{I}_{yy})\right], \tag{10.106}$$

leading to

$$-\left\langle \frac{\mathrm{d}e}{\mathrm{d}t} \right\rangle = \frac{304}{15}\frac{m_1 m_2 M e}{a^4} g(e) \tag{10.107}$$

with

$$g(e) = \frac{1 + \frac{121}{304}e^2}{(1-e^2)^{7/2}}. \tag{10.108}$$

Remark 10.1: Back reaction:—The emission of gravitational waves leads to a back-reaction force, which slows down the the star. We can derive the time-averaged back-reaction force, asking that

$$P = \langle \boldsymbol{F} \cdot \boldsymbol{v} \rangle = -L_{\mathrm{gr}} = -\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{G}{5}\dddot{Q}_{ij}\dddot{Q}^{ij} \tag{10.109}$$

is valid. The time-average of any total derivative $\mathrm{d}F/\mathrm{d}t$ vanishes,

$$\left\langle \frac{\mathrm{d}F}{\mathrm{d}t} \right\rangle = \frac{1}{T}\int_0^T \mathrm{d}t \frac{dF}{\mathrm{d}t} = \frac{1}{T}[F((T) - F(0)] \to 0, \tag{10.110}$$

if F is a bounded function. This implies that we can perform partial integrations in time-averages.

Since $\dot{Q}_{ij}$ is linear in the velocity,

$$\dot{Q}_{ij} = \sum_a \left(x_i v_j + v_i x_j - \frac{2}{3}\delta_{ij}\boldsymbol{r}\cdot\boldsymbol{v} \right) \tag{10.111}$$

we switch the time derivaties as

$$-\left\langle \frac{\mathrm{d}E}{\mathrm{d}t} \right\rangle = \frac{G}{5}\left\langle \dddot{Q}_{ij}\dddot{Q}^{ij} \right\rangle = \frac{G}{5}\left\langle \frac{\mathrm{d}Q_{ij}}{\mathrm{d}t}\frac{\mathrm{d}^5Q^{ij}}{\mathrm{d}t^5} \right\rangle. \tag{10.112}$$

Thus the back-reaction is given by

$$F_i = -\frac{2Gm}{5}\frac{\mathrm{d}^5Q_{ij}}{\mathrm{d}t^5}x^j = \tag{10.113}$$

Now we can obtain the loss of angular momentum as

$$\left\langle \frac{\mathrm{d}L}{\mathrm{d}t} \right\rangle = \langle \boldsymbol{r}\times\boldsymbol{f} \rangle. \tag{10.114}$$

Hulse-Taylor pulsar The binary system found by Hulse and Taylor consists of a pulsar with mass $m_1 = 1.44M_\odot$ and a companion with mass $m_2 = 1.34M_\odot$. Their orbital period is P = 7h40min on an orbit with rather strong eccentricity, $e = 0.617$. In this case, the emission of gravitational radiation is strongly enhanced compared to an circular orbit. Let us now compare the observed change in the orbit of the binary with the prediction of general relativity. The prediction of Einstein's general relativity,

$$\dot{P}(e)\Big|_{\mathrm{th}} = f(e)\,\dot{P}(0) \simeq 11.7\times\dot{P}(0) \simeq (-2.403\pm 0.002)\times 10^{-12}, \tag{10.115}$$

is in excellent agreement with the observed value,

$$\dot{P}(e)\Big|_{\mathrm{obs}} \simeq (-2.4184\pm 0.0009)\times 10^{-12} \tag{10.116}$$

A comparison of the predicted and observed accumulated shift in the period is shown in Fig. **??**.

Example 10.2: Circularisation/some numbers:

10.4.2 Strong field limit and binary merger

Post-Newtonian approximation and beyond In the previous section, we used the orbits obtained in the Newtonian limit. This approximation corresponds to the limit $c\to\infty$ and neglects all retardation effects. Since the energy loss due the gravitational wave emission is of order $\mathcal{O}(1/c^5)$, cf. with (Eq. 10.86), we should be able to improve this approximation using a Lagrange function only of coordinates and velocities but including post-Newtonian (PN) corrections up to order $(v/c)^4$. The first relativistic terms, at the 1PN order, were derived in 1937–39, the 2PN approximation was tackled by Ohta et al. in 1973–74, while results for the 3PN order were obtained starting from 1998. Alternatives to this brûte-de-force approach such as the effective one-body theory have been developped where one maps the two-body problem of GR onto an one-body problem in an effective metric. However, all these approaches are restricted to the inspiral phase of a merger. In contrast, numerical simulations of the merging phase of binaries give accurate results, but can take months even on super-computers. Thus their extension towards early times of the inspiral phase is restricted, and the set of parameters $\{m_1, m_2, \boldsymbol{s}_1, \boldsymbol{s}_2, e, \ldots\}$ for which simulations exist is sparse. For instance, simulations for large mass ratios m_1/m_2 are numerically still prohibitive. As a results, a combination of the different approaches is needed to describe the coalesence of binaries accurately.

Qualitative discussion Let us now discuss qualitatively the final stage in the time evolution of a close binary system. We can assume that the emission of GWs has lead to a circularisation of the orbits. Then

$$L_{\rm gw} = \frac{32}{5}\frac{G^4\mu^2M^3}{a^5}. \tag{10.117}$$

Next we can relate the relative changes per time in the orbital period P, the separation a and the energy E using $E \propto 1/a$ and $P \propto a^{3/2}$ as

$$\frac{\dot{E}}{E} = -\frac{\dot{a}}{a} = -\frac{2}{3}\frac{\dot{P}}{P}. \tag{10.118}$$

Solving first for the change in the period,

$$\dot{P} = \frac{3}{2}\frac{L_{\rm gw}}{E}\,P = -\frac{96}{5}\frac{G^3\mu M^2}{a^4}\,P, \tag{10.119}$$

and eliminating then a (which is not observed) gives

$$\dot{P} = -\frac{96}{5}(2\pi)^{8/3}G^{5/3}\mu M^{2/3}P^{-5/3}\,. \tag{10.120}$$

Combining Eqs. (10.118) and (10.119), we obtain

$$\dot{a} = \frac{2}{3}\frac{\dot{P}}{P}a = -\frac{64}{3}\frac{G^3\mu M^2}{a^3}. \tag{10.121}$$

Separating variables and integrating, we find

$$a^4 = \frac{256}{5}G^3\mu M^2(t - t_c). \tag{10.122}$$

Here, t_c denotes the (theoretical) coalensence time for point-like stars. With the initial condition $a(t=0) = a_0$, it follows

$$a(t) = a_0\left(1 - \frac{t}{t_c}\right)^{1/4} \tag{10.123}$$

and

$$t_c = \frac{5}{256}\frac{a_0^4}{G^3\mu M^2}. \tag{10.124}$$

As a rule of thumb, our approximations (slow velocities and weak fields) break down at $r \simeq r_{\rm ISCO}$. Since the last stage of the merger is fast, the estimate (10.124) is quite reliable.

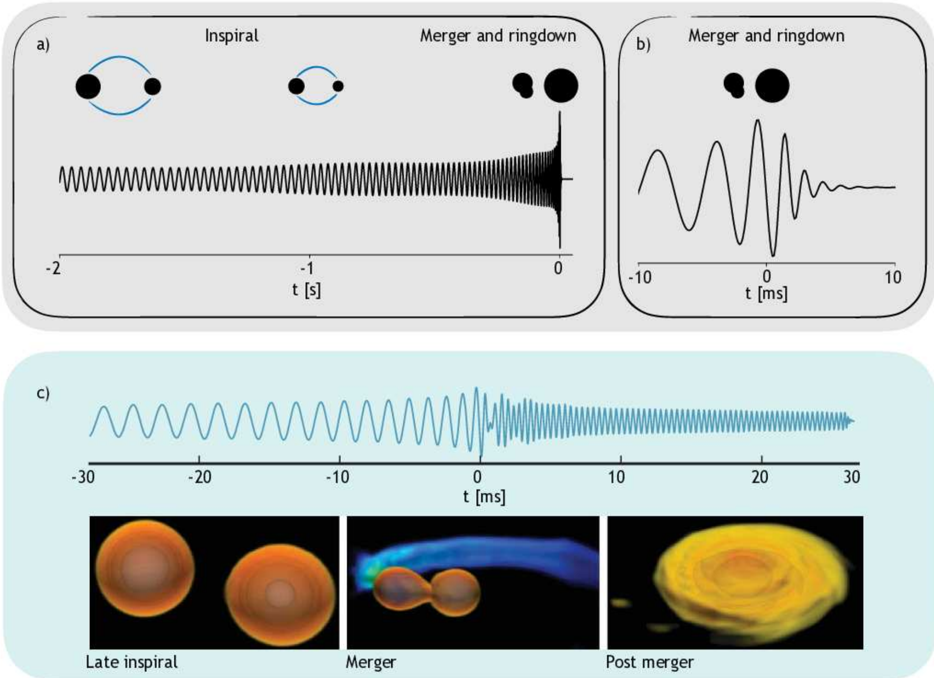


Figure 10.3: Typical waveforms for the merger of black hole (upper) and neutrons star (lower panel) binaries.

From the exercise, we know that the amplitude of the gravitational wave is

$$h_{ij} = \frac{4G\mu M}{ar} A_{ij} = \frac{h}{r} A_{ij}, \tag{10.125}$$

where the factors A_{ij} for the non-zero amplitudes have the form $A_{ij} \propto \sin(2\omega t + \phi)$. Thus the emitted gravitational wave is monochromatic, with frequency twice the orbital frequency of the binaries,

$$\nu_{\rm GW} = \frac{2\omega}{2\pi} = \frac{2}{P} = \frac{(GM)^{1/2}}{\pi a^{3/2}} = \nu_0 \left(1 - \frac{t}{t_c}\right)^{-3/8}. \tag{10.126}$$

This factor two reflects the helicity of gravitational waves. Moreover, the amplitude of the gravitational wave signal increases with time as

$$h(t) \propto \frac{1}{a} \propto (t - t_c)^{-1/4}. \tag{10.127}$$

Expressed as function of the frequency $\nu_{\rm GW}$, the amplitude becomes

$$h(t) = \frac{4G\mu M}{a} = 4G\mu M \frac{\omega^{2/3}}{(GM)^{1/3}} \equiv 4\pi^{2/3} G^{2/3} \mathcal{M}^{5/3}\, \nu_{\rm GW}^{2/3}. \tag{10.128}$$

In the last step, we introduced the chirp mass,

$$\mathcal{M} \equiv \mu^{3/5} M^{2/5} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}, \tag{10.129}$$

which is the combination of the masses m_1 and m_2 easiest to extract from the gravitational wave signal. Finally, we have to replace the instantaneous phase in the polarisation tensor by the time-integrated phase, since ω depends on time,

$$\Phi(t) = \int \mathrm{d}t\, 2\omega = \left(\frac{t - t_c}{5G\mathcal{M}}\right)^{-5/8} + \phi_0. \tag{10.130}$$

Thus both the amplitude and the phase evolution of the gravitational wave signal provide information on the chirp mass $\mathcal{M}$.

Example 10.3: Show that the chirp mass $\mathcal{M}$ provides a lower bound on the total mass M.

A typical wave-form of the merger of a black hole binary is shown in the upper panel of Fig. 10.3. It consists of the waves emitted during the inspiral ("the chirp"), the merger, and the ring-down. In this last phase, oscillations of the BH formed during the merger are damped by the emission of GWs and decay exponentially, leading to standard Kerr BH. The frequencies and the damping times of the eigenmodes of a BH can be calculated, and thus the ring-down provides additional opportunities to test GR. The lower panel of Fig. 10.3 shows a typical wave-form of a neutron star merger: When tidal interactions start to deform the neutron stars, the gravitational wave signal is not monochromatic anymore and the structure of the stars has to be accounted for.

10.A Appendix: Projection operator on TT states

We want to find the trace-less transverse part M_{ik}^{TT} of an arbitrary symmetric tensor M_{ik}. We start by searching for an operator which projects any tensor on the two-dimensional subspace orthogonal to the unit vector $\boldsymbol{n}$. Any (set of two) projection operator should satisfy

$$P_\pm^2 = P_\pm, \qquad P_\pm P_\mp = 0, \quad \text{and} \quad P_+ + P_- = 1.$$

In our case, the desired projection operator is

$$P_i^{\ j} = \delta_i^j - n_i n^j. \tag{10.131}$$

First, we show that this operator satisfies $P^2 = P$,

$$P_i^{\ j} P_j^{\ k} = (\delta_i^j - n_i n^j)(\delta_j^k - n_j n^k) = \delta_i^k - n_i n^k = P_i^{\ k}. \tag{10.132}$$

Morover, it is $n^i P_i^{\ j} v_j = 0$ for all vectors $\boldsymbol{v}$; Thus P projects indeed any vector on the subspace orthogonal to $\boldsymbol{n}$. Since a tensor is a multi-linear map, we have to apply a projection operator on each of its indices,

$$M_{kl}^{\mathrm{T}} = P_k^{\ i} P_l^{\ j} M_{ij}. \tag{10.133}$$

The tensor M_{kl}^{T} is transverse, $n^k M_{kl}^{\mathrm{T}} = n^l M_{kl}^{\mathrm{T}} = 0$, but in general not traceless

$$M_k^{\mathrm{T}\,k} = P_k^{\ i} P^{kj} M_{ij} = P_l^{\ i} M_{il}. \tag{10.134}$$

Subtracting the trace, we obtain the transverse, traceless part of M,

$$M_{kl}^{\mathrm{TT}} = \left(P_k^{\ i} P_l^{\ j} - \frac{1}{2} P_{kl} P^{ij}\right) M_{ij}. \tag{10.135}$$

TT polarisation states We apply Eq. (10.135) to the polarisation tensor

$$\varepsilon_{kl}^{\mathrm{TT}} = P_k{}^i \varepsilon_{ij} P_l{}^j - \frac{1}{2} P_{kl} \varepsilon_{ij} P^{ij} \tag{10.136}$$

of a gravitational wave. Moving to matrix notation,

$$\varepsilon^{\mathrm{TT}} = P \varepsilon P^T - \frac{1}{2} P \mathrm{tr}[\varepsilon P], \tag{10.137}$$

and choosing $\boldsymbol{k} \| \boldsymbol{e}_z$, it follows

$$\varepsilon_{ij}^{\mathrm{TT}} = \begin{pmatrix} \frac{1}{2}(\varepsilon_{11} - \varepsilon_{22}) & \varepsilon_{12} & 0 \\ \varepsilon_{12} & \frac{1}{2}(\varepsilon_{22} - \varepsilon_{11}) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{10.138}$$

If $\boldsymbol{k}$ agrees with one of the axes of a cartesian coordinate system, we can obtain the polarisation tensor in TT gauge simply by setting first the longitudinal part to zero and subtracting then half of the trace.

Evaluation of the quadrupole formulae We insert this projection operator into Eq. (10.85),

$$L_{\mathrm{gr}} = -\frac{G}{8\pi} \int \mathrm{d}\Omega \, \dddot{Q}_{ij}^{\mathrm{TT}} \dddot{Q}^{\mathrm{TT}\, ij} . \tag{10.139}$$

We have to find the projection onto the radial unit vector $\boldsymbol{e}_r$, whose Cartesian components we denote as $(\hat{x}^1, \hat{x}^2, \hat{x}^3)$. Then we obtain for the transverse part

$$\dddot{Q}_{ij}^{\mathrm{T}} \dddot{Q}^{\mathrm{T}\, ij} = \dddot{Q}_{ij} P_k{}^i P_l{}^j \dddot{Q}^{kl} = \dddot{Q}_{ij} \dddot{Q}^{ij} - 2 \dddot{Q}_i{}^j \dddot{Q}^{ik} \hat{x}_j \hat{x}_k + \dddot{Q}^{ij} \dddot{Q}^{kl} \hat{x}_i \hat{x}_j \hat{x}_k \hat{x}_l. \tag{10.140}$$

and for the trace

$$-\frac{1}{2} \dddot{Q}_{ij} P_{kl} P^{ij} \dddot{Q}^{kl} = -\frac{1}{2} \dddot{Q}^{ij} \dddot{Q}^{kl} \hat{x}_i \hat{x}_j \hat{x}_k \hat{x}_l. \tag{10.141}$$

Here, we used $P^2 = P$ and the fact that Q_{ij} is traceless. Since Q_{ij} is an integral over space, it does not depend on $\boldsymbol{e}_r$ and can be taken out of the angular integral. Then it follows

$$\int \mathrm{d}\Omega \, \hat{x}^i \hat{x}^j = \frac{4\pi}{3} \delta^{ij} \quad \text{and} \quad \int \mathrm{d}\Omega \, \hat{x}^i \hat{x}^j \hat{x}^k \hat{x}^l = \frac{4\pi}{15} \left(\delta^{ij} \delta^{kl} + \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk} \right). \tag{10.142}$$

To see the first result, we note that the only available symmetric tensor of rank two is δ^{ij}. Contracting the indices in $\int \mathrm{d}\Omega \, \hat{x}^i \hat{x}^j = A \delta^{ij}$, it follows $A = 4\pi/3$. Using the same line of argument, the integral with four $\hat{x}^i$ is evaluated. Combining everything and recalling that $Q_i^i = 0$,

$$\int \mathrm{d}\Omega \, \dddot{Q}_{ij}^{\mathrm{TT}} \dddot{Q}^{\mathrm{TT\, ij}} = \int \mathrm{d}\Omega \left[\dddot{Q}_{ij} \dddot{Q}^{ij} - 2 \dddot{Q}_i{}^j \dddot{Q}^{ik} \hat{x}_i \hat{x}_j + \frac{1}{2} \dddot{Q}^{ij} \dddot{Q}^{kl} \hat{x}_i \hat{x}_j \hat{x}_k \hat{x}_l \right] \tag{10.143}$$

$$= 4\pi \left[1 - \frac{2}{3} + \frac{1}{15} \right] \dddot{Q}_{ij} \dddot{Q}^{ij} = \frac{24\pi}{15} \dddot{Q}_{ij} \dddot{Q}^{ij}, \tag{10.144}$$

we obtain the quadrupole formuala (10.86) for the emission of gravitational waves.

10.B Appendix: Retarded Green function

We want to find the Green function for the gravitational wave equation (10.13). Starting from Eq. (10.57),

$$-\left(\frac{\partial^2}{\partial t^2} - \Delta\right) G(x - x') = \delta(\boldsymbol{x} - \boldsymbol{x}')\delta(t - t'), \tag{10.145}$$

we introduce relative coordinates, $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}'$ and $\tau = t - t'$. Then we perform an (asymmetric) Fourier transformation in the time t,

$$\int \mathrm{d}t \left(\Delta - \frac{\partial^2}{\partial t^2}\right) G(\boldsymbol{r}, \tau)\mathrm{e}^{\mathrm{i}\omega t} = \delta(\boldsymbol{r}) \int \mathrm{d}t\, \delta(t - t')\mathrm{e}^{\mathrm{i}\omega t}. \tag{10.146}$$

The integral on the RHS is trivial, and the one on the LHS defines $G(\boldsymbol{r}, \omega)$. Next we perform the time derivatives, obtaining

$$(\Delta + \omega^2)G(\boldsymbol{r}, \omega) = \delta(\boldsymbol{r})\mathrm{e}^{\mathrm{i}\omega t}. \tag{10.147}$$

Thus the time dependence of the Green function is

$$G(\boldsymbol{r}, \omega) = G_\omega(\boldsymbol{r})\mathrm{e}^{\mathrm{i}\omega t}. \tag{10.148}$$

We are interested in spherically symmetric solutions, emitted by a source at $r = |\boldsymbol{r}| = 0$. Then

$$\frac{1}{r}\frac{\mathrm{d}^2[rG_\omega(r)]}{\mathrm{d}r^2} + \omega^2 G_\omega(r) = \delta(\boldsymbol{r}). \tag{10.149}$$

For $r > 0$, the solution of

$$\frac{\mathrm{d}^2[rG_\omega(r)]}{\mathrm{d}r^2} + \omega^2 rG_\omega(r) = 0 \tag{10.150}$$

is

$$G_\omega(r) = \frac{A\mathrm{e}^{\mathrm{i}kr}}{r} + \frac{B\mathrm{e}^{-\mathrm{i}kr}}{r} \equiv AG_\omega^{(+)}(r) + BG_\omega^{(-)}(r). \tag{10.151}$$

Thus the solution consists of out- and in-going spherical waves. Next we consider the limit $r \to 0$ (or the static limit) of the wave equation. Integrating over a small sphere of radius r, we obtain

$$\int \mathrm{d}^3x\, \Delta G(\boldsymbol{r}, \omega) = \int \mathrm{d}S_i\, \partial_i G(\boldsymbol{r}, \omega) = 4\pi r^2 \partial_r G(\boldsymbol{r}, \omega) = 1. \tag{10.152}$$

Here, we used Gauss' theorem to convert the volume into a surface integral, while we could neglect $\int \mathrm{d}^3x\omega^2 G(\boldsymbol{r}, \omega) \propto r^3$ for $r \to 0$. Moreover, we used that the integral over the delta function on the RHS gives one. Thus the Green function for small r satisfies

$$G(\boldsymbol{r}, \omega) = -\frac{1}{4\pi r} + C. \tag{10.153}$$

Comparing this to Eq, (10.151) fixes $A + B = -1$ and $C = 0$. Finally, we transform back to time,

$$G^{(\pm)}(r, t) = \int \frac{\mathrm{d}\omega}{2\pi}\, G^{(\pm)}(r, \omega)\mathrm{e}^{-\mathrm{i}\omega t} = -\frac{1}{4\pi r}\int \frac{\mathrm{d}\omega}{2\pi}\mathrm{e}^{-\mathrm{i}\omega(t \mp r)}, \tag{10.154}$$

where we used $\omega = |\boldsymbol{k}|$. Then it follows

$$G^{(\pm)}(r, t) = -\frac{\delta(t \mp r)}{4\pi r}. \tag{10.155}$$

The delta function enforces $t = \pm r$. Since $r > 0$, the Green function $G^{(+)}$ includes only sources with $\tau > 0$, i.e. along the past light-cone of the observer at $\{t, \boldsymbol{x}\}$, while the Green function $G^{(-)}$ includes only sources with $\tau < 0$, i.e. along the past light-cone.

Finally, we comment on the differences between the classical and the quantum treatement of wave propagation:

- In classical physics, we use only positive energy solutions and the causal propagator is the retarded one, which propagates these solutions forward in time. A relativistic quantum theory contains in addition negative energy solutions. The causal or Feynman propagates then positive energy solutions (particles) forward, and negative energy solutions (antiparticles) backward in time, in a way consistent with the CPT theorem.
- In the classical case, one eleminates the gauge freedom completely such that only physical degrees of freedom propagate. Then it is sufficient to use a scalar Green function, which propagates the physical polarisation states in the same way. Such gauges (like the Coulomb or TT gauge) are however valid only in a specific frame. Moreover, the effects of the instantanous Coulomb-like interactions have to added seperately. Therefore one prefers in the quantum case a covariant gauges (like the Lorenz or harmonic gauge) which include also the instantanous Coulomb or Newtonian interactions. The Green function of a tensor of rank n becomes then a tensor of rank $2n$.
- In classical physics, we are mostly interested in the temporal evolution of the energy flux and the polarisation of a wave consisting of many quanta. This dependence is obtained easiest using spherical waves. In the quantum limit, an accelerated electron may emit only a single high-energy photon. Typically the energy and the direction of such a photon is measured. Therefore it is preferable to use plane waves to describe such emission processes.

Problems

10.1 Dynamical stress tensor. Show that the definition of the dynamical stress tensor can be simplified to

$$T_{\mu\nu} = \frac{2}{\sqrt{|g|}} \frac{\delta S_{\mathrm{m}}}{\delta g^{\mu\nu}} = 2\frac{\partial \mathscr{L}}{\partial g^{\mu\nu}} - g_{\mu\nu}\mathscr{L}\,. \tag{10.156}$$

10.2 Variation $\delta g_{\mu\nu}$. The variation of S_{EH} w.r.t. $g_{\mu\nu}$ will lead to different signs in Eq. (8.42). Explain why one obtains the same Einstein equation.

10.3 Cosmological constant Λ ♣. a.) Compare the stress density $T_{\mu\nu} = \kappa\Lambda g_{\mu\nu}$ of the cosmological constant to the one of an ideal fluid and determine thereby its EoS $w = P_\Lambda/\rho_\Lambda$. b.) Confirm the EoS using $U = V\rho_\Lambda$ and thermodynamics. c.) Estimate a bound on ρ_Λ using that the observable universe with size $\sim 3000\,\mathrm{Mpc}$ looks flat.

10.4 Expansion of S_{EH}. Expand $g^{\mu\nu}$ and $\sqrt{|g|}$ up to $\mathcal{O}(\lambda^3)$ around Minkowski space. Show that the $\mathcal{O}(\lambda)$ term of $\mathscr{L}_{\mathrm{EH}}$ is a total derivative which can be dropped.

10.5 Helicity. Show that the unphysical degrees of freedom of an electromagnetic wave transform as helicity 0, and of a a gravitational wave as helicity 0 and 1.

10.6 GWs from a binary system. Consider a binary system of two stars with equal mass M on circular orbits. a.) Calculate the quadrupole moments I_{ab}. b.) Determine the amplitude of the gravitational wave $\bar{h}_{\alpha\beta}(t, \boldsymbol{x})$. c.) Estimate the strength for a Galactic neutron star-neutron star binary with a separation of $r = 0.1\,\mathrm{AU}$.

10.7 GWs from a binary system. The energy flux

$\mathcal{F}$ of a GW is $\mathcal{F} = \frac{c^3}{32\pi G}\omega^2(a^2 + b^2)$, where a and b are the amplitudes of the two polarisation states. a.) Estimate the energy flux for the binary system in 8. b.) Estimate how much energy is dissipated if a GW crosses the interstellar or intergalactic medium: Which processes might be relevant? Use simple dimensional analysis for your estimate.

11 Cosmological models for an homogeneous, isotropic universe

11.1 Friedmann-Robertson-Walker metric for an homogeneous, isotropic universe

Einstein's cosmological principle Einstein postulated that the Universe is homogeneous and isotropic at each moment of its evolution. Note that a space isotropic around at least two points is also homogeneous, while a homogeneous space is not necessarily isotropic. The CMB provides excellent evidence that the universe is isotropic around us. Baring suggestions that we live at a special place, the universe is also homogeneous.

Weyl's postulate In 1923, Hermann Weyl postulated the existence of a privileged class of observers in the universe, namely those following the "average" motion of galaxies. He postulated that these observers follow time-like geodesics that never intersect. They may however diverge from a point in the (finite or infinite) past or converge towards such a point in the future.

Weyl's postulate implies that we can find coordinates such that galaxies are at rest. These coordinates are called *comoving coordinates* and can be constructed as follows: One chooses first a space-like hypersurface. Through each point in this hypersurface lies a unique worldline of a privileged observer. We choose the coordinate time such that it agrees with the proper-time of all observers, $g_{00} = 1$, and the spatial coordinate vectors such that they are constant and lie in the tangent space $\boldsymbol{T}$ at this point. Then $u^\alpha = \delta_0^\alpha$ and for $\boldsymbol{n} \in \boldsymbol{T}$ it follows $n^\alpha = (0, \boldsymbol{n})$ and

$$0 = u_\alpha n^\alpha = g_{\alpha\beta} u^\alpha n^\beta = g_{0b} n^b \,. \tag{11.1}$$

Since $\boldsymbol{n}$ is arbitrary, the metric tensor satisfies $g_{0b} = 0$. Hence as a consequence of Weyl's postulate we may choose the metric as

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}l^2 = \mathrm{d}t^2 - g_{ab}\mathrm{d}x^a \mathrm{d}x^b \,. \tag{11.2}$$

The cosmological principle constrains further the form of $\mathrm{d}l^2$: Homogeneity requires that the g_{ab} can depend on time only via a common factor $S(t)$, while isotropy requires that only $\boldsymbol{x} \cdot \boldsymbol{x} \equiv r^2$, $\mathbf{d}\boldsymbol{x} \cdot \boldsymbol{x}$, and $\mathbf{d}\boldsymbol{x} \cdot \mathbf{d}\boldsymbol{x}$ enter $\mathrm{d}l^2$. Hence

$$\mathrm{d}l^2 = C(r)(\boldsymbol{x} \cdot \mathbf{d}\boldsymbol{x})^2 + D(r)\mathbf{d}\boldsymbol{x} \cdot \mathbf{d}\boldsymbol{x} = C(r)r^2\mathrm{d}r^2 + D(r)[\mathrm{d}r^2 + r^2\mathrm{d}\vartheta^2 + r^2\sin^2\vartheta\mathrm{d}\phi^2]. \tag{11.3}$$

We can eliminate the function $D(r)$ by the rescaling $r^2 \to Dr^2$. Thus the line-element becomes

$$\mathrm{d}l^2 = S(t)\left[B(r)\mathrm{d}r^2 + r^2\mathrm{d}\Omega\right] \tag{11.4}$$

with $\mathrm{d}\Omega = \mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\phi^2$, while $B(r)$ is a function that we have still to specify.

Maximally symmetric spaces are spaces with constant curvature. Hence the Riemann tensor of such spaces can depend only on the metric tensor and a constant K specifying the curvature. The only form that respects the (anti-)symmetries of the Riemann tensor is

$$R_{abcd} = K(g_{ac}g_{bd} - g_{ad}g_{bc})\,. \tag{11.5}$$

Contracting R_{abcd} with g^{ac}, we obtain in three dimensions for the Ricci tensor

$$R_{bd} = g^{ac}R_{abcd} = Kg^{ac}(g_{ac}g_{bd} - g_{ad}g_{bc}) = K(3g_{bd} - g_{bd}) = 2Kg_{bd}\,. \tag{11.6}$$

A final contraction gives as curvature R of a three-dimensional maximally symmetric space

$$R = g^{ab}R_{ab} = 2K\delta^a_a = 6K\,. \tag{11.7}$$

A comparison of Eq. (11.6) with the Ricci tensor for the metric (11.4) will fix the still unknown function $B(r)$. We proceed in the standard way: Calculation of the Christoffel symbols with the help of the geodesic equations, then use of the definition (8.19) for the Ricci tensor,

$$R_{rr} = \frac{1}{rB}\frac{\mathrm{d}B}{\mathrm{d}r} = 2Kg_{rr} = 2KB \tag{11.8}$$

$$R_{\vartheta\vartheta} = 1 + \frac{r}{2B^2}\frac{\mathrm{d}B}{\mathrm{d}r} - \frac{1}{B} = 2Kg_{\vartheta\vartheta} = 2Kr^2\,. \tag{11.9}$$

(The $\phi\phi$ equation contains no additional information.) Integration of (11.8) gives

$$B = \frac{1}{A - Kr^2} \tag{11.10}$$

with A as integration constant. Inserting the result into (11.9) determines A as $A = 1$. Thus we have determined the line-element of a maximally symmetric 3-space with curvature K as

$$\mathrm{d}l^2 = \frac{\mathrm{d}r^2}{1 - Kr^2} + r^2(\sin^2\vartheta\mathrm{d}\phi^2 + \mathrm{d}\vartheta^2)\,. \tag{11.11}$$

Going over to the full four-dimensional line-element, we rescale for $K \neq 0$ the r coordinate by $r \to |K|^{1/2}r$. Then we absorb the factor $1/|K|$ in front of $\mathrm{d}l^2$ by defining the scale factor $R(t)$ as

$$R(t) = \begin{cases} S(t)/|K|^{1/2}, & K \neq 0 \\ S(t), & K = 0 \end{cases} \tag{11.12}$$

As result we obtain the Friedmann-Robertson-Walker (FRW) metric for an homogeneous, isotropic universe

$$\boxed{\mathrm{d}s^2 = \mathrm{d}t^2 - R^2(t)\left[\frac{\mathrm{d}r^2}{1 - kr^2} + r^2(\sin^2\vartheta\mathrm{d}\phi^2 + \mathrm{d}\vartheta^2)\right]} \tag{11.13}$$

with $k = \pm 1$ (positive/negative curvature) or $k = 0$ (flat three-dimensional space). Finally, we give two alternatives forms of the FRW metric that are also often used. The first one uses the conformal time $\mathrm{d}\eta = \mathrm{d}t/R$,

$$\mathrm{d}s^2 = R^2(\eta)\left[\mathrm{d}\eta^2 - \mathrm{d}l^2\right] \tag{11.14}$$

and gives for $k=0$ a conformally flat metric. In the second one, one introduces $r=\sin\chi$ for $k=1$. Then $\mathrm{d}r=\cos\chi\mathrm{d}\chi=(1-r^2)^{1/2}\mathrm{d}\chi$ and

$$\mathrm{d}s^2=\mathrm{d}t^2-R^2(t)\left[\mathrm{d}\chi^2+S_k^2(\chi)(\sin^2\vartheta\mathrm{d}\phi^2+\mathrm{d}\vartheta^2)\right] \tag{11.15}$$

with $S_k(\chi)=\sin\chi=r$. Defining

$$S_k(\chi)=\begin{cases}\sin\chi & \text{for} \quad k=1\,,\\ \chi & \text{for} \quad k=0\,,\\ \sinh\chi & \text{for} \quad k=-1\,.\end{cases} \tag{11.16}$$

the metric (11.15) is valid for all three values of k. Note that the rescaling $r\to|K|^{1/2}r$ makes r dimensionless, while R has the dimension of a length. Therefore one often introduces additionally a dimensionless scale factor $a(t)\equiv R(t)/R_0$ which satisfies $a_0=1$. Here, we denote the value of physical quantities at the present epoch by the subscript zero.

11.2 Geometry of the Friedmann-Robertson-Walker metric

Geometry of the FRW spaces Let us consider a sphere of fixed radius at fixed time, $\mathrm{d}r=\mathrm{d}t=0$. The line-element $\mathrm{d}s^2$ simplifies then to $R^2(t)r^2(\sin^2\vartheta\mathrm{d}\phi^2+\mathrm{d}\vartheta^2)$, which is the usual line-element of a sphere S^2 with radius $rR(t)$. Thus the area of the sphere is $A=4\pi(rR(t))^2=4\pi[S_k(\chi)R(t)]^2$ and the circumference of a circle is $L=2\pi rR(t)$, while $rR(t)$ has the physical meaning of a length.

By contrast, the radial distance between two points (r,ϑ,ϕ) and $(r+\mathrm{d}r,\vartheta,\phi)$ is $\mathrm{d}l=R(t)\mathrm{d}r/\sqrt{1-kr^2}$. Thus the radius of a sphere centered at $r=0$ is

$$l=R(t)\int_0^r\frac{\mathrm{d}r'}{\sqrt{1-kr'^2}}=R(t)\times\begin{cases}\operatorname{arcsin}(r) & \text{for} \quad k=1\,,\\ r & \text{for} \quad k=0\,,\\ \operatorname{arcsinh}(r) & \text{for} \quad k=-1\,.\end{cases} \tag{11.17}$$

Using χ as coordinate, the same result follows immediately

$$l=R(t)\int_0^{\chi(r)}\mathrm{d}\chi=R(t)\chi\,. \tag{11.18}$$

Hence for $k=0$, i.e. a flat space, one obtains the usual result $L/l=2\pi$, while for $k=1$ (spherical geometry) $L/l=2\pi r/\arcsin(r)<2\pi$ and for $k=-1$ (hyperbolic geometry) $L/l=2\pi r/\operatorname{arcsinh}(r)>2\pi$.

For $k=0$ and $k=-1$, l is unbounded, while for $k=+1$ there exists a maximal distance $l_{\max}(t)$. Hence the first two case correspond to open spaces with an infinite volume, while the latter is a closed space with finite volume.

Hubble's law Hubble found empirically that the spectral lines of "distant" galaxies are redshifted, $z=\Delta\lambda/\lambda_0>1$, with a rate proportional to their distance d,

$$cz=H_0d\,. \tag{11.19}$$

If this redshift is interpreted as Doppler effect, $z=\Delta\lambda/\lambda_0=v_r/c$, then the recession velocity of galaxies follows as

$$v=H_0d\,. \tag{11.20}$$

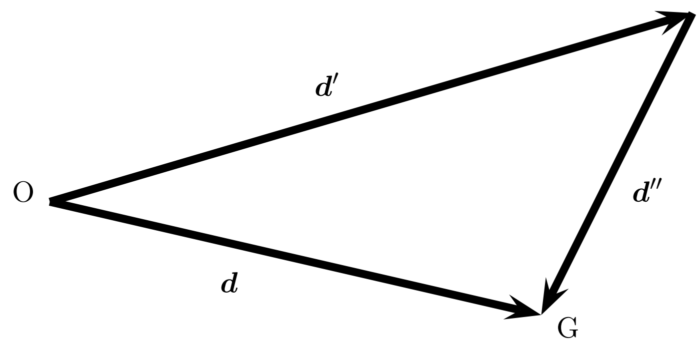


Figure 11.1: An observer at position $\boldsymbol{d}'$ sees the galaxy G recessing with the speed $H(\boldsymbol{d}-\boldsymbol{d}') = H\boldsymbol{d}''$, if the Hubble relation is linear.

The restriction "distant galaxies" means more precisely that $H_0 d \gg v_{\rm pec} \sim$ few $\times$ 100 km/s. In other words, the peculiar motion of galaxies caused by the gravitational attraction of nearby galaxy clusters should be small compared to the Hubble flow $H_0 d$. Note that the interpretation of v as recession velocity is problematic. The validity of such an interpretation is certainly limited to $v \ll c$.

The parameter H_0 is called Hubble constant and has the value $H_0 \approx 71^{+4}_{-3}$ km/s/Mpc. We will see soon that the Hubble law Eq. (11.20) is an approximation valid for $z \ll 1$. In general, the Hubble constant is not constant but depends on time, $H = H(t)$, and we will call it therefore Hubble parameter for $t \neq t_0$.

We can derive Hubble's law by a Taylor expansion of $R(t)$,

$$\begin{aligned} R(t) &= R(t_0) + (t-t_0)\dot{R}(t_0) + \frac{1}{2}(t-t_0)^2\ddot{R}(t_0) + \dots && (11.21) \\ &= R(t_0)\left[1 + (t-t_0)H_0 - \frac{1}{2}(t-t_0)^2 q_0 H_0^2 + \dots\right], && (11.22) \end{aligned}$$

where

$$\boxed{H_0 \equiv \frac{\dot{R}(t_0)}{R(t_0)}} \qquad \text{and} \qquad \boxed{q_0 \equiv -\frac{\ddot{R}(t_0)R(t_0)}{\dot{R}^2(t_0)}} \tag{11.23}$$

is called deceleration parameter: If the expansion is slowing down, $\ddot{R} < 0$ and $q_0 > 0$.

Hubble's law follows now as an an approximation for small redshift: For not too large time-differences, we can use the expansion Eq. (11.21) and write

$$1 - z \approx \frac{1}{1+z} = \frac{R(t)}{R_0} \approx 1 + (t-t_0)H_0 \,. \tag{11.24}$$

Hence Hubble's law, $z = (t_0 - t)H_0 = d/cH_0$, is valid as long as $z \approx H_0(t_0 - t) \ll 1$. Deviations from its linear form arises for $z \gtrsim 1$ and can be used to determine q_0.

Hubble's law as consequence of homogeneity Consider Hubble's law as a vector equation with us at the center of the coordinate system,

$$\boldsymbol{v} = H\boldsymbol{d} \,. \tag{11.25}$$

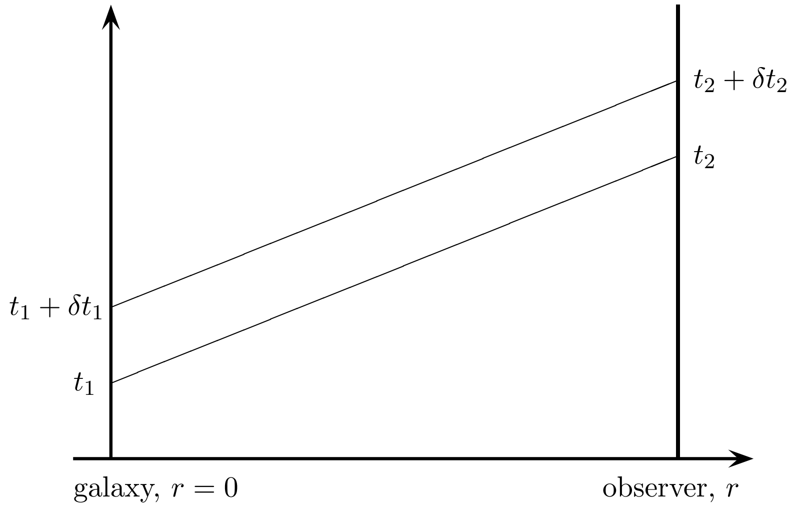


Figure 11.2: World lines of a galaxy emitting light and an observer at comoving coordinates $r = 0$ and r, respectively.

What sees a different observer at position $\boldsymbol{d}'$? He has the velocity $\boldsymbol{v}' = H\boldsymbol{d}'$ relative to us. We are assuming that velocities are small and thus

$$\boldsymbol{v}'' \equiv \boldsymbol{v} - \boldsymbol{v}' = H(\boldsymbol{d} - \boldsymbol{d}') = H\boldsymbol{d}'' \,, \tag{11.26}$$

where $\boldsymbol{v}''$ and $\boldsymbol{d}''$ denote the position relative to the new observer. A linear relation between v and d as Hubble law is the only relation compatible with homogeneity and thus the "cosmological principle".

Lemaitre's redshift formula

A light-ray propagates with $v = c$ or $\mathrm{d}s^2 = 0$. Assuming a galaxy at $r = 0$ and an observer at r, i.e. light rays with $\mathrm{d}\phi = \mathrm{d}\vartheta = 0$, we rewrite the FRW metric as

$$\frac{\mathrm{d}t}{R} = \frac{\mathrm{d}r}{\sqrt{1 - kr^2}} \,. \tag{11.27}$$

We integrate this expression between the emission and absorption times t_1 and t_2 of the first light-ray,

$$\int_{t_1}^{t_2} \frac{\mathrm{d}t}{R} = \int_0^r \frac{\mathrm{d}r}{\sqrt{1 - kr^2}} \tag{11.28}$$

and between $t_1 + \delta t_1$ and $t_2 + \delta t_2$ for the second light-ray (see also Fig. 11.2),

$$\int_{t_1+\delta t_1}^{t_2+\delta t_2} \frac{\mathrm{d}t}{R} = \int_0^r \frac{\mathrm{d}r}{\sqrt{1 - kr^2}} \,. \tag{11.29}$$

The RHS's are the same and thus we can equate the LHS's,

$$\int_{t_1}^{t_2} \frac{\mathrm{d}t}{R} = \int_{t_1+\delta t_1}^{t_2+\delta t_2} \frac{\mathrm{d}t}{R} \,. \tag{11.30}$$

We change the integration limits, subtracting the common interval $[t_1 + \delta t_1 : t_2]$ and obtain

$$\int_{t_1}^{t_1+\delta t_1} \frac{\mathrm{d}t}{R} = \int_{t_2}^{t_2+\delta t_2} \frac{\mathrm{d}t}{R} . \tag{11.31}$$

Now we choose the time intervals δt_i as the time between two wave crests separated by the wave lengths λ_i of an electromagnetic wave. Since these time intervals are extremely short compared to cosmological times, $\delta t_i = \lambda_i / c \ll t_i$, we can assume $R(t)$ as constant performing the integrals and obtain

$$\frac{\delta t_1}{R_1} = \frac{\delta t_2}{R_2} \qquad \text{or} \qquad \frac{\lambda_1}{R_1} = \frac{\lambda_2}{R_2} . \tag{11.32}$$

The redshift z of an object is defined as the relative change in the wavelength between emission and detection,

$$z = \frac{\lambda_2 - \lambda_1}{\lambda_1} = \frac{\lambda_2}{\lambda_1} - 1 \tag{11.33}$$

or

$$\boxed{1 + z = \frac{\lambda_2}{\lambda_1} = \frac{R_2}{R_1} .} \tag{11.34}$$

Typically, the observation happens at the present epoch, and thus we set $1 + z = R_0/R(t)$.

This result is intuitively understandable, since the expansion of the universe stretches all lengths of *unbound* sytems including the wave-length of a photon. For a massless particle like the photon, $\nu = c\lambda$ and $E = cp$, and thus its frequency (energy) and its wave-length (momentum) are affected in the same way. By contrast, the energy of a non-relativistic particle with $E \approx mc^2$ is nearly fixed.

A similar calculation as for the photon can be done for massive particles. Since the geodesic equation for massive particles leads to a more involved calculation, we use in this case however a different approach. We consider two comoving observer separated by the proper distance δl. A massive particle with velocity v needs the time $\delta t = \delta l / v$ to travel from observer one to observer two. The relative velocity of the two observer is

$$\delta u = \frac{\dot{R}}{R} \delta l = \frac{\dot{R}}{R} v \delta t = v \frac{\delta R}{R} . \tag{11.35}$$

Since we assume that the two observes are separated only infinitesimally, we can use the addition law for velocities from special relativity for the calculation of the velocity v' measured by the second observer,

$$v' = \frac{v - \delta u}{1 - v\delta u} = v - (1 - v^2)\delta u + \mathcal{O}(\delta u^2) = v - (1 - v^2) v \frac{\delta R}{R} . \tag{11.36}$$

Introducing $\delta v = v - v'$, we obtain

$$\frac{\delta v}{v(1 - v^2)} = \frac{\delta R}{R} . \tag{11.37}$$

and integrating this equation results in

$$p = \frac{mv}{\sqrt{1 - v^2}} = \frac{\text{const.}}{R} . \tag{11.38}$$

Thus not the energy but the momentum $p = \hbar / \lambda$ of massive particles is red-shifted: The kinetic energy of massive particles goes quadratically to zero, and hence peculiar velocities relative to the Hubble flow are strongly damped by the expansion of the universe.

Luminosity and angular diameter distance

In an expanding universe, the distance to an object depends on the expansion history, i.e. the behaviour of the scale factor $R(t)$ between the time of emission t of a light signal and its reception at t_0. From the metric (11.15), we can define the (radial) coordinate distance

$$\chi = \int_t^{t_0} \frac{\mathrm{d}t}{R(t)} \tag{11.39}$$

as well as the proper distance $d = g_{\chi\chi}\chi = R(t)\chi$. The proper distance is however only for a static metric a measurable quantity and cosmologists use therefore other, operationally defined measures for the distance. The two most important examples are the luminosity and the angular diameter distances. They are useful for standard candles, i.e. for sources with known luminosity, and for standard ruler, i.e. processes happening on a knows physical scales, respectively. Two important examples are supernova explosions of type Ia and baryon-acoustic oscillations in the early universe. Most recently, the possibility of "standard sirenes" has been suggested, using the emission of gravitational waves in binary mergers.

Luminosity distance The luminosity distance d_L is defined such, that the inverse-square law between luminosity $L = \mathrm{d}E/\mathrm{d}t$ of a source at distance d and the received energy flux $\mathcal{F} = \mathrm{d}E/(\mathrm{d}A\mathrm{d}t)$ is valid,

$$d_L = \left(\frac{L}{4\pi\mathcal{F}}\right)^{1/2} . \tag{11.40}$$

Assume now that a (isotropically emitting) source with luminosity $L(t)$ and comoving coordinate χ is observed at t_0 by an observer at O. The cut at O through the forward light cone of the source emitted at t_e defines a sphere S^2 with proper area

$$A = 4\pi R^2(t_0) S_k^2(\chi) \,. \tag{11.41}$$

Two additional effects are that the frequency of a single photon is redshifted, $\nu_0 = \nu_e/(1+z)$, and that the arrival rate of photons is reduced by the same factor due to time-dilation. Hence the received flux is

$$\mathcal{F}(t_0) = \frac{1}{(1+z)^2} \frac{L(t_e)}{4\pi R_0^2 S_k^2(\chi)} \tag{11.42}$$

and the luminosity distance in a FRW universe follows as

$$d_L = (1+z)\, R_0 S_k(\chi) \,. \tag{11.43}$$

Note that d_L depends via χ on the expansion history of the universe between t_e and t_0.

Observable are not the coordinates χ or r, but the redshift z of a galaxy. Differentiating $1 + z = R_0/R(t)$, we obtain

$$\mathrm{d}z = -\frac{R_0}{R^2}\mathrm{d}R = -\frac{R_0}{R^2}\frac{\mathrm{d}R}{\mathrm{d}t}\mathrm{d}t = -(1+z)H\mathrm{d}t \tag{11.44}$$

or

$$t_0 - t = \int_t^{t_0} \mathrm{d}t = \int_0^z \frac{\mathrm{d}z}{H(z)(1+z)} \,. \tag{11.45}$$

This relation connects the redshift z with the "looking-back time" $t_0 - t$, or the age t of th universe.

Inserting the relation (11.44) into Eq. (11.39), we find the coordinate χ of a galaxy at redshift z as

$$\chi = \int_t^{t_0} \frac{\mathrm{d}t}{R(t)} = \frac{1}{R_0} \int_0^z \frac{\mathrm{d}z}{H(z)}. \tag{11.46}$$

This defines χ via the redshift and the expansion history of the universe which is encoded in the function $H(z)$.

For small redshift $z \ll 1$, we can use the expansion (11.22)

$$\chi = \int_t^{t_0} \frac{\mathrm{d}t}{R_0}[1-(t-t_0)H_0+\ldots]^{-1} \tag{11.47}$$

$$\approx \frac{1}{R_0}[(t-t_0)+\frac{1}{2}(t-t_0)^2H_0+\ldots] = \frac{1}{R_0H_0}[z-\frac{1}{2}(1+q_0)z^2+\ldots]. \tag{11.48}$$

Note also that for a flat universe, $k=0$, the parameter R_0 has no physical meaning and drops out of d_L. For $k=\pm 1$, R_0 corresponds to the curvature radius of the spatial 3-space at the present time. We will seein example 11.1 that its value can be determined via a measurement of H_0, q_0 and the current energy density of the universe.

In practise, one observes only the luminosity within a certain frequency range instead of the total (or bolometric) luminosity. A correction for this effect requires the knowledge of the intrinsic source spectrum.

Angular diameter distance Instead of basing a distance measurement on standard candles, one may use standard rods with know proper length l whose angular diameter $\Delta\vartheta$ can be observed. Then we define the angular diameter distance as

$$d_A = \frac{l}{\Delta\vartheta}\,. \tag{11.49}$$

Inserting the length $l = R(t)S_k(\chi)$ of a line-elment of an a sphere in FLRW metric (11.15) gives

$$d_A = \frac{R_0 S_k(\chi)}{1+z}\,. \tag{11.50}$$

Thus at small distances, $z \ll 1$, the two definitions for the distance agree by construction, while for large redshifts the differences increase as $(1+z)^2$.

Cosmological tests For a given cosmological model, $H(z)$ is fixed and we can calculate the value of χ for an event with the measured resdshift z using Eq. (11.46). If the event is a standard candle, we can use the measured energy flux $\mathcal{F}$ to obtain the the luminosity distance via the definition (11.40). Comparing this value to the theoretical predition (11.43) allows to (dis-) favour this model. The same procedure can be applied to a standard ruler.

11.3 Friedmann equations

The FRW metric together with a perfect fluid as energy-momentum tensor gives for the time-time component of the Einstein equation

$$\ddot{R} = -\frac{4\pi G}{3}(\rho+3P)R\,, \tag{11.51}$$

for the space-time components

$$R\ddot{R} + 2\dot{R}^2 + 2K = 4\pi G(\rho - P)\,, \tag{11.52}$$

and $0 = 0$ for the space-space components. Eliminating $\ddot{R}$ and showing explicitly the contribution of a cosmological constant to the energy density ρ, the usual Friedmann equation follows as

$$\boxed{H^2 \equiv \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi}{3}G\rho - \frac{k}{R^2} + \frac{\Lambda}{3}\,.} \tag{11.53}$$

while the "acceleration equation" is

$$\boxed{\frac{\ddot{R}}{R} = \frac{\Lambda}{3} - \frac{4\pi G}{3}(\rho + 3P)\,.} \tag{11.54}$$

This equation determines the (de-) acceleration of the Universe as function of its matter and energy content. "Normal" matter is characterized by $\rho > 0$ and $P \geq 0$. Thus a static solution is impossible for a universe with $\Lambda = 0$. Such a universe is decelerating and since today $\dot{R} > 0$, $\ddot{R}$ was always negative and there was a "big bang".

We define the *critical density* $\rho_{\rm cr}$ as the density for which the spatial geometry of the universe is flat. From $k = 0$, it follows

$$\rho_{\rm cr} = \frac{3H_0^2}{8\pi G} \tag{11.55}$$

and thus $\rho_{\rm cr}$ is uniquely fixed by the value of H_0. One "hides" this dependence by introducing h,

$$H_0 = 100\,h\,{\rm km/(s\,Mpc)}\,.$$

Then one can express the critical density as function of h,

$$\rho_{\rm cr} = 2.77 \times 10^{11} h^2 M_\odot/{\rm Mpc}^3 = 1.88 \times 10^{-29} h^2 {\rm g/cm}^3 = 1.05 \times 10^{-5} h^2 {\rm GeV/cm}^3\,.$$

Thus a flat universe with $H_0 = 100h$ km/s/Mpc requires an energy density of ~ 10 protons per cubic meter. We define the abundance Ω_i of the different players in cosmology as their energy density relative to $\rho_{\rm cr}$, $\Omega_i = \rho/\rho_{\rm cr}$.

In the following, we will often include Λ as other contributions to the energy density ρ via

$$\frac{8\pi}{3}G\rho_\Lambda = \frac{\Lambda}{3}\,. \tag{11.56}$$

Thereby one recognizes also that the cosmological constant acts as a constant energy density

$$\rho_\Lambda = \frac{\Lambda}{8\pi G} \qquad \text{or} \qquad \Omega_\Lambda = \frac{\Lambda}{3H_0^2}\,. \tag{11.57}$$

We can understand better the physical properties of the cosmological constant by replacing Λ by $(8\pi G)\rho_\Lambda$. Now we can compare the effect of normal matter and of the Λ term on the acceleration,

$$\frac{\ddot{R}}{R} = \frac{8\pi G}{3}\rho_\Lambda - \frac{4\pi G}{3}(\rho + 3P) \tag{11.58}$$

Thus Λ is equivalent to matter with an E.o.S. $w_\Lambda = P/\rho = -1$. This property can be checked using only thermodynamics: With $P = -(\partial U/\partial V)_S$ and $U_\Lambda = \rho_\Lambda V$, it follows $P = -\rho$.

The borderline between an accelerating and decelerating universe is given by $\rho = -3P$ or $w = -1/3$. The condition $\rho < -3P$ violates the so-called strong energy condition for "normal" matter in equilibrium. An accelerating universe requires therefore a positive cosmological constant or a dominating form of matter that is not in equilibrium.

Note that the energy contribution of relativistic matter, photons and possibly neutrinos, is today much smaller than the one of non-relativistic matter (stars and cold dark matter). Thus the pressure term in the acceleration equation can be neglected at the present epoch. Measuring $\ddot{R}/R$, $\dot{R}/R$ and ρ fixes therefore the geometry of the universe.

Thermodynamics The first law of thermodynamics becomes for a perfect fluid with $\mathrm{d}S = 0$ simply

$$\mathrm{d}U = T\mathrm{d}S - P\mathrm{d}V = -P\mathrm{d}V \tag{11.59}$$

or

$$\mathrm{d}(\rho R^3) = -P\mathrm{d}(R^3)\,. \tag{11.60}$$

Dividing by $\mathrm{d}t$,

$$R\dot{\rho} + 3(\rho + P)\dot{R} = 0\,, \tag{11.61}$$

we obtain our old result,

$$\dot{\rho} = -3(\rho + P)H\,. \tag{11.62}$$

This result could be also derived from $\nabla_a T^{ab} = 0$. Moreover, the three equations are not independent.

11.4 Scale-dependence of different energy forms

The dependence of different energy forms as function of the scale factor R can derived from energy conservation, $\mathrm{d}U = -P\mathrm{d}V$, if an E.o.S. $P = P(\rho) = w\rho$ is specified. For $w =$ const., it follows

$$\mathrm{d}(\rho R^3) = -3PR^2\mathrm{d}R \tag{11.63}$$

or eliminating P

$$\frac{\mathrm{d}\rho}{\mathrm{d}R}R^3 + 3\rho R^2 = -3w\rho R^2\,. \tag{11.64}$$

Separating the variables,

$$-3(1+w)\frac{\mathrm{d}R}{R} = \frac{\mathrm{d}\rho}{\rho}\,, \tag{11.65}$$

we can integrate and obtain

$$\boxed{\rho \propto R^{-3(1+w)}} = \begin{cases} R^{-3} & \text{for matter} \quad (w=0)\,, \\ R^{-4} & \text{for radiation} \quad (w=1/3)\,, \\ \text{const.} & \text{for } \Lambda \quad (w=-1)\,. \end{cases} \tag{11.66}$$

This result can be understood also from heuristic arguments:

- (Non-relativistic) matter means that $kT \ll m$. Thus $\rho = nm \gg nT = P$ and non-relativistic matter is pressure-less, $w = 0$. The mass m is constant and $n \propto 1/R^3$, hence ρ is just diluted by the expansion of the universe, $\rho \propto 1/R^3$.

- Radiation is not only diluted but the energy of each single photon is additionally redshifted, $E \propto 1/R$. Thus the energy density of radiation scales as $\propto 1/R^4$. Alternatively, one can use that $\rho = aT^4$ and $T \propto \langle E \rangle \propto 1/R$.
- Cosmological constant Λ: From $\frac{8\pi}{3}G\rho_\lambda = \frac{\Lambda}{3}$ one obtains that the cosmological constant acts as an energy density $\rho_\lambda = \frac{\Lambda}{8\pi G}$ that is constant in time, independent from a possible expansion or contraction of the universe.
- Note that the scaling of the different energy forms is very different. It is therefore surprising that "just today", the energy in matter and due to the cosmological constant is of the same order ("coincidence problem").

Let us rewrite the Friedmann equation for the present epoch as

$$\frac{k}{R_0^2} = H_0^2\left(\frac{8\pi G}{3H_0^2}\rho_0 + \frac{\Lambda}{3H_0^2} - 1\right) = H_0^2\left(\Omega_{\mathrm{tot},0} - 1\right) . \tag{11.67}$$

We express the curvature term for arbitrary times through $\Omega_{\mathrm{tot},0}$ and the redshift z as

$$\frac{k}{R^2} = \frac{k}{R_0^2}(1+z)^2 = H_0^2(\Omega_{\mathrm{tot},0} - 1)(1+z)^2 . \tag{11.68}$$

Dividing the Friedmann equation (11.53) by $H_0^2 = 8\pi G\rho_{\mathrm{cr}}/3$, we obtain

$$\begin{aligned} \frac{H^2(z)}{H_0^2} &= \sum_i \Omega_i(z) - (\Omega_{\mathrm{tot},0} - 1)(1+z)^2 \\ &= \Omega_{\mathrm{rad},0}(1+z)^4 + \Omega_{\mathrm{m},0}(1+z)^3 + \Omega_\Lambda - (\Omega_{\mathrm{tot},0} - 1)(1+z)^2 \end{aligned} \tag{11.69}$$

This expression allows us to calculate the age of the universe (11.45), distances (11.43), etc. for a given cosmological model, i.e. specifying the energy content $\Omega_{i,0}$ and the Hubble parameter H_0 at the present epoch.

11.5 Cosmological models

11.5.1 Single energy component

We consider a flat universe, $k = 0$, with one dominating energy component with E.o.S $w = P/\rho = \mathrm{const.}$. With $\rho = \rho_{\mathrm{cr}}\,(R/R_0)^{-3(1+w)}$, the Friedmann equation becomes

$$\dot{R}^2 = \frac{8\pi}{3}G\rho R^2 = H_0^2 R_0^{3+3w}\, R^{-(1+3w)} , \tag{11.70}$$

where we inserted the definition of $\rho_{\mathrm{cr}} = 3H_0^2/(8\pi G)$. Separating variables we obtain

$$R_0^{-(3+3w)/2} \int_0^{R_0} \mathrm{d}R\, R^{(1+3w)/2} = H_0 \int_0^{t_0} \mathrm{d}t = t_0 H_0 \tag{11.71}$$

and hence the age of the Universe follows as

$$\boxed{t_0 H_0 = \frac{2}{3+3w}} = \begin{cases} 2/3 & \text{for matter} \quad (w=0) , \\ 1/2 & \text{for radiation} \quad (w=1/3) , \\ \to \infty & \text{for } \Lambda \quad (w=-1) . \end{cases} \tag{11.72}$$

Models with $w > -1$ needed a finite time to expand from the initial singularity $R(t = 0) = 0$ to the current size R_0, while a Universe with only a Λ has no "beginning".

In models with a hot big-bang, $\rho, T \to \infty$ for $t \to 0$, and we should expect that classical gravity breaks down at some moment t_*. As long as $R \propto t^\alpha$ with $\alpha < 1$, most time elapsed during the last fractions of $t_0 H_0$. Hence our result for the age of the universe does not depend on unknown physics close to the big-bang as long as $w > -1/3$.

If we integrate (11.71) to the arbitrary time t, we obtain the time-dependence of the scale factor,

$$\boxed{R(t) \propto t^{2/(3+3w)}} = \begin{cases} t^{2/3} & \text{for matter} \quad (w = 0)\,, \\ t^{1/2} & \text{for radiation} \quad (w = 1/3)\,, \\ \exp(t) & \text{for } \Lambda \quad (w = -1)\,. \end{cases} \tag{11.73}$$

11.5.2 Matter- or radiation dominated universe

In the case of a matter- or radiation dominated universe with non-zero curvature, we start again from the Friedmann equation,

$$\dot{R}^2 = \frac{8\pi G}{3}\rho R^2 - k. \tag{11.74}$$

Then we introduce conformal time,

$$R'^2 \equiv \left(\frac{\mathrm{d}R}{\mathrm{d}\eta}\right)^2 = \frac{8\pi G}{3}\rho R^4 - kR^2, \tag{11.75}$$

Next we differentiate again w.r.t. η and divide by $2R'$,

$$R'' + -kR = \frac{4\pi G}{3}\left(\frac{\mathrm{d}\rho}{\mathrm{d}R}R + 4\rho\right)R^3. \tag{11.76}$$

Using now (11.64) in the form

$$\mathrm{d}\rho = -3(\rho + P)\frac{\mathrm{d}R}{R} \tag{11.77}$$

gives

$$R'' + kR = \frac{4\pi G}{3}\left(\rho - 3P\right)R^3. \tag{11.78}$$

This equation can be also obtained calculating the trace of the Einstein equation.

For a radiation-dominated universe, $P = \rho/3$, and thus $R'' + kR = 0$. Fixing one of the two integration constants by setting $a(0) = 0$, it follows

$$a(\eta) = a_m S_k(\eta). \tag{11.79}$$

The remaining integration constant can be determined by considering the Friedmann equation at $\eta = 0$,

$$R'^2(\eta = 0) = \frac{8\pi G}{3}\rho R^4 = \frac{8\pi G}{3}\rho_0 \equiv a_m, \tag{11.80}$$

where we used also $\rho R^4 = \rho_0 = \text{const.}$. Integrating $\mathrm{d}t = R\mathrm{d}\eta$, the time dependence follows as

$$t(\eta) = a_m \begin{cases} (1 - \cos\eta) & \text{for} \quad k = 1\,, \\ \eta^2/2 & \text{for} \quad k = 0\,, \\ (1 - \cosh\eta) & \text{for} \quad k = -1\,. \end{cases} \tag{11.81}$$

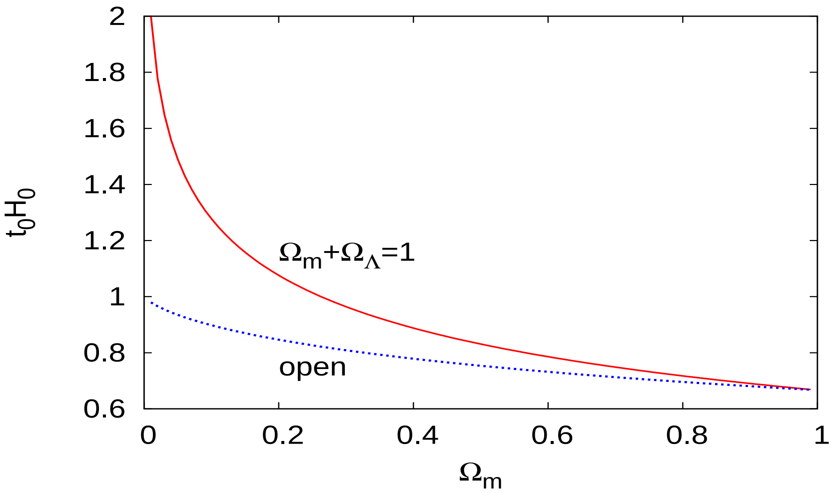


Figure 11.3: The product t_0H_0 for an open universe containing only matter (dotted blue line) and for a flat cosmological model with $\Omega_\Lambda + \Omega_m = 1$ (solid red line).

For a matter=dominated universe, $P = 0$, the RHS of (11.78) is constant. Adding a particular solution to the homogenous solution derived above gives

$$a(\eta) = a_r \begin{cases} (1-\cos\eta) & \text{for} \quad k=1\,, \\ \eta^2 & \text{for} \quad k=0\,, \\ (\cosh\eta - 1) & \text{for} \quad k=-1\,. \end{cases} \tag{11.82}$$

and

$$t(\eta) = a_r \begin{cases} (\eta-\sin\eta) & \text{for} \quad k=1\,, \\ \eta^3/2 & \text{for} \quad k=0\,, \\ (1-\sinh\eta) & \text{for} \quad k=-1\,. \end{cases} \tag{11.83}$$

with $a_r = a_m/2$.

Age problem of the universe The age of a matter-dominated universe is (expanded around $\Omega_0 = 1$)

$$t_0 = \frac{2}{3H_0}\left[1 - \frac{1}{5}(\Omega_0 - 1) + \ldots\right]. \tag{11.84}$$

Globular cluster ages require $t_0 \geq 13\,\text{Gyr}$. Using $\Omega_0 = 1$ leads to $H_0 \leq 2/3 \times 13\,\text{Gyr} = 1/19.5\,\text{Gyr}$ or $h \leq 0.50$. Thus a flat universe with $t_0 = 13\,\text{Gyr}$ without cosmological constant requires a too small value of H_0. Choosing $\Omega_m \approx 0.3$ increases the age by just 14%.

We derive the age t_0 of a flat Universe with $\Omega_m + \Omega_\Lambda = 1$ in the next section as

$$\frac{3t_0H_0}{2} = \frac{1}{\sqrt{\Omega_\Lambda}} \ln \frac{1+\sqrt{\Omega_\Lambda}}{\sqrt{1-\Omega_\Lambda}}\,. \tag{11.85}$$

Requiring $H_0 \geq 65$ km/s/Mpc and $t_0 \geq 13\,\text{Gyr}$ means that the function on the RHS should be larger than $3 \times 13\text{Gyr} \times 0.65/(2 \times 9.8\text{Gyr} \approx 1.3$ or $\Omega_\Lambda \geq 0.55$.

11.5.3 The ΛCDM model

We consider a flat Universe containing as its only two components pressure-less matter and a cosmological constant, $\Omega_m + \Omega_\Lambda = 1$. Then the curvature term in the Friedmann equation and the pressure term in the deceleration equation play no role and we can hope to solve these equations for $a(t)$. Multiplying the deceleration equation (11.54) by two and adding it to the Friedmann equation (11.53), we eliminate ρ_m,

$$2\,\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = \Lambda\,. \tag{11.86}$$

Next we rewrite first the LHS and then the RHS as total time derivatives: With

$$\frac{\mathrm{d}}{\mathrm{d}t}(a\dot{a}^2) = \dot{a}^3 + 2a\dot{a}\ddot{a} = \dot{a}a^2\left[\left(\frac{\dot{a}}{a}\right)^2 + 2\,\frac{\ddot{a}}{a}\right], \tag{11.87}$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(a\dot{a}^2) = \dot{a}a^2\Lambda = \frac{1}{3}\,\frac{\mathrm{d}}{\mathrm{d}t}(a^3)\Lambda\,. \tag{11.88}$$

Integrating is now trivial,

$$a\dot{a}^2 = \frac{\Lambda}{3}\,a^3 + C\,. \tag{11.89}$$

The constant C can be determined most easily by setting $a(t_0) = 1$ and comparing the Friedmann equation (11.53) with (11.89) for $t = t_0$ as $C = 8\pi G\rho_{m,0}/3$.

Next we introduce the new variable $x = a^{3/2}$. Then

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\,\frac{\mathrm{d}a}{\mathrm{d}x} = \frac{\mathrm{d}x}{\mathrm{d}t}\,\frac{2x^{-1/3}}{3}\,, \tag{11.90}$$

and we obtain as new differential equation

$$\dot{x}^2 - 3\Lambda x^2/4 + 9C/4 = 0\,. \tag{11.91}$$

Inserting the solution $x(t) = A\sinh(\sqrt{3\Lambda}t/2)$ of the homogeneous equation fixes the constant A as $A = \sqrt{3C/\Lambda}$. We can express A also by the current values of Ω_i as $A = \Omega_m/\Omega_\Lambda = (1-\Omega_\Lambda)/\Omega_\Lambda$. Hence the time-dependence of the scale factor is

$$a(t) = A^{2/3}\sinh^{2/3}(\sqrt{3\Lambda}t/2)\,. \tag{11.92}$$

The time-scale of the expansion is set by $t_\Lambda = 2/\sqrt{3\Lambda}$.

The present age t_0 of the universe follows by setting $a(t_0) = 1$ as

$$t_0 = t_\Lambda \mathrm{arctanh}(\sqrt{\Omega_\Lambda})\,. \tag{11.93}$$

The deceleration parameter $q = -\ddot{a}/aH^2$ is an important quantity for observational tests of the ΛCDM model. We calculate first the Hubble parameter

$$H(t) = \frac{\dot{a}}{a} = \frac{2}{3t_\Lambda}\coth(t/t_\Lambda) \tag{11.94}$$

and then

$$q(t) = \frac{1}{2}[1 - 3\tanh^2(t/t_\Lambda)\,. \tag{11.95}$$

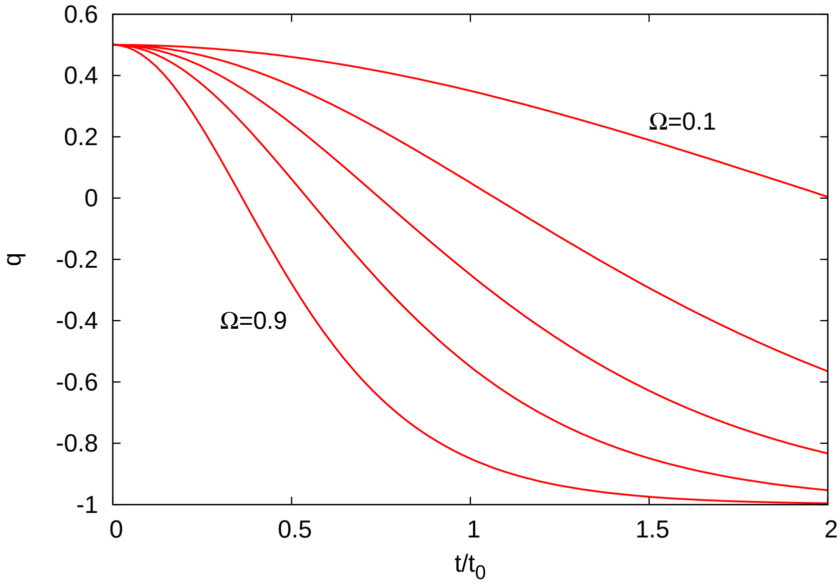


Figure 11.4: The deceleration parameter q as function of t/t_0 for the ΛCDM model and various values for Ω_Λ (0.1, 0.3, 0.5, 0.7 and 0.9 from the top to the bottom).

The limiting behavior of q corresponds with $q = 1/2$ for $t \to 0$ and $q = -1$ for $t \to \infty$ as expected to the one of a flat $\Omega_m = 1$ and a $\Omega_\Lambda = 1$ universe. More interesting is the transition region and, as shown in Fig. 11.4, the transition from a decelerating to an accelerating universe happens for $\Omega_\Lambda = 0.7$ at $t \approx 0.55 t_0$. This can easily converted to redshift, $z_* = a(t_0)/a(t_*) - 1 \approx 0.7$, that is directly measured in supernova observations.

11.6 Cosmological tests

11.6.1 Determining Λ and the curvature R_0 from $\rho_{m,0}$, H_0, q_0

General discussion: We apply now the Friedmann and the acceleration equation to the present time. Thus $\dot{R}_0 = R_0 H_0$, $\ddot{R} = -q_0 H_0^2 R_0$ and we can neglect the pressure term in Eq. (11.54),

$$\frac{\ddot{R}_0}{R_0} = -q_0 H_0^2 = \frac{\Lambda}{3} - \frac{4\pi G}{3}\rho_{m,0} \,. \tag{11.96}$$

Thus we can determine the value of the cosmological constant from the observables $\rho_{m,0}$, H_0 and q_0 via

$$\Lambda = 4\pi G \rho_{m,0} - 3 q_0 H_0^2 \,. \tag{11.97}$$

Solving next the Friedmann equation (11.53) for k/R_0^2,

$$\frac{k}{R_0^2} = \frac{8\pi G}{3}\rho_{m,0} + \frac{\Lambda}{3} - H_0^2 \,, \tag{11.98}$$

we write $\rho_{m,0} = \Omega_m \rho_{\rm cr}$ and insert Eq. (11.97) for Λ. Then we obtain for the curvature term

$$\frac{k}{R_0^2} = \frac{H_0^2}{2}\left(3\Omega_m - 2q_0 - 2\right) \,. \tag{11.99}$$

Hence the sign of $3\Omega_m - 2q_0 - 2$ decides about the sign of k and thus the curvature of the universe. For a universe without cosmological constant, $\Lambda = 0$, equation (11.97) gives $\Omega_m = 2q_0$ and thus

$$\begin{aligned} k = -1 \quad &\Leftrightarrow \quad \Omega_m < 1 \quad \Leftrightarrow \quad q_0 < 1/2\,, \\ k = 0 \quad &\Leftrightarrow \quad \Omega_m = 1 \quad \Leftrightarrow \quad q_0 = 1/2\,, \\ k = +1 \quad &\Leftrightarrow \quad \Omega_m > 1 \quad \Leftrightarrow \quad q_0 > 1/2\,. \end{aligned} \tag{11.100}$$

For a flat universe with $\Lambda = 0$, $\rho_{m,0} = \rho_{\rm cr}$ and $k = 0$,

$$0 = 4\pi G\,\frac{3H_0^2}{8\pi G} + H_0^2(q_0 - 1) = H_0^2\left(\frac{3}{2} + q_0 - 1\right)\,, \tag{11.101}$$

and thus $q_0 = 1/2$. In this special case, $q_0 < 1/2$ means $k = -1$ and thus an infinite space with negative curvature, while a finite space with positive curvature has $q > 1/2$.

Example 11.1: Comparison with observations:: Use the Friedmann equations applied to the present time to derive central values of Λ and k, R_0 from the observables $H_0 \approx (71 \pm 4)$ km/s/Mpc and $\rho_0 = (0.27 \pm 0.04)\rho_{\rm cr}$, and $q_0 = -0.6$.
We evaluate first

$$H_0^2 \approx \left(\frac{7.1 \times 10^6 \text{cm}}{\text{s}\; 3.1 \times 10^{24}\text{cm}}\right)^2 \approx 5.2 \times 10^{-36}\text{s}^{-2}\,.$$

The value of the cosmological constant Λ follows as

$$\Lambda = 4\pi G\rho_{m,0} - 3q_0H_0^2 = 3H_0^2\left(\frac{\rho}{2\rho_{\rm cr}} - 3q_0\right) \approx 3H_0^2 \times (\frac{1}{2} \times 0.27 + 0.6) \approx 0.73 \times 3H_0^2$$

or $\Omega_\Lambda = 0.73$. The curvature radius R follows as

$$\begin{aligned} \frac{k}{R_0^2} &= 4\pi G\rho_{m,0} - H_0^2(q_0 + 1) = 3H_0^2\left(\frac{\rho}{2\rho_{\rm cr}} - \frac{q_0 + 1}{3}\right) \\ &= 3H_0^2\,(0.135 \pm 0.02 - 0.4/3) = 3H_0^2\,(0.002 \pm 0.02) \end{aligned}$$

thus a flat universe ($k = 0$) is consistent with the given values.

11.6.2 Standard candles, rulers and sirenes

11.7 Particle horizons

The particle horizon l_H is defined as distance out to which one can observe a particle by exchange of a light signal, i.e. it is the border of the region causally connected to the observer. Without expansion, $l_H = ct_0$, where t_0 is the age of the universe. In an expanding universe, the path the light has to travel will be stretched, $dl_H = R_0/R(t)cdt$, and thus

$$l_H = cR_0\int \frac{dt'}{R(t')}$$

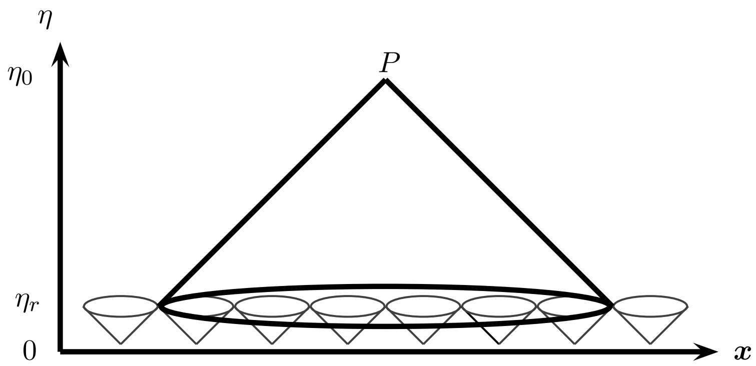


Figure 11.5: Causal structure of a spacetime with big bang: particle horizons at the time η_r are shown in grey together with the event horizon of point P (black cone).

For a matter- or radiation-dominated universe $R(t) = R_0(t/t_0)^\alpha$ with $\alpha = 2/3$ and $1/2$, respectively. Both models start with an initial singularity $t = 0$, and thus

$$l_H(t_0) = c\int_0^{t_0} dt \left(\frac{t}{t_0}\right)^{-\alpha} = \frac{ct_0}{1-\alpha}\,.$$

The ratio

$$\frac{l_H(t)}{R(t)} \propto \frac{t}{t^\alpha} \propto t^{1-\alpha}$$

gives the fraction of the Hubble horizon that was causally connected at time $t < t_0$. Since $0 < \alpha < 1$, this fraction decreases going back in time.

For an universe dominated by a cosmological constant $\Lambda > 0$, $R(t) = R_0 \exp(\sqrt{\Lambda/3}t) = R_0 \exp(Ht)$ and thus

$$l_H(t_2) = cR_0 \int_t^{t_0} \frac{dt'}{\exp(Ht')} = \frac{cR_0}{H}[\exp(-Ht) - \exp(-Ht_0)]$$

With $R(t_0) = R_0$ and thus $t_0 = 0$,

$$\frac{l_H(t)}{R_0} = \frac{c}{H}[\exp(-Ht) - 1]\,.$$

Since $t < t_0 = 0$, the expression in the bracket is always larger than one and the causally connected region is larger than the Hubble horizon. If exponential expansion would have persisted for all times, then $l_H(t) \to \infty$ for $t \to -\infty$ and thus the whole universe would be causally connected.

12 Cosmic relics

12.1 Time-line of important dates in the early universe

Different energy form today. Let us summarize the relative importance of the various energy forms today. The critical density $\rho_{\rm cr} = 3H_0^2/(8\pi G)$ has with $h = 0.7$ today the numerical value $\rho_{\rm cr} \approx 7.3 \times 10^{-6}$ GeV/cm^3. This would corresponds to roughly 8 protons per cubic meter. However, main player today is the cosmological constant with $\Omega_\Lambda \approx 0.73$. Next comes (pressure-less) matter with $\Omega_m \approx 0.27$ that consists mostly of non-baryonic dark matter, while only $\Omega_b = 4\%$ of the total energy density of the universe consists of matter that we know. The energy density of cosmic microwave background (CMB) photons with temperature $T = 2.7\,\text{K} = 2.3 \times 10^{-4}\,\text{eV}$ is $\rho_\gamma = aT^4 = 0.4$ eV/cm^3 or $\Omega_\gamma \approx 5 \times 10^{-5}$.

The contribution of the three neutrino flavors to the energy density depends on the unknown absolute neutrino mass scale, $5 \times 10^{-5} \lesssim \Omega_\nu \lesssim 0.05$. The lower bound corresponds to three (effectively) massless neutrinos, the upper to one massive neutrino flavor with $m_\nu \sim 0.3\,\text{eV}$.

Different energy forms as function of time The scaling of Ω_i with redshift z, $1 + z = R_0/R(t)$ is given by

$$H^2(z)/H_0^2 = \Omega_{\rm m,0}(1+z)^3 + \Omega_{\rm rad,0}(1+z)^4 + \Omega_\Lambda - \underbrace{(\Omega_{\rm tot,0} - 1)(1+z)^2}_{\approx 0} . \tag{12.1}$$

Thus the relative importance of the different energy forms changes: Going back in time, one enters first the matter-dominated and then the radiation-dominated epoch.

The cosmic triangle shown in Fig. 12.1 illustrates the evolution in time of the various energy components and the resulting coincidence problem: Any universe with a non-zero positive cosmological constant will be driven with time to a fix-point with $\Omega_m, \Omega_k \to 0$. The only other non-evolving state is a flat universe containing only matter—however, this solution is unstable. Hence, the question arises why we live in an epoch where all energy components have comparable size.

Temperature increase as $T \sim 1/R$ has three main effects: Firstly, bound states like atoms and nuclei are dissolved when the temperature reaches their binding energy, $T \gtrsim E_b$. Secondly, particles with mass m_X can be produced, when $T \gtrsim 2m_X$, in reactions like $\gamma\gamma \to \bar{X}X$. Thus the early Universe consists of a plasma containing more and more heavier particles that are in thermal equilibrium. Finally, most reaction rates $\Gamma = n\sigma v$ increase faster than the expansion rate of the universe for $t \to 0$, since $n \propto T^3$ for relativistic particles, while $H \propto \rho_{\rm rad}^{1/2} \propto T^2$. Therefore, reactions that have became ineffective today were important in the early Universe.

Matter-radiation equilibrium $z_{\rm eq}$: The density of matter decreases slower than the energy density of radiation. Going backward in time, there will be therefore a time when the density

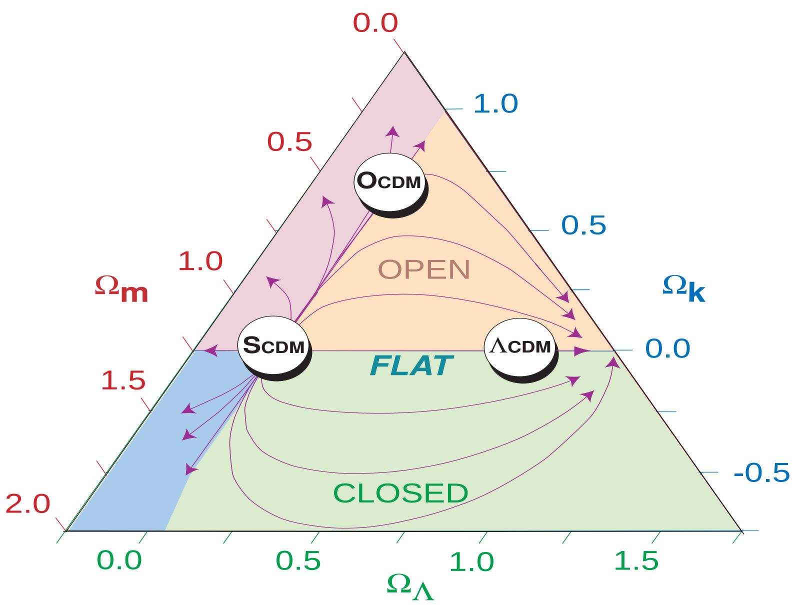


Figure 12.1: The cosmic triangle showing the time evolution of the various energy components.

of matter and radiation were equal. Before that time with redshift $z_{\rm eq}$, the universe was *radiation-dominated*,

$$\Omega_{\rm rad,0}(1+z_{\rm eq})^4 = \Omega_{\rm m,0}(1+z_{\rm eq})^3 \tag{12.2}$$

or

$$z_{\rm eq} = \frac{\Omega_{\rm m,0}}{\Omega_{\rm rad,0}} - 1 \approx 5400\,. \tag{12.3}$$

This time is important, because i) the time-dependence of the scale factor changes from $R \propto t^{2/3}$ for a matter to $R \propto t^{1/2}$ for a radiation dominated universe, ii) the E.o.S. and thus the speed of sound changed from $w \approx 1/3$, $v_s^2 = (\partial P/\partial\rho)_S = c^2/3$ to $w \approx 0$, $v_s^2 = 5kT/(3m) \ll c^2$. The latter quantity determines the Jeans length and thus which structures in the Universe can collapse.

Recombination $z_{\rm rec}$: Today, most hydrogen and helium in the interstellar and intergalactic medium is neutral. Increasing the temperature, the fraction of ions and free electron increases, i.e. the reaction $H+\gamma \leftrightarrow H^+ + e^-$ that is mainly controlled by the factor $\exp(-E_b/kT)$ will be shifted to the right. By definition, we call recombination the time when 50% of all atoms are ionized. A naive estimate gives $kT \sim E_b \approx 13.6$ eV$\approx 160.000K$ or $z_{\rm rec} = 60.000$. However, there are many more photons than hydrogen atoms, and therefore recombination happens latter: A more detailed calculation gives $z_{\rm rec} \sim 1000$.
Since the interaction probability of photons with neutral hydrogen is much smaller than with electrons and protons, recombination marks the time when the Universe became transparent to light.

Big Bang Nucleosynthesis At $T_{ns} \sim \Delta \equiv m_n - m_p \approx 1.3$ MeV or $t \sim 1$ s, part of protons and neutrons forms nuclei, mainly ^{4}He. As in the case of recombination, the large number of photons delays nucleosynthesis relative to the estimate $T_{ns} \approx \Delta$ to $T_{ns} \approx 0.1\,\mathrm{MeV}$.

Quark-hadron or QCD transition Above $T \sim m_\pi \sim 100$ MeV, hadrons like protons, neutrons or pions dissolve into their fundamental constituents, quarks q and gluons g.

Baryogenesis All the matter observed in the Universe consists of matter (protons and electrons), and not of anti-matter (anti-protons an positrons). Thus the baryon-to-photon ratio is

$$\eta = \frac{n_b - n_{\bar{b}}}{n_\gamma} = \frac{n_b}{n_\gamma} = \frac{\Omega_b \rho_{\mathrm{cr}}/m_N}{2\zeta(3)T_\gamma^3/\pi^2} \approx 7 \times 10^{-10}\,. \tag{12.4}$$

The early plasma of quarks q and anti-quarks $\bar{q}$ contained a tiny surplus of quarks. After all anti-matter annihilated with matter, only the small surplus of matter remained. The tiny asymmetry can be explained by interactions in the early Universe that were not completely symmetric with respect to an exchange of matter-antimatter.

12.2 Equilibrium statistical physics in a nut-shell

The distribution function $f(p)$ of a free gas of fermions or bosons in *kinetic equilibrium* are are

$$f(p) = \frac{1}{\exp[\beta(E-\mu)] \pm 1} \tag{12.5}$$

where $\beta = 1/T$ denotes the inverse temperature, $E = \sqrt{m^2 + p^2}$, and +1 refers to fermions and -1 to bosons, respectively. As we will see later, photons as massless particles stay also in an expanding universe in equilibrium and may serve therefore as a thermal bath for other particles. A species X stays in kinetic equilibrium, if e.g. in the reaction $X + \gamma \to X + \gamma$ the energy exchange with photons is fast enough.

The chemical potential μ is the average energy needed, if an additional particles is added, $\mathrm{d}U = \sum_i \mu \mathrm{d}N_i$. If μ is zero, If the species X is also in *chemical equilibrium* with other species, e.g. via the reaction $X + \bar{X} \leftrightarrow \gamma + \gamma$ with photons, then their chemical potentials are related by $\mu_X + \mu_{\bar{X}} = 2\mu_\gamma = 0$.

The number density n, energy density ρ and pressure P of a species X follows as

$$n = \frac{g}{(2\pi)^3} \int d^3p\, f(p)\,, \tag{12.6}$$

$$\rho = \frac{g}{(2\pi)^3} \int d^3p\, E f(p)\,, \tag{12.7}$$

$$P = \frac{g}{(2\pi)^3} \int d^3p\, \frac{p^2}{3E}\, f(p)\,. \tag{12.8}$$

The factor g takes into account the internal degrees of freedom like spin or color. Thus for a photon, a massless spin-1 particle $g = 2$, for an electron $g = 4$, etc.

Derivation of the pressure integral for free quantum gas:
Comparing the 1. law of thermodynamics, $dU = TdS - PdV$, with the total differential $dU = (\partial U/\partial S)_V dS + (\partial U/\partial V)_S dV$ gives $P = -(\partial U/\partial V)_S$.
Since $U = V\int Ef(p)$ and $S \propto \ln(Vf(p))$, differentiating U keeping S constant means $P = -V\int(\partial E/\partial V)f(p)$.
We write $\partial E/\partial V = (\partial E/\partial p)(\partial p/\partial L)(\partial L/\partial V)$. To evaluate this we note that $\partial E/\partial p = p/E$, that from $V = L^3$ it follows $\partial L/\partial V = 1/(3L^2)$ and that finally the quantization conditions of free particles, $p_k = 2\pi k/L$ implies $\partial p/\partial L = -p/L$. Combined this gives $\partial E/\partial V = -p^2/(3EV)$.

In the non-relativistic limit $T \ll m$, $\mathrm{e}^{\beta(m-\mu)} \gg 1$ and thus differences between bosons and fermions disappear,

$$
\begin{aligned}
n &= \frac{g}{2\pi^2}\mathrm{e}^{-\beta(m-\mu)}\int_0^\infty dp\, p^2 \mathrm{e}^{-\beta\frac{p^2}{2m}} = g\left(\frac{mT}{2\pi}\right)^{3/2}\exp[-\beta(m-\mu)]\,, && (12.9)\\
\rho &= mn\,, && (12.10)\\
P &= nT \ll \rho\,. && (12.11)
\end{aligned}
$$

These expressions correspond to the classical Maxwell-Boltzmann statistics[1]. The number of non-relativistic particles is exponentially suppressed, if their chemical potential is small. Since the number of protons per photons is indeed very small in the universe, cf. Eq. (12.4), and therefore also the number of electron (the universe should be neutral), the chemical potential μ can be neglected in cosmology at least for protons and electron.

In the relativistic limit $T \gg m$ with $T \gg \mu$ all properties of a gas are determined by its temperature T,

$$
\begin{aligned}
n &= \frac{g\,T^3}{2\pi^2}\int_0^\infty dx \frac{x^2}{\mathrm{e}^x \pm 1} = \varepsilon_1 \frac{\zeta(3)}{\pi^2}\, gT^3\,, && (12.12)\\
\rho &= \frac{g\,T^4}{2\pi^2}\int_0^\infty dx \frac{x^3}{\mathrm{e}^x \pm 1} = \varepsilon_2 \frac{\pi^2}{30}\, gT^4\,, && (12.13)\\
P &= \rho/3\,, && (12.14)
\end{aligned}
$$

where for bosons $\varepsilon_1 = \varepsilon_2 = 1$ and for fermions $\varepsilon_1 = 3/4$ and $\varepsilon_2 = 7/8$, respectively.

Since the energy density and the pressure of non-relativistic species is exponentially suppressed, the total energy density and the pressure of all species present in the universe can be well-approximated including only relativistic ones,

$$
\begin{aligned}
\rho_{\mathrm{rad}} &= \frac{\pi^2}{30}\, g_* T^4\,, && (12.15)\\
P_{\mathrm{rad}} &= \rho_{\mathrm{rad}}/3 = \frac{\pi^2}{90}\, g_* T^4\,, && (12.16)
\end{aligned}
$$

where

$$
g_* = \sum_{\text{bosons}} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8}\sum_{\text{fermions}} g_i \left(\frac{T_i}{T}\right)^4 . \qquad (12.17)
$$

Here we took into account that the temperature of different particle species can differ.

[1] Integrals of the type $\int_0^\infty dx x^{2n} \mathrm{e}^{-ax^2}$ can be reduced to a Gaussian integral by differentiating with respect to the parameter a.

Entropy Rewriting the first law of thermodynamics, $\mathrm{d}U = T\mathrm{d}S - P\mathrm{d}V$, as

$$\mathrm{d}S = \frac{\mathrm{d}U}{T} + \frac{P}{T}\mathrm{d}V = \frac{\mathrm{d}(V\rho)}{T} + \frac{p}{T}\mathrm{d}V = \frac{V}{T}\frac{d\rho}{\mathrm{d}T}\mathrm{d}T + \frac{\rho + P}{T}\mathrm{d}V \tag{12.18}$$

and comparing this expression with the total differential $\mathrm{d}S(T,V)$, one obtains

$$\frac{\partial S}{\partial V}_T = \frac{\rho + P}{T}. \tag{12.19}$$

Since the RHS is independent of V for constant T, we can integrate and obtain

$$S = \frac{\rho + P}{T}V + f(T). \tag{12.20}$$

The integration constant $f(T)$ has to vanish to ensure that S is an extensive variable, $S \propto V$.

The total entropy density $s \equiv S/V$ of the universe can again approximated by the relativistic species,

$$s = \frac{2\pi^2}{45}\, g_{*S} T^3 , \tag{12.21}$$

where now

$$g_{*,S} = \sum_{\text{bosons}} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{\text{fermions}} g_i \left(\frac{T_i}{T}\right)^3 . \tag{12.22}$$

The entropy S is an important quantity because it is conserved during the evolution of the universe. Conservation of S implies that $S \propto g_{*,S}R^3T^3 = \text{const.}$ and thus the temperature of the Universe evolves as

$$\boxed{T \propto g_{*,S}^{-1/3} R^{-1} .} \tag{12.23}$$

When g_* is constant, the temperature $T \propto 1/R$. Consider now the case that a particle species, e.g. electrons, becomes non-relativistic at $T \sim m_e$. Then the particles annihilate, $e^+ + e^- \to \gamma\gamma$, and its entropy is transferred to photons. Formally, $g_{*,S}$ decreases and therefore the temperature decreases for a short period less slowly than $T \propto 1/R$.

Since $s \propto R^{-3}$ and also the net number of particles with a conserved charge, e.g. $n_B \equiv n_B - n_{\bar{B}} \propto R^{-3}$ if baryon number B is conserved, the ratio n_B/s remains constant.

Relativistic degrees of freedom. To obtain the number of relativistic degrees of freedom g_* in the universe as function of T, we have to know the degrees of freedom of the various particle species:

- The spin degrees of freedom of massive particles with spin s are $2s+1$, and of neutrinos 1, where we count particles and anti-particles separately. Massless bosons like photons and gravitons are their own anti-particle and have 2 spin states.
- Below $T_{\text{QCD}} \sim 250$ MeV strongly interacting particles are bound in hadrons, while above T_{QCD} free quarks and gluons exist.
- Quarks have as additional label 3 colors, there are eight gluons.
- We assume that all species have the same temperature and approximate their contribution to g_* by a step function $\vartheta(T - m)$.

Using the "Particle Data Book" to find the masses of the various particles, we can construct g_* as function of T as shown in table 12.1.

Temperature	new particles	$4\Delta g_*$	$4g_*$
$T < m_e$	$\gamma + \nu_i$	$4 \times (2 + 3 \times 2 \times 7/8)$	29
$m_e < T < m_\mu$	$e^\pm$	14	43
$m_\mu < T < m_\pi$	$\mu\pm$	14	57
$m_\pi < T < T_c$	$\pi^\pm, \pi^0$	12	69
$T_c < T < m_s$	$u, \bar{u}, d, \bar{d}, g$	$6 \times 14 + 4 \times 8 \times 2 - 12$	205
$m_s < T < m_c$	$s, \bar{s}$	$3 \times 14 = 42$	247
$m_c < T < m_\tau$	$c, \bar{c}$	42	289
$m_\tau < T < m_b$	$\tau^\pm$	14	303
$m_b < T < m_{W,Z}$	$b, \bar{b}$	42	345
$m_{W,Z} < T < m_h$	$W^\pm, Z$	$4 \times 3 \times 3 = 36$	381
$m_h < T < m_t$	h	4	384
$m_t < T <?$	$t, \bar{t}$	42	426

Table 12.1: The number of relativistic degrees of freedom g_* present in the universe as function of its temperature.

12.3 Big Bang Nucleosynthesis

Nuclear reactions in stars are supposed to produce all the observed heavier elements. However, stellar reaction can explain at most a fraction of 5% of ^{4}He, while the production of the weakly bound deuterium and Lithium-7 in stars is impossible. Thus the light elements up to Li-7 are primordial: $Y(D) = \text{few} \times 10^{-5}$, $Y(^3H) = \text{few} \times 10^{-5}$ $Y(^4\text{He}) \approx 0.25$, $Y(^7Li) \approx (1-2) \times 10^{-7}$. Observational challenge is to find as "old" stars/gas clouds as possible and then to extrapolate back to primordial values.

Example 12.2: Estimate the amount of ^{4}He producted by stars:
The binding energy of ^{4}He is $E_b = 28.3\,\text{MeV}$. If 1/4 of all nucleons were fused into ^{4}He during $t \sim 10\,\text{Gyr}$, the luminosity-mass ratio would be

$$\frac{L}{M_b} = \frac{1}{4}\,\frac{E_b}{4m_p t} = 5\,\frac{\text{erg}}{\text{g s}} \simeq 2.5\frac{L_\odot}{M_\odot}\,.$$

The observed luminosity-mass ratio is however only $L/M_b \leq 0.05 L_\odot/M_\odot$. Assuming a roughly constant luminosity of stars over time, they can produce only $0.05/2.5 \simeq 2\%$ of the observed ^{4}He.

Big Bang Nucleosynthesis (BBN) is controlled by two parameters: The mass difference between protons and neutrons, $\Delta \equiv m_n - m_p \simeq 1.3\,\text{MeV}$ and the freeze-out temperature T_f of reaction converting protons into neutrons and vice versa.

12.3.1 Equilibrium distributions

In the non-relativistic limit $T \ll m$, the number density of the nuclear species with mass number A and charge Z is

$$n_A = g_A \left(\frac{m_A T}{2\pi}\right)^{3/2} \exp[\beta(\mu_A - m_A)]\,. \tag{12.24}$$

In chemical equilibrium, $\mu_A = Z\mu_p + (A-Z)\mu_n$ and we can eliminate μ_A by inserting the equivalent expression of (12.24) for protons and neutrons,

$$\exp(\beta\mu_A) = \exp[\beta(Z\mu_p + (A-Z)\mu_n)] = \frac{n_p^Z n_n^{A-Z}}{2^A}\left(\frac{2\pi}{m_N T}\right)^{3A/2} \exp[\beta(Zm_p + (A-Z)m_n)]\,. \tag{12.25}$$

Here and in the following we can set in the pre-factors $m_p \approx m_n \approx m_N$ and $m_A \approx Am_N$, keeping the exact masses only in the exponentials. Inserting this expression for $\exp(\beta\mu_A)$ together with the definition of the binding energy of a nucleus, $B_A = Zm_p + (A-Z)m_n - m_A$, we obtain

$$n_A = g_A \left(\frac{2\pi}{m_N T}\right)^{3(A-1)/2} \frac{A^{3/2}}{2^A}\, n_p^Z n_n^{A-Z} \exp(\beta B_A)\,. \tag{12.26}$$

The mass fraction X_A contributed by a nuclear species is

$$X_A = \frac{An_A}{n_B} \qquad \text{with} \qquad n_B = n_p + n_n + \sum_i A_i n_{A_i} \qquad \text{and} \qquad \sum_i X_i = 1\,. \tag{12.27}$$

With $n_p^Z n_n^{A-Z}/n_N = X_p^Z X_n^{A-Z} n_N^{A-1}$ and $\eta \propto T^3$ and thus $n_B^{A-1} \propto \eta^{A-1} T^{3(A-1)}$, we have

$$X_A \propto \left(\frac{T}{m_N}\right)^{3(A-1)/2} \eta^{A-1} X_p^Z X_n^{A-Z} \exp(\beta B_A)\,. \tag{12.28}$$

The fact that $\eta \ll 1$, i.e. that the number of photons per baryon is extremely large, means that nuclei with $A > 1$ are much less abundant and that nucleosynthesis takes place later than naively expected. Let us consider the particular case of deuterium in Eq. (12.28),

$$\frac{X_D}{X_p X_n} = \frac{24\zeta(3)}{\sqrt{\pi}} \left(\frac{T}{m_N}\right)^{3/2} \eta\, \exp(\beta B_D) \tag{12.29}$$

with $B_D = 2.23$ MeV. The start of nucleosynthesis could be defined approximately by the condition $X_D/(X_p X_n) = 1$, or $T \approx 0.1\,\text{MeV}$ according to the left panel in Fig. 12.2. The right panel of the same figure shows the results, if the equations (12.28) together with $\sum_i X_i = 1$ are solved for the lightest and stablest nuclei. Now it becomes clear that in thermal equilibrium between $0.1 \lesssim T \lesssim 0.2\,\text{MeV}$ essentially all free neutrons will bind to ^{4}He. For low temperatures one cannot expect that the true abundance follows the equilibrium abundance, Eq. (12.28), shown in Fig. 12.2. First, in the expanding universe the weak reactions that convert protons and nucleons will freeze out as soon as their rate drops below the expansion rate of the universe. This effect will discussed in the following in more detail. Second, the Coulomb barrier will prevent the production of nuclei with $Z \gg 1$. Third, neutrons are not stable and decay.

12.3.2 Proton-neutron ratio

Gamov criterion The interaction depth $\tau = nl\sigma$ gives the probability that a test particle interacts with cross section σ in a slab of length l filled with targets of density n. If $\tau \gg 1$, interactions are efficient and the test particle is in thermal equilibrium with the surrounding. We can apply the same criteria to the Universe: We say a particle species A is in thermal equilibrium, as long as $\tau = nl\sigma = n\sigma vt \gg 1$. The time t corresponds to the typical time-scale

Figure 12.2: Relative equilibrium abundance $X_D/(X_p X_n)$ of deuterium as function of temperature T (left) and equilibrium mass fractions of nucleons, D, ^{3}He, ^{2}He and ^{12}C (right).

for the expansion of the universe, $\tau = (\dot{R}/R)^{-1} = H^{-1}$. Note that this is also the typical time-scale for changes in the temperature T. Thus we can rewrite this condition as

$$\Gamma \equiv n\sigma v \gg H\,. \tag{12.30}$$

A particle species "goes out of equilibrium" when its interaction rate Γ becomes smaller than the expansion rate H of the universe.

Decoupling of neutrinos The interaction rates of neutrinos in processes like $n \leftrightarrow p + e^- + \nu_e$ or $e^+e^- \leftrightarrow \bar{\nu}\nu$ is $\sigma \sim G_F^2 E^2$. If we approximate the energy of all particle species by their temperature T, their velocity by c and their density by $n \sim T^3$, then the interaction rate of weak processes is

$$\Gamma \approx \langle v\sigma n_\nu \rangle \approx G_F^2 T^5 \tag{12.31}$$

The early universe is radiation-dominated with $\rho_{\rm rad} \propto 1/R^4$, $H = 1/(2t)$ and negligible curvature k/R^2. Thus the Friedmann equation simplifies to $H^2 = (8\pi/3)G\rho$ with $\rho = g_*\pi^2/30T^4$, or

$$\frac{1}{2t} = H = 1.66\sqrt{g_*}\,\frac{T^2}{M_{\rm Pl}}\,. \tag{12.32}$$

Here, we introduced also the Planck mass $M_{\rm Pl} = 1/\sqrt{G_N} \approx 1.2 \times 10^{19}$ GeV. Requiring $\Gamma(T_{\rm fr}) = H(T_{\rm fr})$ gives as freeze-out temperature $T_{\rm fr}$ of weak processes

$$T_{\rm fr} \approx \left(\frac{1.66\sqrt{g_*}}{G_F^2 M_{\rm Pl}}\right)^{1/3} \approx 1\text{MeV} \tag{12.33}$$

with $g_* = 10.75$. The relation between time and temperature follows as

$$\frac{t}{\rm s} = \frac{2.4}{\sqrt{g_*}}\left(\frac{\rm MeV}{T}\right)^2\,. \tag{12.34}$$

Thus the time-sequence is as follows

- at $T_{\rm fr} \approx 1$ MeV: the neutron-proton ratio freezes-in and can be approximated by the ratio of their equilibrium distribution in the non-relativistic limit.
- as the universe cools down from $T_{\rm fr}$ to $T_{\rm ns}$, neutrons decay with half-live $\tau_n \approx 886$ s.
- at $T_{\rm ns}$, practically all neutrons are bound to ^{4}He, with only small admixture of other elements.

Proton-neutron ratio Above T_f, reactions like $\nu_e + n \leftrightarrow p + e^-$ keep nucleons in thermal equilibrium. As we have seen, $T_f \sim 1$ MeV and thus we can treat nucleons in the non-relativistic limit. Then their relative abundance is given by the Boltzmann factor $\exp(-\Delta/T)$ for $T \gtrsim T_f$ with $\Delta = m_N - m_P = 1.29\,\text{MeV}$ for the mass difference of neutrons and protons. Hence for T_f,

$$\left.\frac{n_n}{n_p}\right|_{t=t_{\rm fr}} = \exp\left(-\frac{\Delta}{T_f}\right) \approx \frac{1}{6}\,. \tag{12.35}$$

As the universe cools down to $T_{\rm ns}$, neutrons decay with half-live $\tau_n \approx 886$ s,

$$\left.\frac{n_n}{n_p}\right|_{t=t_{\rm ns}} \approx \frac{1}{6}\exp\left(-\frac{t_{\rm ns}}{\tau_n}\right) \approx \frac{2}{15}\,. \tag{12.36}$$

12.3.3 Estimate of helium abundance

The synthesis if ^{4}He proceeds though a chain of reactions, $pn \to d\gamma$, $dp \to {}^3\text{He}\gamma$, $d\,{}^3\text{He} \to {}^4\text{He}p$. Let's assume that ^{4}He formation takes place instantaneously. Moreover, we assume that all neutrons are bound in ^{4}He. We need two neutrons to form one helium atom, $n(^4\text{He}) = n_n/2$, and thus

$$Y(^4\text{He}) \equiv \frac{M(^4\text{He})}{M_{\rm tot}} = \frac{4m_N \times n_n/2}{m_N(n_p+n_n)} = \frac{2n_n/n_p}{1+n_n/n_p} = \frac{4}{17} \sim 0.235 \tag{12.37}$$

Our naive estimate not too far away from $Y \sim 0.245$.

The dependence of $Y(^4\text{He})$ on the input physics is rather remarkable.

- The helium abundance dependence exponentially on Δ and T_f:
  - The mass difference Δ depends on both electromagnetic and strong interactions. BBN tests therefore the time-dependence of fundamental interaction expected e.g. in string theories.
  - The freeze-out temperature $T_{\rm fr}$ depends on number of relativistic degrees of freedom g_* and restricts thereby additional light particle.
  - a non-zero chemical potential of neutrinos.
- A weaker dependence on start of nucleosynthesis $T_{\rm ns}$ and thus η_b or Ω_b.

12.3.4 Results from detailed calculations

Detailed calculations predict not only the relative amount of light elements produced, but also their absolute amount as function of e.g. the baryon-photon ratio η. Requiring that the relative fraction of helium-4, deuterium and lithium-7 compared to hydrogen is consistent with observation allows one to determine η or equivalently the baryon content, $\Omega_b h^2 = 0.019 \pm 0.001$.

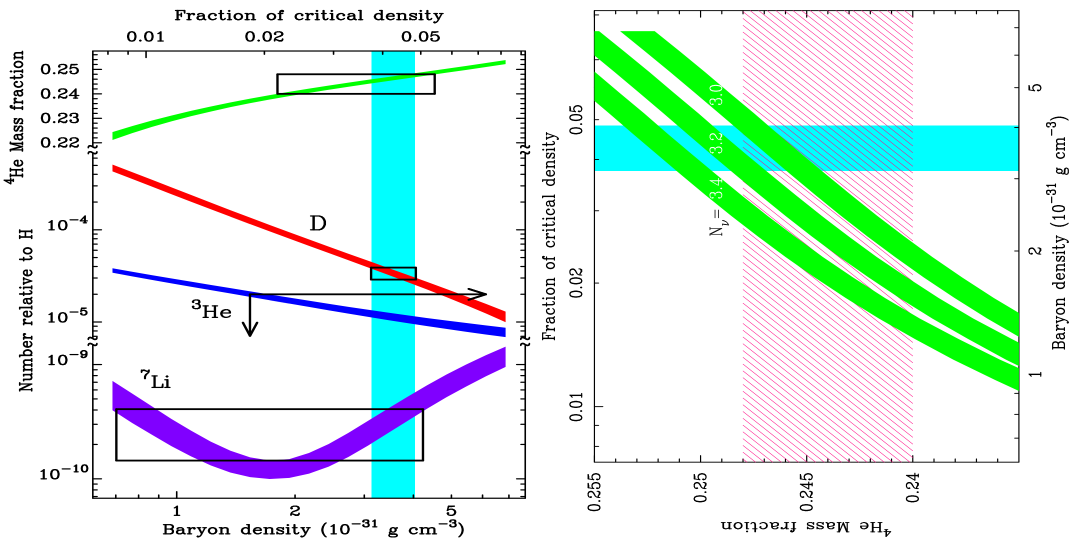


Figure 12.3: Abundances of light-elements as function of η (left) and of the number of light neutrino species (right).

Although the binding energy per nucleon of Carbon-12 and Oxygen-16 is higher than the of ^{4}He, they are not produced: at time of ^{4}He production Coulomb barrier prevents already fusion. Also, stable element with $A = 5$ is missing.

12.4 Dark matter

12.4.1 Freeze-out of thermal relic particles

When the number density n_X of a particle species X is not changed by interactions, then it is diluted just by the expansion of space, $n_X \propto R^{-3}$. It is convenient to account for this trivial expansion effect by dividing n_X through the entropy density $s \propto R^{-3}$, i.e. to use the quantity $Y = n/s$. We first consider again the equilibrium distribution $Y_{\rm eq}$ for $\mu_X = 0$,

$$Y_{\rm eq} = \frac{n_X}{s} = \begin{cases} \frac{45}{2\pi^4}\left(\frac{\pi}{8}\right)^{1/2} \frac{g_X}{g_{*S}}\, x^{3/2}\exp(-x) = 0.145\, \frac{g_X}{g_{*S}}\, x^{3/2}\exp(-x) & \text{for } x \gg 3, \\ \frac{45\zeta(3)}{2\pi^4}\, \frac{\varepsilon g_X}{g_{*S}} = 0.278\, \frac{\varepsilon g_X}{g_{*S}} & \text{for } x \ll 3 \end{cases} \tag{12.38}$$

where $x = T/m$ and $g_{\rm eff} = 3/4$ ($g_{\rm eff} = 1$) for fermions (bosons). If the particle X is in chemical equilibrium, its abundance is determined for $T \gg m$ by its contribution to the total number of degrees of freedom of the plasma, while $Y_{\rm eq}$ is exponentially suppressed for $T \ll m$ (assuming $\mu_X = 0$). In an expanding universe, one may expect that the reaction rate Γ for processes like $\gamma\gamma \leftrightarrow \bar{X}X$ drops below the expansion rate H mainly for two reasons: i) Cross sections may depend on energy as, e.g., weak processes $\sigma \propto s \propto T^2$ for $s \lesssim m_W^2$, ii) the density n_X decreases at least as $n \propto T^3$. Around the freeze-out time x_f, the true abundance Y starts to deviate from the equilibrium abundance $Y_{\rm eq}$ and becomes constant, $Y(x) \approx Y_{\rm eq}(x_{\rm f})$ for $x \gtrsim x_{\rm f}$. This behavior is illustrated in Fig. 12.4.

Boltzmann equation When the number $N = nV$ of a particle species is not changed by interactions, then the expansion of the Universe dilutes their number density as $n \propto R^{-3}$.

The corresponding change in time is connected with the expansion rate of the universe, the Hubble parameter $H = \dot{R}/R$, as

$$\frac{\mathrm{d}n}{\mathrm{d}t} = \frac{\mathrm{d}n}{\mathrm{d}R}\frac{\mathrm{d}R}{\mathrm{d}t} = -3n\frac{\dot{R}}{R} = -3Hn\,. \tag{12.39}$$

Additionally, there might be production and annihilation processes. While the annihilation rate $\beta n^2 = \langle\sigma_{\rm ann}v\rangle\, n^2$ has to be proportional to n^2, we allow for an arbitrary function as production rate ψ,

$$\frac{\mathrm{d}n}{\mathrm{d}t} = -3Hn - \beta n^2 + \psi\,. \tag{12.40}$$

In a static Universe, $\mathrm{d}n/\mathrm{d}t = 0$ defines equilibrium distributions $n_{\rm eq}$. Detailed balance requires that the number of X particles produced in reactions like $e^+e^- \to \bar{X}X$ is in equilibrium equal to the number that is destroyed in $\bar{X}X \to e^+e^-$, or $\beta n_{\rm eq}^2 = \psi_{\rm eq}$. Since the reaction partners (like the electrons in our example) are assumed to be in equilibrium, we can replace $\psi = \psi_{\rm eq}$ by $\beta n_{\rm eq}^2$ and obtain

$$\frac{\mathrm{d}n}{\mathrm{d}t} = -3Hn - \langle\sigma_{\rm ann}v\rangle(n^2 - n_{\rm eq}^2)\,. \tag{12.41}$$

This equation together with the initial condition $n \approx n_{\rm eq}$ for $T \to \infty$ determines $n(t)$ for a given annihilation cross section $\sigma_{\rm ann}$.

Next we rewrite the evolution equation for $n(t)$ using the dimensionless variables Y and x. Changing from $n = sY$ to Y we can eliminate the $3Hn$ term,

$$\frac{\mathrm{d}n}{\mathrm{d}t} = -3Hn + s\,\frac{\mathrm{d}Y}{\mathrm{d}t}\,. \tag{12.42}$$

With $(2t)^{-2} = H^2 \propto \rho \propto T^4 \propto x^{-4}$ or $t = t_* x^2$, we obtain

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = -\frac{sx}{H}\langle\sigma_{\rm ann}v\rangle\left(Y^2 - Y_{\rm eq}^2\right)\,. \tag{12.43}$$

Finally we recast the Boltzmann equation in a form that makes our intuitive Gamov criterion explicit,

$$\frac{x}{Y_{\rm eq}}\frac{\mathrm{d}Y}{\mathrm{d}x} = -\frac{\Gamma_A}{H}\left[\left(\frac{Y}{Y_{\rm eq}}\right)^2 - 1\right] \tag{12.44}$$

with $\Gamma_A = n_{\rm eq}\langle\sigma_{\rm ann}v\rangle$: The relative change of Y is controlled by the factor Γ_A/H times the deviation from equilibrium. The evolution of $Y = n_X/s$ is shown schematically in Fig. 12.4: As the universe expands and cools down, n_X decreases at least as R^{-3}. Therefore, the annihilation rate $\propto n^2$ quenches and the abundance "freezes-out:" The reaction rates are not longer sufficient to keep the particle in equilibrium and the ratio n_X/s stays constant.

For the discussion of approximate solutions to this equation, it is convenient to distinguish according to the freeze-out temperature: hot dark matter (HDM) with $x_{\rm f} \ll 3$, cold dark matter (CDM) with $x_{\rm f} \gg 3$ and the intermediate case of warm dark matter with $x_{\rm f} \sim 3$.

12.4.2 Hot dark matter

For $x_{\rm f} \ll 3$, freeze-out occurs when the particle is still relativistic and $Y_{\rm eq}$ is not changing with time. The asymptotic value of Y, $Y(x \to \infty) \equiv Y_\infty$, is just the equilibrium value at freeze-out,

$$Y_\infty = Y_{\rm eq}(x_{\rm f}) = 0.278\,\frac{g_{\rm eff}}{g_{*S}}\,, \tag{12.45}$$

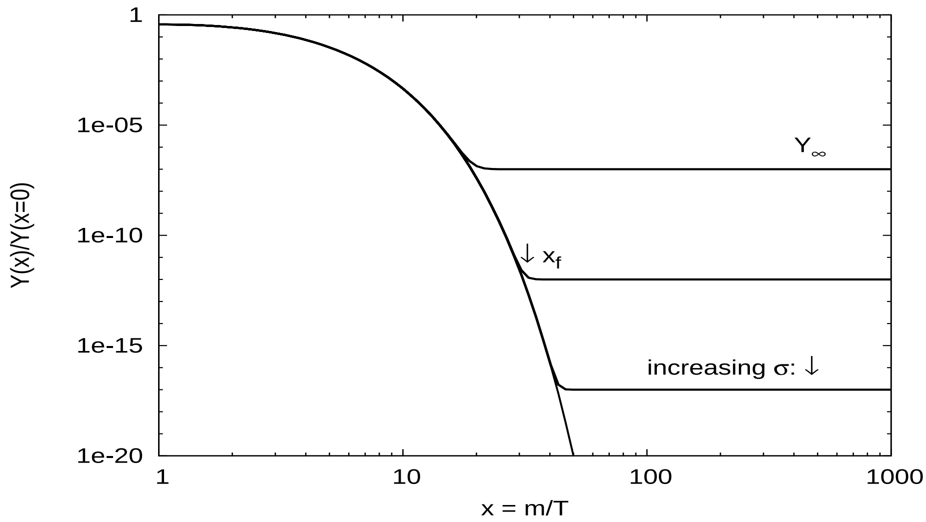


Figure 12.4: Illustration of the freeze-out process. The quantity $Y = n_X/s$ is n_X divided by the entropy density $s \propto R^{-3}$ to scale out the trivial effect of expansion.

where the only temperature-dependence is contained in g_{*S}. The number density today is then

$$n_0 = s_0 Y_\infty = 2970\, Y_\infty \mathrm{cm}^{-3} = 825\, \frac{g_{\mathrm{eff}}}{g_{*S}}\, \mathrm{cm}^{-3}\,. \tag{12.46}$$

The numerical value of s_0 used will be discussed in the next paragraph. Although a HDM particle was relativistic at freeze-out, it is today non-relativistic if its mass m is $m \gg 3\mathrm{K} \approx 0.2\mathrm{meV}$. In this case its energy density is simply $\rho_0 = m s_0 Y_\infty$ and its abundance $\Omega h^2 = \rho_0/\rho_{\mathrm{cr}}$ or

$$\Omega h^2 = 7.8 \times 10^{-2}\, \frac{m}{\mathrm{eV}}\, \frac{g_{\mathrm{eff}}}{g_{*S}}\,. \tag{12.47}$$

Hence HDM particles heavier than $O(100\mathrm{eV})$ overclose the universe.

12.4.3 Cold dark matter

Abundance of CDM For CDM with $x_{\mathrm{f}} \gg 3$, freeze-out occurs when the particles are already non-relativistic and Y_{eq} is exponentially changing with time. Thus the main problem is to find x_{f}, for late times we use again $Y(x \to \infty) \equiv Y_\infty \approx Y(x_{\mathrm{f}})$, i.e. the equilibrium value at freeze-out. We parametrize the temperature-dependence of cross section as $\langle \sigma_{\mathrm{ann}} \rangle = \sigma_0 (T/m)^n = \sigma_0/x^n$. For simplicity, we consider only the most relevant case for CDM, $n = 0$ or s-wave annihilation. Then the Gamov criterion becomes with $H = 1.66\sqrt{g_*}\, T^2/M_{\mathrm{Pl}}$ and $\Gamma_A = n_{\mathrm{eq}} \langle \sigma_{\mathrm{ann}} v \rangle$,

$$g \left(\frac{m T_{\mathrm{f}}}{2\pi} \right)^{3/2} \exp(-m/T_{\mathrm{f}})\, \sigma_0 = 1.66 \sqrt{g_*}\, \frac{T_{\mathrm{f}}^2}{M_{\mathrm{Pl}}} \tag{12.48}$$

or

$$x_{\mathrm{f}}^{-1/2} \exp(x_{\mathrm{f}}) = 0.038 \frac{g}{\sqrt{g_*}} M_{\mathrm{Pl}} m \sigma_0 \equiv C\,. \tag{12.49}$$

To obtain an approximate solution, we neglect first in

$$\ln C = -\frac{1}{2}\, \ln x_{\mathrm{f}} + x_{\mathrm{f}} \tag{12.50}$$

the slowly varying term $\ln x_{\rm f}$. Inserting next $x_{\rm f} \approx \ln C$ into Eq. (12.50) to improve the approximation gives then

$$x_{\rm f} = \ln C + \frac{1}{2}\ln(\ln C)\,. \tag{12.51}$$

The relic abundance for CDM follows from $n(x_{\rm f}) = 1.66\sqrt{g_*}\,T_{\rm f}^2/(\sigma_0 M_{\rm Pl})$ and $n_0 = n(x_{\rm f})[R(x_{\rm f})/R_0]^3 = n(x_{\rm f})[g_{*,f}/g_{*,0}][T_0/T(x_{\rm f})]^3$ as

$$\rho_0 = mn_0 \approx 10\frac{x_{\rm f}T_0^3}{\sqrt{g_{*,f}}\sigma_0 M_{\rm Pl}} \tag{12.52}$$

or

$$\Omega_X h^2 = \frac{mn_0}{\rho_{\rm cr}} \approx \frac{4\times 10^{-39}{\rm cm}^2}{\sigma_0}\,x_{\rm f} \tag{12.53}$$

Thus the abundance of a CDM particle is inverse proportionally to its annihilation cross section, since a more strongly interacting particle stays longer in equilibrium. Note that the abundance depends only logarithmically on the mass m via Eq. (12.51) and implicitly via $g_{*,{\rm f}}$ on the freeze-out temperature $T_{\rm f}$. Typical values of $x_{\rm f}$ found numerically for weakly interacting massive particles (WIMPs) are $x_{\rm f} \sim 20$. Partial-wave unitarity bounds $\sigma_{\rm ann}$ as $\sigma_{\rm ann} \leq c/m^2$. Requiring $\Omega < 0.3$ leads to $m < 20-50$ TeV. This bounds the mass of any stable particle that was once in thermal equilibrium.

Baryon abundance from freeze-out:
We can calculate the expected baryon abundance for a zero chemical potential using the formulas derived above. Nucleon interact via pions; their annihilation cross section can be approximated as $\langle\sigma v\rangle \approx m_\pi^{-2}$. With $C \approx 2\times 10^{19}$, it follows $x_{\rm f} \approx 44$, $T_f \sim 22\,{\rm MeV}$ and $Y_\infty = 7\times 10^{-20}$. The observed baryon abundance is much larger and can be not explained as a usual freeze-out process.

Cold dark matter candidates

A particle suitable as CDM candidate should interact according Eq. (12.53) with $\sigma \sim 10^{-37}$ cm^2. It is surprising that the numerical values of T_0 and $M_{\rm Pl}$ conspire in Eq. (12.53) to lead to numerical value of σ_0 typical for weak interactions. Cold dark matter particles with masses around the weak scale and interaction strengths around the weak scale were dubbed "WIMP". An obvious candidate was a heavy neutrino, $m_\nu \sim 10\,{\rm GeV}$, excluded early by direct DM searches, neutrino mass limits, and accelerator searches. Presently, the candidate with most supporters is the lightest supersymmetric particle (LSP). Depending on the details of the theory, it could be a neutralino (most favorable for detection) or other options. The mass range open of thermal CDM particles is rather narrow: If it is too light, it becomes a warm or hot dark matter particle. If it is too heavy, it overcloses the universe. There exists however also the possibility that DM was never in thermal equilibrium. Two examples are the axion (a particle proposed to solve the CP problem of QCD) and superheavy particle (generically produced at the end of inflation). An overview of different CDM candidates is given in Fig. 12.5.

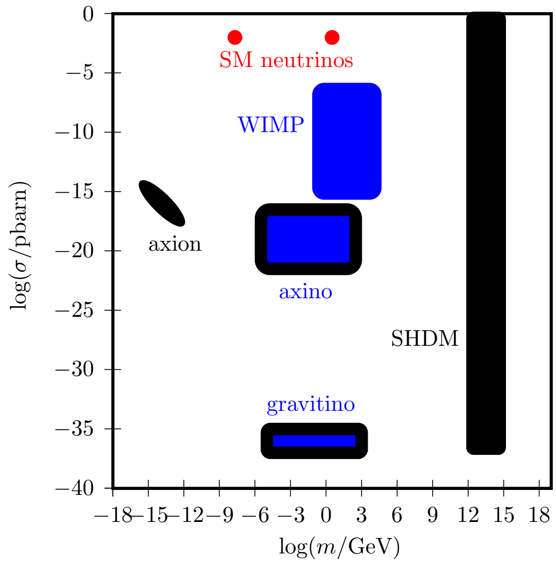


Figure 12.5: Particles proposed as DM particle with $\Omega \sim 1$, the expected size of their cross section and their mass. Red excluded; blue thermally and black non-thermally produced.

13 Inflation and structure formation

The hot big bang cosmology described in the previous two chapters is a remarkably successful theory, explaining the evolution of the Universe from at least the time of BBN ($t \sim s$ and $T \sim$MeV) until today ($t \sim 13\,$Gyr and $T \sim 2.7\,$keV). Complementing the hot big bang cosmology by a phase of exponential expansion in the early universe is an attempt to avoid some shortcomings of the standard big-bang model.

13.1 Inflation

Shortcomings of the standard big-bang model

- Causality or horizon problem: why are even causally disconnected regions of the universe homogeneous, as we discussed for CMB?

  The horizon grows like t, but the scale factor in radiation or matter dominated epoch only as $t^{2/3}$ or $t^{1/2}$, respectively. Thus for any scale l contained today completely inside the horizon, there exists a time $t < t_0$ where it crossed the horizon. A solution to the horizon problem requires that R grows faster than the horizon t. Since $R \propto t^{2/[3(1+w]}$, we need $w < -1/3$ or ($q < 0$, accelerated expansion of the universe).
- Flatness problem: the curvature term in the Friedmann equation is k/R^2. Thus this term decreases slower than matter ($\propto 1/R^3$) or radiation ($1/R^4$), but faster than vacuum energy. Let us rewrite the Friedmann equation as

$$\frac{k}{R^2} = H^2\left(\frac{8\pi G}{3H^2}\rho + \frac{\Lambda}{3H^2} - 1\right) = H^2\left(\Omega_{\rm tot} - 1\right) . \tag{13.1}$$

  The LHS scales as $(1+z)^2$, the Hubble parameter for MD as $(1+z)^3$ and for RD as $(1+z)^4$. General relativity is supposed to be valid until the energy scale $M_{\rm Pl}$. Most of time was RD, so we can estimate $1 + z_{\rm Pl} = (t_0/t_{\rm Pl})^{1/2} \sim 10^{30}$ ($t_{\rm Pl} \sim 10^{-43}$s). Thus if today $|\Omega_{\rm tot} - 1| \lesssim 1\%$, then the deviation had too be extremely small at $t_{\rm Pl}$, $|\Omega_{\rm tot} - 1| \lesssim 10^{-2}/(1 + z_{\rm Pl})^2 \approx 10^{-62}$!

  Taking the time-derivative of

$$|\Omega_{\rm tot} - 1| = \frac{|k|}{H^2R^2} = \frac{|k|}{\dot{R}^2} \tag{13.2}$$

  gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\,|\Omega_{\rm tot} - 1| = \frac{\mathrm{d}}{\mathrm{d}t}\,\frac{|k|}{\dot{R}^2} = -\frac{2|k|\ddot{R}}{\dot{R}^3} < 0 \tag{13.3}$$

  for $\ddot{R} > 0$. Thus $\Omega_{\rm tot} - 1$ increases if the universe decelerates, i.e. $\dot{R}$ decreases (radiation/matter dominates), and decreases if the universe accelerates , i.e. $\dot{R}$ increases (or vacuum energy dominates). Thus again $q < 0$ (or $w < -1/3$) is needed.
- The standard big-bang model contains no source for the initial fluctuations required for structure formation.

Solution by inflation Inflation is a modification of the standard big-bang model where a phase of accelerated expansion in the very early universe is introduced. For the expansion a field called inflaton with E.o.S $w < -1/3$ is responsible. We discuss briefly how the inflation solves the short-comings of standard big-bang model for the special case $w = -1$:

- Horizon problem: In contrast to the radiation or matter-dominated phase, the scale factor grows during inflation faster than the horizon scale, $R(t_2)/R(t_1) = \exp[(t_2 - t_1)H] \gg t_2/t_1$. Thus one can blow-up a small, at time t_1 causally connected region, to superhorizon scales.
- Flatness problem: During inflation $\dot{R} = HR$, $R = R_0 \exp(Ht)$ and thus

$$\Omega_{\text{tot}} - 1 = \frac{k}{\dot{R}^2} \propto \exp(-2Ht)\,. \tag{13.4}$$

  Thus $\Omega_{\text{tot}} - 1$ drives exponentially towards zero.

- Inflation blows-up quantum fluctuation to astronomical scales, generating initial fluctuation without scale, $P_0(k) = k^{n_s}$ with $n_s \approx 1$, as required by observations.

13.1.1 Scalar fields in the expanding universe

Equation of state A scalar field ϕ sitting at the minimum of its potential $V(\phi)$ has the desired EoS $w = -1$ to drive accelerated expansion. The simplest dynamical model for an inflationary phase is a single, scalar field which is initially displaced from its minimum. A necessary condition for accelerated expansion is $w < -1/3$, and thus we should find the EoS of a scalar field ϕ evolving in an FLRW background.

The stress tensor for a scalar field with $\mathscr{L} = \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi)$ follows from

$$T_{\mu\nu} = 2\frac{\partial \mathscr{L}}{\partial g^{\mu\nu}} - g_{\mu\nu}\mathscr{L} = \nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}g^{\rho\sigma}\nabla_\rho\phi\nabla_\sigma\phi - V(\phi)\right], \tag{13.5}$$

where we used the relation (13.91) derived in problem 19.1. We can describe the scalar field also as an ideal fluid. Equating the two expressions for the stress tensor gives

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}\mathscr{L} \stackrel{!}{=} (\rho + P)u_\mu u_\nu - P g_{\mu\nu}. \tag{13.6}$$

Comparing the two independent tensor structures we can identify $P = \mathscr{L}$ and

$$\nabla_\mu\phi\nabla_\nu\phi = (\rho + P)u_\mu u_\nu. \tag{13.7}$$

Contracting the indices with $g^{\mu\nu}$, remembering $u_\mu u^\mu = 1$ and using $\nabla_\mu\phi\nabla^\mu\phi = 2\mathscr{L} + 2V$ results in

$$\rho = P + 2V. \tag{13.8}$$

Now we have to calculate only the energy density $\rho = T_{00}$ in order to determine the (isotropic) pressure P and the equation of state $w = P/\rho$. In an FLRW background, the energy density of the field ϕ is given by

$$\rho = T_{00} = \dot{\phi}^2 - \left[\frac{1}{2}\dot{\phi}^2 - \frac{1}{2a^2}(\boldsymbol{\nabla}\phi)^2 - V(\phi)\right] = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2a^2}(\boldsymbol{\nabla}\phi)^2 + V(\phi). \tag{13.9}$$

Thus the pressure[1] follows as

$$P = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2a^2}(\boldsymbol{\nabla}\phi)^2 - V(\phi). \tag{13.10}$$

If we require the field ϕ to respect the symmetries of the FLRW background then ϕ has to be homogeneous and the $(\boldsymbol{\nabla}\phi)^2$ term vanishes. As result, the equation of state simplifies to

$$w = \frac{P}{\rho} = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)} \in [-1:1]. \tag{13.11}$$

Thus a classical scalar field may act as dark energy, $w < -1/3$, leading to an accelerated expansion of the universe. A necessary condition is that the field is "slowly rolling", that is, that its kinetic energy is sufficiently smaller than its potential energy, $\dot{\phi}^2 < 2V/3$.

Field equation in a FRW background We use Eq. (7.53) including a potential $V(\phi)$ (that could be also a mass term, $V(\phi) = m^2\phi^2/2$),

$$\mathscr{L} = \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\,, \tag{13.12}$$

to derive the equation of motion for a scalar field in a flat FRW metric with $g_{\mu\nu} = \mathrm{diag}(1, -a^2, -a^2, -a^2)$, $g^{\mu\nu} = \mathrm{diag}(1, -a^{-2}, -a^{-2}, -a^{-2})$, and $\sqrt{|g|} = a^3$. Varying the action

$$S_{\mathrm{KG}} = \int_\Omega \mathrm{d}^4x\, a^3 \left\{ \frac{1}{2}\dot{\phi}^2 - \frac{1}{2a^2}(\nabla\phi)^2 - V(\phi) \right\} \tag{13.13}$$

gives

$$\begin{aligned} \delta S_{\mathrm{KG}} &= \int_\Omega \mathrm{d}^4x\, a^3 \left\{ \dot{\phi}\delta\dot{\phi} - \frac{1}{a^2}(\nabla\phi)\cdot\delta(\nabla\phi) - V'\delta\phi \right\} \\ &= \int_\Omega \mathrm{d}^4x \left\{ -\frac{\mathrm{d}}{\mathrm{d}t}(a^3\dot{\phi}) + a\nabla^2\phi - a^3V' \right\}\delta\phi \\ &= \int_\Omega \mathrm{d}^4x\, a^3 \left\{ -\ddot{\phi} - 3H\dot{\phi} + \frac{1}{a^2}\nabla^2\phi - V' \right\}\delta\phi \stackrel{!}{=} 0\,. \end{aligned} \tag{13.14}$$

Thus the field equation for a Klein-Gordon field in a FRW background is

$$\boxed{\ddot{\phi} + 3H\dot{\phi} - \frac{1}{a^2}\nabla^2\phi + V' = 0\,.} \tag{13.15}$$

The term $3H\dot{\phi}$ acts in an expanding universe as a friction term for the oscillating ϕ field. Moreover, the gradient of ϕ is also suppressed for increasing a; this term can be therefore often neglected in an expanding universe.

[1] For an alternative definition of the pressure see problem 24.??.

Number of e-foldings and slow roll conditions We can integrate $\dot{R} = RH$ for an arbitrary time-evolution of H,

$$R(t) = R(t_0) \exp\left(\int \mathrm{d}t H(t)\right) . \tag{13.16}$$

If we define the number N of e-foldings as $N = \ln(R_2/R_1)$, then

$$N = \ln \frac{R_2}{R_1} = \int \mathrm{d}t\, H(t) = \int \frac{\mathrm{d}\phi}{\dot{\phi}} H(t) . \tag{13.17}$$

With $\ddot{\phi} + 3H\dot{\phi} + V' = 0$ or

$$\dot{\phi} = -\frac{\ddot{\phi} + V'}{3H} \approx -\frac{V'}{3H} \tag{13.18}$$

and the Friedmann equation $H^2 = 8\pi GV/3$ it follows

$$N = \int \mathrm{d}\phi\, \frac{3H^2}{V'} = \int \mathrm{d}\phi\, \frac{8\pi GV}{V'} \gg 1 . \tag{13.19}$$

Successful inflation requires $N \gtrsim 40$ and thus

$$\varepsilon \equiv \frac{1}{2}\left(\frac{V'}{8\pi GV}\right)^2 \ll 1 . \tag{13.20}$$

Additionally to large V and a flat slope V', the potential energy can dominate only, if $|\ddot{\phi}| \ll |V'|$. Then the field equation reduces to $V' \approx -3H\dot{\phi}$, or after differentiating to $V''\dot{\phi} \approx -3H\ddot{\phi}$. Thus another condition for inflation is

$$1 \gg \frac{|\ddot{\phi}|}{|V'|} \approx \frac{|V''\dot{\phi}|}{|3HV'|} \approx \frac{|V''|}{24\pi GV} \tag{13.21}$$

and one defines as second slow-roll condition

$$\eta \equiv \frac{V''}{8\pi GV} \ll 1 . \tag{13.22}$$

Hence inflation requires large V, a flat slope V' and small curvature V'' of the potential.

Mode solutions of the KG equation in a FRW background Next we want to rewrite the KG equation as the one for an harmonic oscillator with a time-dependent oscillation frequency. We introduce first the conformal time $\mathrm{d}\eta = \mathrm{d}t/a$,

$$\dot{\phi} = \frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\mathrm{d}\phi}{\mathrm{d}\eta}\frac{\mathrm{d}\eta}{\mathrm{d}t} = \frac{1}{a}\phi' , \tag{13.23}$$

$$\ddot{\phi} = \frac{1}{a}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{1}{a}\phi'\right) = \frac{1}{a^2}\phi'' - \frac{a'}{a^3}\phi' , \tag{13.24}$$

and express also the Hubble parameter as function of η,

$$H = \frac{\dot{a}}{a} = \frac{a'}{a^2} \equiv \frac{\mathscr{H}}{a} . \tag{13.25}$$

Inserting these expressions into Eq. (13.15) and multiplying with a^2 gives

$$\phi'' + 2\mathscr{H}\phi' - \nabla^2\phi + V' = 0\,. \tag{13.26}$$

Performing then a Fourier transformation,

$$\phi(\boldsymbol{x}, t) = \sum_k \phi_k(t)\mathrm{e}^{\mathrm{i}\boldsymbol{kx}}\,, \tag{13.27}$$

and using as potential a mass term, $V' = m^2\phi$, we obtain

$$\phi_k'' + 2H\phi_k' + (k^2 + m^2a^2)\phi_k = 0\,. \tag{13.28}$$

Finally, we can eliminate the friction term $2H\phi_k'$ by introducing $\phi_k(\eta) = u_k(\eta)/a$. Then a harmonic oscillator equation for u_k,

$$u_k'' + \omega_k^2 u_k = 0\,, \tag{13.29}$$

with the time-dependent frequency

$$\omega_k^2(\eta) = k^2 + m^2a^2 - \frac{a''}{a} \tag{13.30}$$

results. You can check that the action for the field u using conformal coordinates $\eta, \boldsymbol{x}$ is mathematically equivalent to the one of a scalar field in Minkowski space with time-dependent mass $m_{\text{eff}}^2(\eta) = m^2a^2 - a''/a$. This time-dependence appears, because the gravitational field can perform work on the field u. Alternatively, we could show that "the" vacuum at different times η is not the same, because we compare the vacuum for fields with different effective masses, leading to particle production. For an excellent introduction into this subject see the book by V. F. Mukhanov and S. Winitzki, "Introduction to quantum fields in gravity;" for a free pdf file of the draft version see `http://sites.google.com/site/winitzki/`. We consider now as two limiting cases the short and the long-wavelength limit. In the first case, $k^2 + m^2a^2 \gg \frac{a''}{a}$, the field equation is conformally equivalent to the one in normal Minkowski space, with solution

$$u_k(\eta, \boldsymbol{x}) = \frac{1}{\sqrt{2k}}(A_k\mathrm{e}^{-\mathrm{i}kx} + A_k\mathrm{e}^{\mathrm{i}kx})\,. \tag{13.31}$$

In the opposite limit, $a''u_k = au_k''$, with the solution $\phi_k = \text{const}$. The complete solution is given by Hankel functions $H_{3/2}(\eta)$,

$$u_k(\eta) = A_k\mathrm{e}^{-\mathrm{i}kx}\left(1 - \frac{\mathrm{i}}{k\eta}\right) + B_k\mathrm{e}^{\mathrm{i}kx}\left(1 + \frac{\mathrm{i}}{k\eta}\right)\,. \tag{13.32}$$

Modes outside the horizon are frozen in with amplitude

$$|\phi_k| = \left|\frac{u_k}{a}\right| = \frac{H}{\sqrt{2k^3}}\,. \tag{13.33}$$

We observe modes which exited the horizon $\Delta N \sim 60$ e-folding before the end of inflation. If there is no special choice of the initial conditions for inflation, we expect that the total number N_{tot} of e-folding is much larger. Thus the physical momentum of these modes at the beginning of inflation, $p(t_i) \sim H\mathrm{e}^{N_{\text{tot}}-\Delta N}$, is extremely high. A natural assumption is

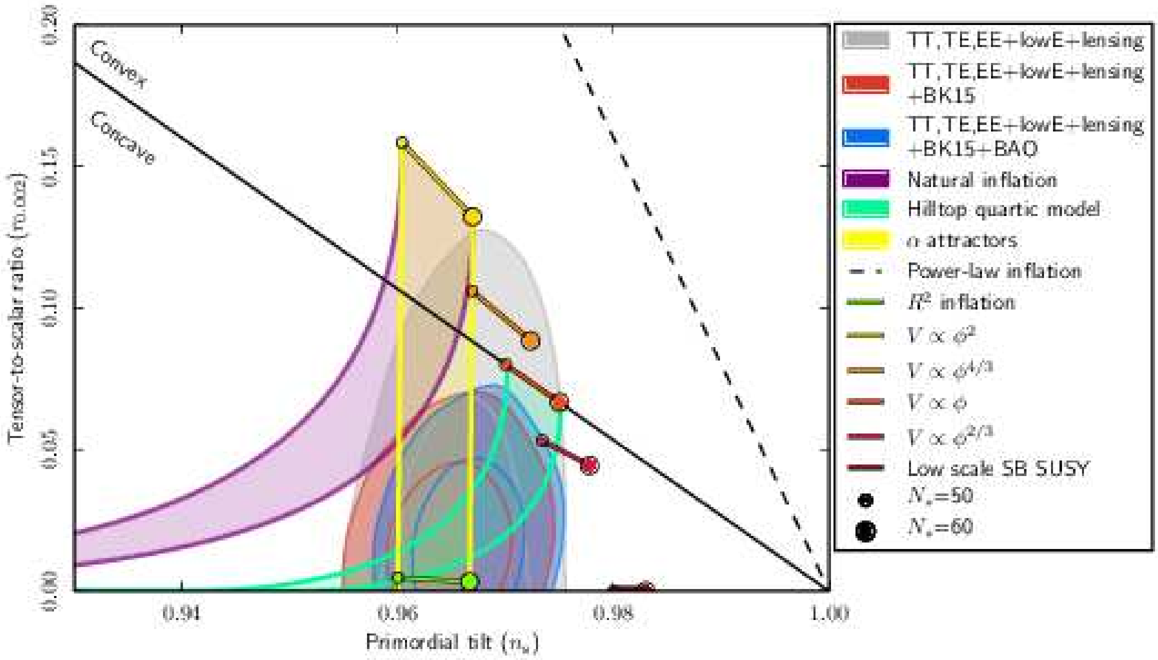


Figure 13.1: Marginalized joint 68 % and 95 % CL regions for n_{s} and r at $k = 0.002\,\mathrm{Mpc}^{-1}$ from alone and in combination with BK15 or BK15+BAO data, compared to the theoretical predictions of selected inflationary models. Note that the marginalized joint 68 % and 95 % CL regions assume $dn_{\mathrm{s}}/d\ln k = 0$,

therefore that these modes were empty at the start of inflation. Thus we should require that for early times, $\eta \to -\infty$, only positive frequencies survive, what implies $B_k = 0$ and $A_k = 1$. This choice is called the *Bunch–Davies vacuum*. Inserting this choice into (**??**), we find for the fluctuations inside the horizon, $k \gg |1/\eta| = aH$,

$$|\delta\phi_k| = \left|\frac{\chi_k}{a}\right| = \frac{H\eta}{\sqrt{2k}}\,. \tag{13.34}$$

Modes outside the horizon, $k \ll aH$, are frozen-in with amplitude

$$|\delta\phi_k| = \left|\frac{\chi_k}{a}\right| = \frac{H}{\sqrt{2k^3}}\,. \tag{13.35}$$

Power spectrum of perturbations The two-point correlation function of the field ϕ is

$$\langle\phi(\boldsymbol{x}',t')\phi(\boldsymbol{x},t)\rangle = \sum_k \langle\phi(\boldsymbol{x}',t')|k\rangle\langle k|\phi(\boldsymbol{x},t)\rangle = \int \frac{\mathrm{d}^3k}{(2\pi)^3}|\phi_k|^2\, e^{ik(x'-x)}\,. \tag{13.36}$$

We introduce spherical coordinates in Fourier space and choose $x = x'$,

$$\langle\phi^2(\boldsymbol{x},t)\rangle = \int \frac{4\pi k^2\mathrm{d}k}{(2\pi)^3}|\phi_k|^2 = \int \mathrm{d}k\, \underbrace{\frac{k^2}{2\pi^2}|\phi_k|^2}_{\equiv P(k)} = \int \frac{\mathrm{d}k}{k}\,\Delta^2_\phi(k)\,. \tag{13.37}$$

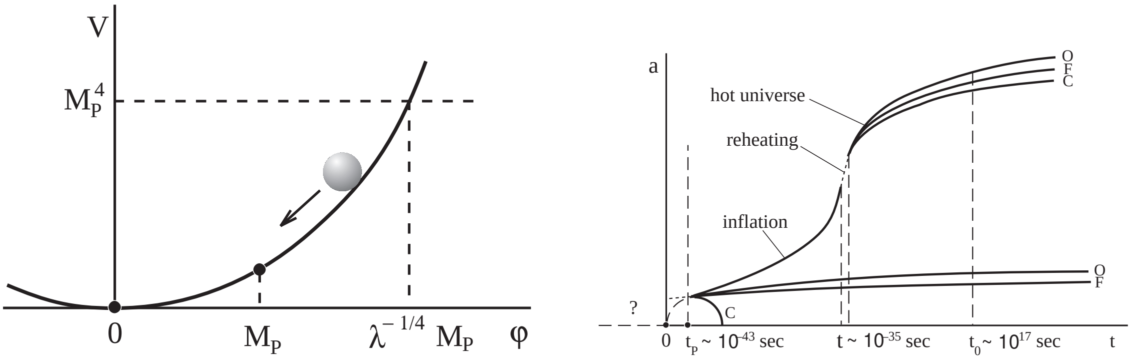


Figure 13.2: Left: A slowly rolling scalar field as model for inflation. Right: The evolution of the scale factor R including an inflationary phase in the early universe.

The functions $P(k)$ is the *power spectrum*, but often one calls also $\Delta^2_\phi(k)$ with the same name.

The spectrum of fluctuations $\Delta^2_\phi(k)$ outside of the horizon is

$$\Delta^2_\phi(k) = \frac{k^3}{2\pi^2}|\phi_k|^2 = \frac{H^2}{4\pi^2} \tag{13.38}$$

Hence, the power-spectrum of superhorizon fluctuations is independent of the wave-number in the approximation that H is constant during inflation. The total area below the function $\Delta^2_\phi(k) = \text{const.}$ plotted versus $\ln(k)$ gives $\langle\phi^2(\boldsymbol{x}, t)\rangle$, as shown by the last part of Eq. (13.90). Hence a spectrum with $\Delta^2_\phi(k) = \text{const.}$ contains the some amount of fluctuation on all angular scales. Such a spectrum of fluctuations is called a *Harisson-Zel'dovich* spectrum, and is produced by inflation in the limit of infinitely slow-rolling of the inflaton.

Fluctuations in the inflaton field, $\phi = \phi_0 + \delta\phi$, lead to fluctuations in the energy-momentum tensor $T^{ab} = T^{ab}_0 + \delta T^{ab}$, and thus to metric perturbations $g^{ab} = g^{ab}_0 + \delta g^{ab}$. These metric perturbations h^{ab} affect in turn all matter fields present.

13.1.2 Models for inflation

Inflation has to start and to stop ("graceful exit problem"). In order to start inflation, the inflaton has to be displaced from its equilibrium position.

Original idea of Guth: Symmetry restoration at a first order (discontinuous) phase transition, bubble creation or 2.order. Latent heat of phase transition is used to reheat universe (expansion lead to cool, empty state!) and to create particles. Too inhomogenous.

Modern ideas: Chaotic inflation: quantum fluctuation in a patch of the universe. Field rolls back, inflation ends when ϕ back in minimum. Oscillates around minimum, coupling to other particles leads to particle production.

If the coupling to other particles is "large", then (instantaneous) reheating $aT^4_{\rm rh} = V$. Generically, the coupling should be small. Delay leads to $aT^4_{\rm rh} = V(R/R')^3$.

Models for inflation Up to now we have discussed the, for a particle physicist, perhaps most natural option that a scalar field drives inflation. Moreover, we have restricted our

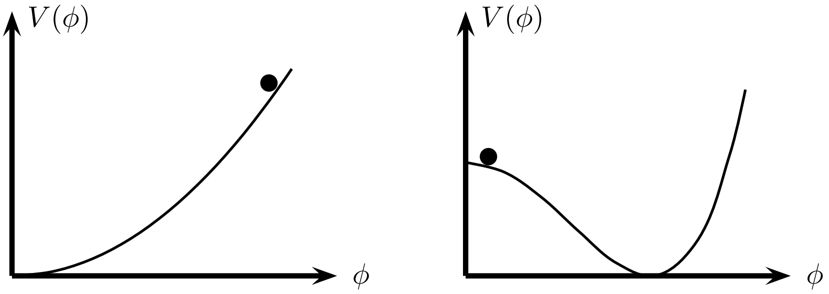


Figure 13.3: Typical potential of a large field (left) and of a small field model (right) for inflation.

attention to the case of a *single* scalar field. Single-field models can be characterised by two parameters, for example, the width and the height of the potential. Generically, we can divide these models into large and small field models as shown in Fig. 13.3. In the first class, inflation requires trans-Planckian field values as, for example, for the $m^2\phi^2$ or the $\lambda\phi^4$ models. The potential of these models has positive curvature, $V'' > 0$, and thus $\varepsilon_V > 0$.

The trans-Planckian field values required to start inflation clearly lead to the question of how the inflaton was displaced from its equilibrium position. A suggestion by Linde is "chaotic inflation". The inflaton field ϕ acquires random values due to quantum fluctuations. In a region of initial size $1/M_{\rm Pl}^4$ with $\phi \gg M_{\rm Pl}$, inflation starts and produces a homogeneous patch inside the universe. Other regions do not undergo inflation at all, or with only a few e-foldings. Thus on scales much larger than our successfully inflated patch, the universe is very inhomogeneous. In a variant, called "stochastic inflation", quantum fluctuations disturb the classical slow-roll trajectory so strongly that the volume filled with large quantum fluctuations $\phi \gg M_{\rm Pl}$ grows exponentially. As a result, new patches of inflating "miniverses" are generated continuously, leading to an eternal self-reproduction of the inflationary universe. A controversial question in these types of models is how generic is the observed universe, and how such a statement can be made precise.

Typical examples for a small field model are potentials like $V(\phi) = \lambda(\eta^2 - \phi^2)^2/4$ or of the Coleman–Weinberg type. Such potentials are generically much flatter than those of large field models. They often are connected to SSB, and the inflaton sits initially at the unstable equilibrium position $\phi = 0$. The potential has negative curvature, $V'' < 0$, and thus $\varepsilon_V < 0$. The idea of small field models implies that we choose parameters such that inflation starts at sub-Planckian field values, and thus, for example, $\lambda\phi_{\rm f}^4 \ll V_0 \lesssim M_{\rm Pl}^4$ for $V(\phi) = V_0 - \lambda\phi^4/4$. Inflation may be realised in this class of models as follows. Before the start of inflation, the universe is at high temperature in a potentially inhomogeneous state. The symmetry of the temperature-dependent effective potential $V_{\rm eff}(\phi, T)$ is restored. Thus the field ϕ sits initially at $\phi = 0$. As the universe expands, it cools down and the size of the temperature-dependent corrections becomes smaller, $V(\phi, T) \simeq V(\phi, 0)$. Finally, the symmetry is broken, the field starts to roll down the potential and inflation starts.

Using a scalar field as the driving force of inflation, a natural question to address is if we can identify the inflaton with a Higgs field. During the 1980s, one tried to connect the GUT phase transition and GUT Higgs fields to inflation. However, combining the slow-roll conditions and the size of density fluctuations (which will be discussed in the after next section) restricts the potential severely. Generically, loop corrections destroy the flatness of the potential if the inflation is not extremely weakly coupled to the SM fields. Therefore, it seems natural to consider the inflaton as a gauge singlet. The discovery of the SM Higgs has created nevertheless interest in the question if the SM Higgs can act as inflaton. First, we know that the Higgs potential flattens for large values of the renormalisation scale. Second, its coupling $\xi\phi^2 R$ to the curvature scalar is unconstrained. A large enough number of e-foldings can be achieved, if the coupling ξ is large, $\xi \approx 50\,000$. Such a term flattens the Higgs potential below $M_{\rm Pl}/\sqrt{\xi}$ sufficiently to lead to slow-roll inflation. This scenario faces however two problems: first, perturbative unitarity is violated below $M_{\rm Pl}$, requiring the existence of new degrees of freedom. Thus predictions in "Higgs inflation" depend on the unknown UV completion. More severely, we have seen that the SM Higgs potential (for the values of m_h and m_t currently favoured) develops an instability below $M_{\rm Pl}$. Thus the SM Higgs cannot be the main agent of inflation but may play some role during inflation.

The range of options widens drastically as soon as one uses several inflaton fields, reducing at the same time, however, the predictive power of the models. We comment here only briefly on another option, namely abandoning the idea that the inflaton is a fundamental field. In particular, during the very early universe higher derivative terms in the gravitational action may have played an important role. As a specific possibility, one can modify gravity by generalising the Einstein–Hilbert action as $\mathscr{L}_{\rm EH} = R \to f(R)$. Here, the function $f(R)$ should be chosen such that observational constraints are obeyed in the $R \to 0$ limit, while for large R modified gravity may lead to inflation. An example of this approach is the Starobinsky model proposed in 1979 that uses $f(R) = R - R^2/(6M^2)$. It represents the first working theory of inflation and is still in excellent agreement with data. Neglecting other matter fields, this theory is equivalent to standard gravity with a scalar field. Changing first the metric as $g_{\mu\nu} \to g_{\mu\nu}/\chi$ with $\chi \equiv \partial f(R)/\partial R$, and using then $\chi = \exp[4\sqrt{\pi}\phi/(\sqrt{3}M_{\rm Pl})]$ in order to obtain a canonically normalised kinetic term gives the scalar potential

$$V(\phi) = \frac{3M^2 M_{\rm Pl}^2}{32\pi^2}\,(1-1/\chi)^2 = \frac{3M^2 M_{\rm Pl}^2}{32\pi^2}\left[1-\exp\left(\frac{4\sqrt{\pi}\phi}{\sqrt{3}M_{\rm Pl}}\right)\right]^2 . \tag{13.39}$$

Thus one can analyse the Starobinsky model using $V(\phi)$ and standard gravity. However, the transformation $g_{\mu\nu} \to g_{\mu\nu}/\chi$ induces couplings of gravitational strength between ϕ and all other SM fields. These additional gravitational couplings indicate that it is more natural to see this class of models as a modification of gravity.

In order to distinguish between these various possibilities, we have to work out the fluctuations predicted by these models. Prior to that, we consider first the transition from the inflationary phase to the standard radiation dominated universe.

13.2 Structure formation

13.2.1 Overview and data

- Structure formation operates via gravitational instability, but needs as starting point a seed of primordial fluctuations (generated in inflation)
- Growth of structure is inhibited by many factors, e.g. pressure.
  The distance travelled by a freely falling particle is $R \sim gt^2/2$ with $g = GM/R^2$; or $t \sim \sqrt{R^3/GM} \sim \sqrt{1/G\rho}$. Thus $\tau_{\rm ff} \sim 1/\sqrt{G\rho}$.

Pressure can balance gravity, if $\tau_{\rm ff} \gtrsim \lambda/v_s$. This defines a critical length ("Jeans length")

$$\lambda \sim \frac{v_s}{\sqrt{G\rho}},$$

below which pressure can counteract density perturbations (resulting in acoustic oscillations), above the density perturbation grows. Shows already that structure formation is sensitive to E.o.S. (compare e.g. radiation with $v_s^2 = 1/3$ with baryonic matter $v_s^2 = 5T/(3m)$).

- If growth of perturbation leads to $\Omega \geq 1$ in a region, the region decouples from the Hubble expansion and collapses.

- Assume $\rho = \rho_m + \rho_\gamma$. If perturbations in ρ are adiabatic, i.e. the entropy per baryon is conserved, $\delta(\rho_m/s) = 0$ or $\delta \ln(\rho_m/T^3) = 0$, then $\delta \ln \rho_m - 3\delta \ln T = 0$ or

$$\frac{\delta\rho_m}{\rho_m} = 3\,\frac{\delta T}{T}\,.$$

[Another possibility would be $\delta\rho = 0$ or $\delta\rho_m = -\delta\rho_\gamma = -4aT^3\delta T = -4\rho_\gamma\delta T/T$ and $4\delta T/T = -\delta\rho_m/\rho_\gamma = -(\rho_m/\rho_\gamma)\,(\delta\rho_m/\rho_m)$. In the radiation epoch $\rho_m/\rho_\gamma \ll 1$ and temperature fluctuations are suppressed.]

⇒ Temperature fluctuation in CMB at $z \approx 1100$ and matter fluctuation today $0 \leq z \lesssim 5$ have the same origin, if primordial fluctuations are adiabatic.

- Basics of structure formation:
  assume initial fluctuations and examine how they are transformed by gravitational instability, interactions and free-streaming of different particle species

- comparison with observations via i) power-spectrum $P(k) = |\delta_k|^2$, where

$$\delta_k \propto \int d^3x\, e^{-ikx}\delta(x) \propto k^{n_s} \quad \text{with} \quad \delta(x) \equiv \frac{\rho(x) - \bar\rho}{\bar\rho}$$

or ii) correlation function $\int d^3x\, n(x)n(x+x_0)$ or normalized

$$\xi(x_0) \equiv \frac{1/V \int d^3x\, n(x)n(x+x_0)}{(1/V \int d^3x\, n(x)n(x+x_0))^2} - 1$$

The correlation function is the Fourier-transform of the power spectrum.
Typical $\xi \approx (r/r_0)^\gamma$ with $\gamma \sim 1.8$ for $0.1 \lesssim r \lesssim 10$Mpc.

- An example of the status in 1995 is shown in Fig. 13.4. The field is driven by a tremendous growth of data:
  galaxy catalogues: Hubble '32: 1250, Abell '58: 2712 cluster, 2dF : 250.000, SDSS (-'08): 10^6.
  CMB experiments: '65 detection, COBE '92: anisotropies, towards '00: first peak, ...
  N-body simulations: Peeble '70: $N = 100$, Efstahiou, Eastwood '81: $N = 20.000$, 2005: Virgo: Millenium simulation $N = 10^6$.

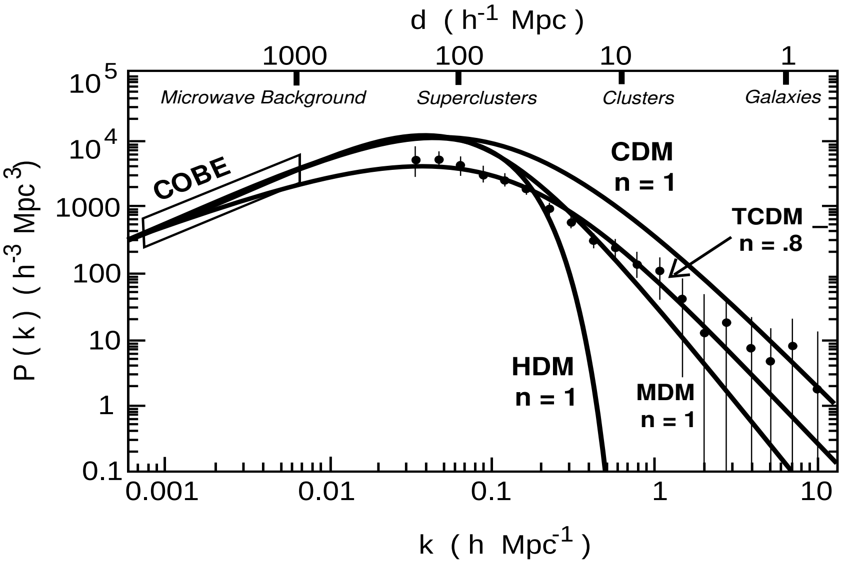


Figure 13.4: Comparison of the predicted power spectrum normalized to COBE data in several models popular around '95 with observations: HDM ($\Omega_\nu = 1$), CDM ($\Omega_m = 1$) and MDM ($\Omega_m = 0.8$, $\Omega_\nu = 0.2$).

13.2.2 Linear growth of perturbations

Matter-dominated universe We restrict ourselves to the simplest case: perturbations in a pressure-less, expanding medium. Starting from a homogeneous universe, we add matter inside a sphere of radius R, $\bar{\rho} \to \bar{\rho}(1+\delta)$. Then the acceleration on the surface of this sphere is

$$\frac{\ddot{R}}{R} = -\frac{4\pi}{3} G\bar{\rho}(1+\delta)\,. \tag{13.40}$$

The time evolution of the mass density is

$$\rho(t) = \bar{\rho}(t)[1+\delta(t)] = \bar{\rho}_0/a^3(t)[1+\delta(t)] \tag{13.41}$$

and thus mass conservation

$$M = \frac{4\pi}{3}\,\bar{\rho}_0\,[1+\delta]\frac{R^3}{a^3} = \text{const.} \tag{13.42}$$

implies

$$R(t) \propto a(t)[1+\delta(t)]^{-1/3}\,. \tag{13.43}$$

Expanding for $\delta \ll 1$ and differentiating twice gives

$$\frac{\ddot{R}}{R} = \frac{\ddot{a}}{a} - \frac{2}{3}\frac{\dot{a}}{a}\dot{\delta} - \frac{1}{3}\ddot{\delta}. \tag{13.44}$$

Combined with the usual acceleration equation for a pressureless universe,

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3} G\bar{\rho}, \tag{13.45}$$

we obtain

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\bar{\rho}\delta = 0. \tag{13.46}$$

General case Our simplified derivation assumed a matter-dominated universe. In general, the same equation holds if one includes into ρ all contributions to the energy density, while δ contains only the contast in the density of matter. Introducing therefore

$$\Omega_m = \frac{\rho_m}{\rho_{\rm cr}} = \frac{8\pi G\bar{\rho}_m}{3H^2},$$

we can re-write Eq. (13.46) in the general case as

$$\boxed{\ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}\Omega_m H^2 \delta = 0.} \tag{13.47}$$

We can now apply this equation to the evolution of the density contrast in various models with a single dominating energy component. For a matter-dominated universe, it is $H = 2/(3t)$ and $\rho = 1/(6\pi G t^2)$. Inserting as trial solution $\delta \propto t^\alpha$ gives

$$\alpha(\alpha-1)t^{\alpha-2} + \frac{4}{3}\,\alpha t^{\alpha-2} - \frac{2}{3}\,t^{\alpha-2} = 0 \tag{13.48}$$

or

$$\alpha^2 + \frac{1}{3}\alpha - \frac{2}{3} = 0 \tag{13.49}$$

and finally $\alpha = -1$ and $2/3$. Thus the general solution $\delta(t) = At^{-1} + Bt^{2/3}$ consists of a decaying mode $\delta \propto 1/t$ and a mode growing like the scale factor, $\delta \propto t^{2/3} \propto R$. Compared to the static universe, we see the the expansion of the universe slows down the growth from an exponential to a power-law growth.

During the radiation-dominated epoch, $\Omega_m \ll 1$, and we can neglect the term linear in $\Omega_m\delta$. With $H = 1/(2t)$, it follows then

$$\ddot{\delta} + \frac{1}{t}\dot{\delta} = 0 \tag{13.50}$$

with solution $\delta(t) = \delta(t_i)[1 + a\ln(t/t_i)]$. Thus sub-horizon perturbations essentially do not grow until $z_{\rm eq}$.

In the late universe, $z \lesssim 0.6$, the cosmological constant dominates, $H \simeq H_\Lambda = \text{const.}$, and

$$\ddot{\delta} + 2H_\Lambda\dot{\delta} = 0 \tag{13.51}$$

with $\delta(t) = C_1 + C_2\exp(-2H_\Lambda t)$. Thus the (fractional) fluctuations in the matter density do not grow further, while the average density decreases exponentially, $\bar{\rho}_m \propto \exp(-3H_\Lambda t)$.

Fourier expansion and super-horizon modes In a flat universe, $k = 0$, we can perform for wavelengths small compared to the Hubble radius a usual Fourier expansion,

$$\delta_{\boldsymbol{k}} = \int \mathrm{d}^3x\ \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\delta(\boldsymbol{x}) \quad \text{with} \quad \delta(\boldsymbol{x}) \equiv \frac{\rho(\boldsymbol{x}) - \bar{\rho}}{\bar{\rho}}. \tag{13.52}$$

The Fourier modes δ_k satisfy then the same equation,

$$\ddot{\delta}_k + 2H\dot{\delta}_k - \frac{3}{2}\Omega_m H^2 \delta_k = 0, \tag{13.53}$$

as long as $2\pi a(t)/k \ll 1/H$. Note that the evolution of different modes is decoupled. Thus we can still use the linear equation (13.53) for all modes with $\delta_k \ll 1$, even if other modes became already non-linear.

As the universe expands deaccelerating, modes which had initially wave-lengths larger than the horizon will cross the horizon and become sub-horizon modes. This implies that we need to know the behaviour of super-horizon modes as initial conditions for our Newtonian analysis. For the study of super-horizon modes, the use of the full perturbed Einstein equations is necessary. The key-point can be uderstood however from the following simple argument: Starting from the Friedmann equation for a flat, homogeneous universe, $H^2 = 8\pi G\bar{\rho}/3$, we consider a perturbed model with the same expansion rate H but higher density $\rho > \bar{\rho}$. Thus this model has positive curvature,

$$H^2 = \frac{8\pi G}{3}\rho - \frac{1}{R^2}. \tag{13.54}$$

Comparing now the perturbed "sub-universe" to the average one, the density contrast follows as

$$\delta \equiv \frac{\rho - \bar{\rho}}{\bar{\rho}} = \frac{3}{8\pi G\bar{\rho}\, R^2}. \tag{13.55}$$

Thus the evolution of the density contrast is controlled by the ratio of the curvature term to the energy density. For small δ, we can assume the same evolution of the scale factor in both models. Using then $\rho \propto 1/R^3$ and $\rho \propto 1/R^4$ in a matter and radiation-dominated universe, respectively, it follows

$$\delta \propto \frac{1}{R^2\bar{\rho}} \propto \begin{cases} R^2 & \text{radiation,} \\ R & \text{matter.} \end{cases} \tag{13.56}$$

Transfer function We are now in the position to sketch in the left panel of Fig. 13.5 the time evolution of the density contrast $\delta_{\boldsymbol{k}}(t)$ as function of the wave-number $\boldsymbol{k}$, if all effects except the expansion of the universe can be neglected. For small enough k, the mode is always super-horizon. Thus the density contrast grows until $a_{\rm eq}$ as $\delta_{\boldsymbol{k}}(t) \propto a^2$, and then as a. Increasing k, the mode crosses the horizon at $\delta_{\boldsymbol{k}}(t) \propto a$ and the growth is then until $a_{\rm eq}$ only logarithmically, after it becomes linear. Thus after $a_{\rm eq}$ the slope of the modes is the same, but the normalisation is k-dependent.

Next we want to relate the density contrast at some initial time t_i with the one observed at later time t. The choice of t_i ia arbitrary, as long it is early enough that all relevant scales are super-horizon. We consider first the range $k < k_{\rm eq}$. Then

$$\delta_{\boldsymbol{k}}(t) = \left(\frac{a_{\rm eq}}{a_i}\right)^2 \left(\frac{a}{a_{\rm eq}}\right) \delta_{\boldsymbol{k}}(t_i) \equiv D(t, t_i)\delta_{\boldsymbol{k}}(t_i), \tag{13.57}$$

where we introduced the scale-independent growth function D. In addition, we define the transfer function $T(k)$ which includes all scale-dependent effects as

$$T(k,t) = D(t, t_i)^{-1}\frac{\delta_{\boldsymbol{k}}(t)}{\delta_{\boldsymbol{k}}(t_i)}. \tag{13.58}$$

As required, it is $T(k,t) = 1$ for $k < k_{\rm eq}$.

Now we consider the range $k > k_{\text{eq}}$, where

$$\delta_{\boldsymbol{k}}(t) = \left(\frac{a_k}{a_i}\right)^2 \left(\frac{a}{a_{\text{eq}}}\right) \delta_{\boldsymbol{k}}(t_i). \tag{13.59}$$

We rwrite the pre-factors such that we can introduce the scale-independent growth function D,

$$\delta_{\boldsymbol{k}}(t) = \left(\frac{a_k}{a_{\text{eq}}}\right)^2 \left(\frac{a_{\text{eq}}}{a_i}\right)^2 \left(\frac{a}{a_{\text{eq}}}\right) \delta_{\boldsymbol{k}}(t_i) = \left(\frac{a_k}{a_{\text{eq}}}\right)^2 D(t, t_i)\delta_{\boldsymbol{k}}(t_i). \tag{13.60}$$

Thus the scale dependence of the transfer function is determined by the ratio $(a_k/a_{\text{eq}})^2$. Since at horizon crossing $k = a_k H_k$ and $H \propto 1/a^2$ in the radiation dominated epoch, it follows $T \propto 1/k^2$. Then the transfer function is given by

$$T(k) = \begin{cases} 1 & k < k_{\text{eq}}, \\ (k_{\text{eq}}/k)^2 & k > k_{\text{eq}}. \end{cases} \tag{13.61}$$

Let us now assume that the initial power spectrum follows a power law, $P_i(k) \propto k^n$. Then the linear power spectrum at late times is

$$P(k) \propto \begin{cases} k^n & k < k_{\text{eq}}, \\ k^{n-4} & k > k_{\text{eq}}. \end{cases} \tag{13.62}$$

Thus this model predicts a break by $\Delta n = 4$ at $k = k_{\text{eq}} \simeq 0.015h/\text{Mpc}$ in the linear power spectrum.

Example 13.1: Determine the numerical value of k_{eq}:

This prediction can be compared only indirectly to observations of the CMB and galaxy clustering. In the former case, the temperature fluctuations can be simply rescaled using Eq. (**??**). In the latter case, however, the fluctuations are non-linear. Moreover, the transfer function is the one for a universe consisting purely of CDM, since we neglected all effects like free-streaming or damping. Thus one has to take into account that the distribution of galaxies has a bias relative to the one of CDM. Accounting for these effects, one can convert the observed power spectrum at late times into the linear power spectrum. The latter is shown in Fig, 13.4; both the position of the break and the change of the slope agree surprisingly well with predictions, if the primordial power spectrum has $n = 1$.

Harisson-Zel'dovich spectrum We have still to find the connection between the power spectrum $\mathcal{P}(k)$ of matter fluctuations and the spectrum of fluctuations $\Delta_\phi^2(k)$ predicted by inflation. On sub-horizon scales, the latter are fluctuations in the Newtonian gravitational potential. Thus we can use the Poisson equation $\Delta\delta\phi(\boldsymbol{x}) = 4\pi G\bar{\rho}\delta(\boldsymbol{x})$ to find first $k^2\delta\phi_k \propto \delta_k$ and then $\mathcal{P}(k) \propto k^4\mathcal{P}_\phi(k)$. Thus the primordial slope $n \simeq 1$ in the matter fluctuations corresponds to

$$\Delta_\phi^2(k) \propto k^3\mathcal{P}_\phi(k) \propto \mathcal{P}(k)/k \sim \text{const.}$$

consistent with the scale-invariant spectrum predicted by inflation.

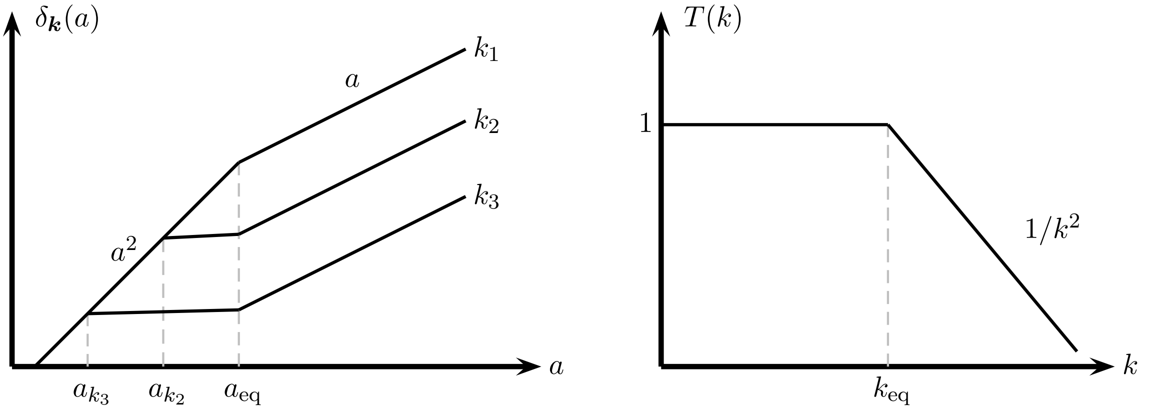


Figure 13.5: Left panel: The growth of $\delta_{\boldsymbol{k}}(a)$ as function of scale factor a for different k values. Right panel: The tranfer function $T(k)$.

Harisson and Zel'dovich argued long before the invention of inflation that the spectrum of initial fluctuations should be scale-invariant: Otherwise we would not see structures in the universe on a broad range of scales. Instead, collapsed objects would be concentrated either on the largest scales close to the Hubble horizon for $n \gg 1$, or on small scales for $n \ll 1$. Moreover, the power spectrum should contain a cutoff for $n \neq 1$, i.e. a scale, to be finite. Therefore a scale invariant spectrum of fluctuations with $n = 1$ is often called a *Harisson-Zel'dovich* spectrum.

Hierarchical structure formation We can now address the question how large the density contrast is when a mode crosses the horzion, given some initial condition for the power-spectrum of density fluctuation. More specifically, we want to determine the mass contained in a fluctuation. Starting from (see appendix)

$$\xi(r) = \frac{1}{2\pi^2} \int \mathrm{d}k\, k^2 P(k) j_0(kr) \tag{13.63}$$

and $P(k) \propto k^n$, it follows

$$\xi(r) = \frac{1}{2\pi^2} \int \mathrm{d}k\, k^{2+n} j_0(kr). \tag{13.64}$$

Since the integrand is approximately constant until $k \simeq 1/r$ and then decreases reapidly, we can integrate up to $k_{\max} \simeq 1/r$, obtaining

$$\xi(r) \propto r^{-(3+n)}. \tag{13.65}$$

With $M \propto r^3$,the mass contained in a fluctuation scales as $\xi(M) \propto M^{-(3+n)/3}$. The density contrast follows as

$$\frac{\delta\rho}{\rho} \equiv \Delta(M) \propto M^{-(n+3)/6} \propto M^{-2/3} \text{ for } n = 1. \tag{13.66}$$

From (13.56), it follows that the density contrast of super-horizon modes scales as $\Delta(M) \propto R^2$ in the radiation-dominated epoch. Thus

$$\Delta(M) \propto R^2 M^{-(n+3)/6}. \tag{13.67}$$

Next we consider modes which cross the horizon and assume that the mass is dominated by CDM. Then with $a \propto t^{1/2}$, it follows

$$M_{\rm CDM}|_{\rm horizon} \propto m_{\rm CDM} n_{\rm CDM} t_H^3 \propto a^{-3} a^6 \propto a^3 \tag{13.68}$$

Thus the scale factor scales as $a \propto M_h^{1/3}$ and

$$\Delta(M) \propto M^{2/3} M^{-(n+3)/6} \propto M^{-(n-1)/6} \tag{13.69}$$

is constant for $n = 1$. Thus for a Harisson-Zel'dovich spectrum, the density contrast of all fluctuations has the same amplitude at horizon crossing, independent of their scale. Since fluctuations with large k cross earlier the horizon, they start to grow earlier, reaching therefore also earlier the non-linear stage, $\delta_k \simeq 1$. Thus in this picture, structures at small scales are formed first.

13.2.3 Mass and damping scales

Jeans mass of baryons Consider a mixture of radiation and non-relativistic nucleons after e^+e^- annihilations, i.e. $T \approx 0.5\,\text{MeV}$. With $\rho = \rho_m + \rho_\gamma$ and $P \approx P_\gamma = \rho_\gamma/3$, we have

$$v_s = \left(\frac{\partial \rho}{\partial P}\right)_S^{-1/2} = \frac{1}{\sqrt{3}} \left(1 + \frac{\partial \rho_m}{\partial \rho_\gamma}_S\right)^{-1/2} = \frac{1}{\sqrt{3}} \left(1 + \frac{3\rho_m}{4\rho_\gamma}\right)^{-1/2} \tag{13.70}$$

where we used $\frac{\delta\rho_m}{\rho_m} = 3\,\frac{\delta T}{T} = \frac{3\delta\rho_\gamma}{4\rho_\gamma}$.
For $t \ll t_{\rm eq}$, the adiabatic sound speed is close to $v_s = 1/\sqrt{3}$, while $v_s = 0.76/\sqrt{3}$ for $t = t_{\rm eq}$. The Jeans mass of baryons is close to the horizon size until recombination. Then v_s drops to the value for a mono-atomic gas, $v_s^2 = \frac{5T_b}{3m}$, where $m \sim m_H \sim 1$ GeV.

The total mass M_J contained within a sphere of radius $\lambda_J/2 = \pi/k_J$ is

$$M_J = \frac{4\pi}{3}\left(\frac{\pi}{k_J}\right)^3 \rho_0 = \frac{\pi^{5/2}}{6} \frac{v_s^3}{G^{3/2}\rho_0^{1/2}} \tag{13.71}$$

is called the Jeans mass. It is unchanged by the expansion of the universe, since the wavenumber $k_J \propto R$ and $\rho_0 \propto 1/R^3$.

Let us compare the Jeans mass just before and after recombination,

$$M_J(z_{{\rm eq},>}) = \frac{\pi^{5/2}}{6} \frac{v_s^3}{G^{3/2}\rho^{1/2}} \sim 10^{15} \left(\Omega h^2\right)^{-2} M_\odot \tag{13.72}$$

and

$$M_J(z_{{\rm eq},<}) \sim 10^5 \left(\Omega h^2\right)^{-1/2} M_\odot \tag{13.73}$$

The Jeans mass of baryons does not coincide with the observed mass of galaxies, neither fits the corresponding length scale the break in the power spectrum around $k \approx 0.04h/\,\text{Mpc}$. Thus the mass scales for structure formation are not controlled by baryons—this was one of the first evidence for the importance of non-baryonic matter.

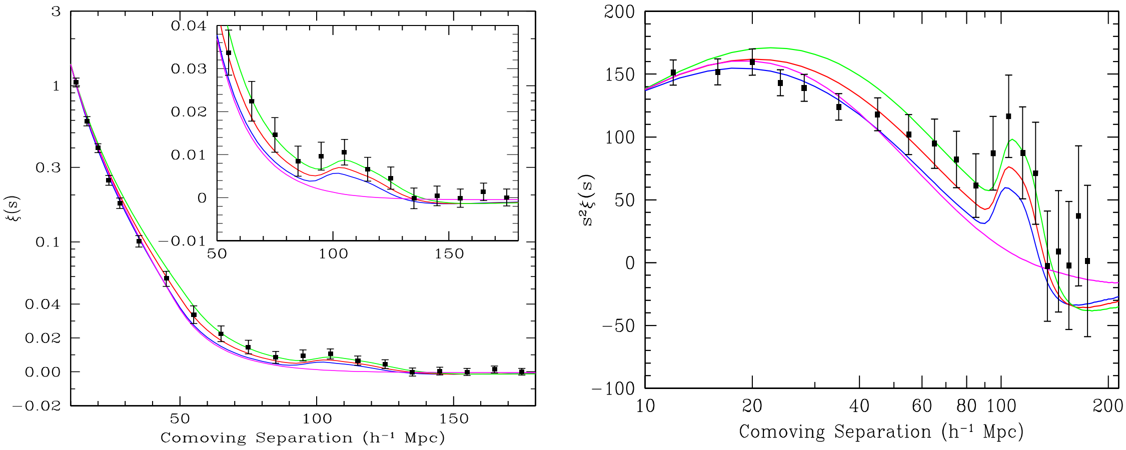


Figure 13.6: Acoustic baryon oscillation in the correlation function of galaxies with large redshift of the SDSS, astro-ph/0501171.

Damping scales

Up to now, we have approximated the matter and radiation content in the universe as an ideal fluid. While this is in most cases an excellent approximatins, deviations may lead to observable imprints. Examples are dissipative processes or the case of particles so weakly interacting that the fluid approximations is invalid.

Collisional or Silk damping Consider which fluctuations are damped by dissipative processes, e.g. by Thomson scattering. We note first that the mean free path of photons is always much larger than the one of electrons, $l_\gamma = (n_e \sigma_T)^{-1} \gg l_e = (n_\gamma \sigma_T)^{-1}$, since $n_\gamma \sim 10^{10} n_e$. Thus photon diffusion is much more important then electron diffusion.

A sound wave with wavelength λ can be damped if the diffusion time $\tau_{\rm diff}$ is smaller than the Hubble time t_H. We can estimate $\tau_{\rm diff}$ by a random-walk with $N = \lambda^2/l_{\rm int}^2$ steps, each with size $l_{\rm int} = 1/n_e \sigma_{th}$,

$$\tau_{\rm diff} = N l_{\rm int} = \lambda^2 / l_{\rm int} < t_H \,. \tag{13.74}$$

Thus the damping scale is $\lambda_D = (l_{\rm int} l_H)^{1/2}$. If there would be a baryon-dominated epoch, then $n_e \propto \rho$ and $\rho \propto 1/t^2$, hence $l_{\rm int} l_H \propto \rho^{-1-1/2}$ and $\lambda_D \propto \rho^{-3/4}$. Finally, the corresponding mass scale is $M_D \propto \lambda^3 \rho \propto \rho^{-9/4} \rho \propto \rho^{-5/4}$. Numerically,

$$M_D = 10^{12} (\Omega h^2)^{-5/4} M_\odot \tag{13.75}$$

and the corresponding length scale is (taking into account $\Omega_b \approx 0.04 < \Omega_m \approx 0.27$ and $h \approx 0.7$)

$$\lambda_D = 3.5 (\Omega_m/\Omega_b)^{1/2} (\Omega h^2)^{-3/4} \,\mathrm{Mpc} \simeq 40 \,\mathrm{Mpc} \,. \tag{13.76}$$

Thus the Silk scale λ_D has the right numerical value to explain the break in the power spectrum at $k \approx 0.04 h/\,\mathrm{Mpc}$. Fluctuations are damped on scales $\lambda \gtrsim \lambda_D$, the stronger the larger Ω_b. Since $M_D \gg M_J$, acoustic oscillation should be visible for $k \gtrsim k_D$ in the power spectrum of galaxies. The first evidence for this baryon-acoustic oscillations was found around 2005, cf. Fig. 13.6.

Collisionless damping or freestreaming Relativistic collisionless particles like neutrinos are streaming out of over-dense regions, smoothing density inhomogenities. Let us estimate the corresponding damping scale λ_{FS}, choosing values appropriate for the (hypothetical) case that neutrinos constitute a significant part of (hot) DM:

- choose $\mathrm{d}\phi = \mathrm{d}\vartheta = 0$, then for a freely falling particle $R(t)\mathrm{d}r = v(t)\mathrm{d}t$
- coordinate distance traversed from $t_i = 0$ to t is

$$\lambda_{FS}(t) = r(t) - r(0) = \int_0^t \mathrm{d}t' \, \frac{v(t')}{R(t')}$$

- divide in relativistic ($v = 1$) and non-relativistic regime ($v \ll 1$), with transition at t_{nr}:

$$\lambda_{FS} = \int_0^{t_{\mathrm{nr}}} \frac{\mathrm{d}t'}{R(t')} + \int_{t_{\mathrm{nr}}}^{t_{\mathrm{eq}}} \mathrm{d}t' \, \frac{v(t')}{R(t')}$$

- Since $p \propto 1/R$, non-relativistically $p = mv \Rightarrow v \propto 1/R$
  assume radiation-dominance $R \propto t^{1/2}$:

$$\lambda_{FS} = \frac{2t_{\mathrm{nr}}}{R_{\mathrm{nr}}} + \int_{t_{\mathrm{nr}}}^{t_{\mathrm{eq}}} \frac{R_{\mathrm{nr}}}{R^2(t')} \, dt' = \frac{t_{\mathrm{nr}}}{R_{\mathrm{nr}}} \left[2 + \ln(t_{\mathrm{eq}}/t_{\mathrm{nr}})\right]$$

- neutrinos become non-relativistic when $3T_\nu \approx m_\nu$, corresponding to $z_{\mathrm{nr}} \approx 6 \times 10^4 h^2 \Omega_\nu$

$$\Rightarrow \lambda_{FS}(t) = 30 \text{ Mpc } (\Omega_\nu h^2)^{-1}$$

- using $M(\lambda) \approx 1.5 \times 10^{11} M_\odot (\Omega_0 h^2) \lambda^3_{\mathrm{Mpc}}$

$$M_{FS} \approx 4 \times 10^{14} M_\odot \left(\frac{30\mathrm{eV}}{m_\nu}\right)^2$$

- pure HDM, assuming $\Omega_\nu \sim 1$:
  $\Rightarrow$ structure formations starts at supercluster scales. Smaller structure are produced later by fragmentation. This is opposite to the case of CDM.
- horizon scale $R_h(z_{\mathrm{nr}} \sim z_{\mathrm{eq}}) = 14\Omega_0^{-1} h^{-2}$ Mpc
  $\Rightarrow$ HDM freestreams and wipes out all density perturbation on galactic scales
- we consider now a mixed dark matter model, HCDM: Then HDM can fall into potential walls created earlier by CDM, if the scale of these walls is $\gtrsim M_{J,\nu}$.
  - mean velocity $v_\nu = c$ for $z \gtrsim z_{\mathrm{nr}}$
  - $v_\nu \propto 1/a$ for $z \lesssim z_{\mathrm{nr}}$
- comoving Jeans length $k_J = 2\pi a/\lambda_j$ is

$$k_J^{-1} \approx \frac{2.3 \times 10^{-2} (1+z)^{1/2}}{h^3 \Omega_\nu} \text{ Mpc}$$

- on scales $1/(h^3\Omega_\nu)$ Mpc ($\gtrsim$ 100 Mpc): HDM clusters already at $z_{\rm nr} \Rightarrow$ small difference to CDM
- on scales $\lesssim$ 1 Mpc: clustering has not started yet
- intermediate scales: surpression of density fluctuation
- CDM:
  if $\lambda_D \ll$ 1Mpc (i.e. CDM becomes non-relativistic much earlier than $t_{\rm eq}$), the first bound structure are of galactic size or smaller $\Rightarrow$ hierarchical structure formation.

13.2.4 Non-linear regime

Spherical collapse model When perturbations reach the non-linear stage, $\delta_k \simeq 1$, our linear analysis is not valid anymore. A useful model for the subsequent evolution is an ovderdense sphere which evolves in the same way as an closed sub-universe. Thus the equation of motion for the radius R of a sphere containing the mean mass density ρ is the same as the one for the scale factor of a closed universe. Thus

$$r = A(1-\cos\vartheta) \tag{13.77}$$

$$t = B(\vartheta - \sin\vartheta) \tag{13.78}$$

with $A = (GmB^2)^{1/3}$ (from $\ddot{r} = -GM/r^2$). An expansion of $r(\vartheta)$ for small ϑ gives

$$r \simeq \frac{A}{2}\left(\frac{6t}{B}\right)^{2/3}\left[1 - \frac{1}{20}\left(\frac{6t}{B}\right)^{2/3}\right] \tag{13.79}$$

The density perturbation is

$$\delta \simeq \frac{3}{20}\left(\frac{6t}{B}\right)^{2/3} \tag{13.80}$$

Thus intially, the perturbation grow as expected as $\delta \propto t^{2/3} \propto a$.

The radius sphere reaches a maximum at $\vartheta = \pi$, or $t = \pi B$. At this point, the sphere decouples completely from the Hubble flow and starts to collapse. The density has increased by a factor $\delta = 9\pi^2/16 \simeq 5.5$, while the linear theory predicts $\delta \simeq 1.1$. Neglecting any dissipation, the collapes ends in a singularity within the time $\vartheta = 2\pi$. In practise, the particles inside the sphere will *viralize:* collisions will convert the ordered kinetic energy of the "Hubble flow" into random motion.

Add: example PBHs?

Zeldovich pancakes

N-**body simulations** Once the a non-linear structure formed and virialized, it will continue to accrete matter. Moreover merger. Formation of the cosmic web.

N-body simulations are mainly used to study structure formation on the smallest scale, e.g. the dark matter profile of a galaxy.

13.2.5 Summary

Recipes for structure formation

Summary of different effects

- On sub-horizon scales our Newtonian analysis applies. During the radiation-dominated epoch, perturbations do not grow. During the matter-dominated epoch, perturbations grow on scales larger than the Jeans scale as $\delta \propto t^{2/3} \propto R$. Perturbations on smaller scales oscillate as acoustic waves.
- Before recombination, baryons are tightly coupled to radiation. The baryon Jeans scale is of order of the horizon size. After recombination, it drops by a factor 10^{10}.
- Silk damping reduces power on scales smaller than 40 Mpc.
- Free-streaming of HDM suppresses exponentially suppress power on scales smaller than few Mpc (for $\Omega_\nu = 1$).
- CDM with baryons would be affected only by Silk damping.

Recipe

- The connection between the initial perturbation spectrum $P_i(k) = |\delta_{k,i}|^2$ and the observed power spectrum $P(k)$ today is formally given by the transfer function $T(k)$,

$$P(k) = T^2(k)P_i(k). \tag{13.81}$$

- Inflation predicts that an initial perturbation spectrum $P_i(k) \propto k^{n_s}$ with $n_s \approx 1$, generally adiabatic ones.
- Normalize $P_i(k)$ to the COBE data.
- Choose a set of cosmological parameters $\{h, \Omega_{CDM}, \Omega_b, \Omega_\Lambda, \Omega_\nu, n_s\}$.
- Calculate $T(k)$.
- Fix a prescription to convert $\rho \approx \rho_{\rm CDM}$ with ρ_b measured in observation ("bias").
- Derive statistical quantities to be compared to observations; perform a likelihood analysis.

Results

- The three models without cosmological constant shown in Fig. 13.4 all fail.
- The exponential suppression on small scales typical for HDM is not observed, can be used to derive limit on $\Omega_\nu \lesssim 0.05$ or $\sum m_{\nu_i} \lesssim 2.2\,\text{eV}$.
- Acoustic baryon oscillations are only a tiny sub-dominant effect, but are now observed, cf. Fig. 13.6.

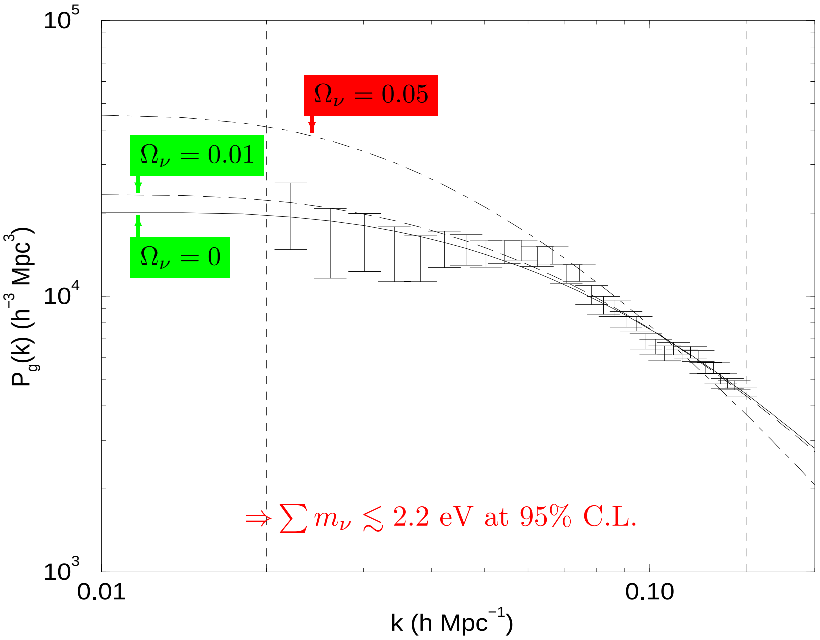


Figure 13.7: Neutrino mass limits from the 2dF galaxy survey: For $\Omega_\nu \gtrsim 0.05$ there is too less power on scales smaller than (or since normalization is arbitrary) slope too steep).

13.A Appendix: Power spectrum and correlation functions

Our convention for the Fourier transformation is asymmetric, putting the factor $V/(2\pi)^3$ into

$$f(\boldsymbol{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3} f(\boldsymbol{k}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\boldsymbol{x}}.$$

If no borders are specified in definite integrals, integration from $-\infty$ to ∞ is assumed. Then the normalisation factor $V/(2\pi)^3$ becomes the factor $1/(2\pi)^3$. Moreover, we often write $f(\boldsymbol{k}) = f_{\boldsymbol{k}}$.

The FLRW spaces are homogenous. Translation invariance implies thus that, e.g., the two-point auto-correlation function $\langle \phi(\boldsymbol{x}')\phi(\boldsymbol{x})\rangle$ can depend only on the relative distance $r = |\boldsymbol{x}' - \boldsymbol{x}|$, and we define the two-point auto-correlation function $\xi(r)$ as

$$\xi(r) = \langle \phi(\boldsymbol{x}')\phi(\boldsymbol{x})\rangle. \tag{13.82}$$

Power spectrum We start by defining the (three-dimensional) power-spectrum $P(\boldsymbol{k})$ of a random field $\phi(\boldsymbol{x})$ as

$$\langle \phi_{\boldsymbol{k}'}\phi_{\boldsymbol{k}}\rangle = (2\pi)^3\delta(\boldsymbol{k}' - \boldsymbol{k})P(\boldsymbol{k}). \tag{13.83}$$

Next we want to show that the power spectrum $P(\boldsymbol{k})$ is the Fourier transform of the two-point auto-correlation function $\xi(r)$. To derive this relation, we Fourier transform $\langle \phi_{\boldsymbol{k}'}\phi_{\boldsymbol{k}}\rangle$ and introduce then relative coordinates $\boldsymbol{r} = \boldsymbol{x}' - \boldsymbol{x}$,

$$\langle \phi_{\boldsymbol{k}'}\phi_{\boldsymbol{k}}\rangle = \langle \int \mathrm{d}^3x' \phi(\boldsymbol{x}')\mathrm{e}^{\mathrm{i}\boldsymbol{k}'\boldsymbol{x}'} \int \mathrm{d}^3x \phi(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\boldsymbol{x}}\rangle \tag{13.84a}$$

$$= \int \mathrm{d}^3x' \mathrm{e}^{\mathrm{i}\boldsymbol{k}'\boldsymbol{x}'} \int \mathrm{d}^3x \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\boldsymbol{x}} \langle \phi(\boldsymbol{x}')\phi(\boldsymbol{x})\rangle \tag{13.84b}$$

$$= \int \mathrm{d}^3x \mathrm{e}^{-\mathrm{i}(\boldsymbol{k}-\boldsymbol{k}')\boldsymbol{x}} \int \mathrm{d}^3r \mathrm{e}^{-\mathrm{i}\boldsymbol{k}'\boldsymbol{r}} \langle \phi(\boldsymbol{x}')\phi(\boldsymbol{x})\rangle \tag{13.84c}$$

$$= (2\pi)^3\delta(\boldsymbol{k}' - \boldsymbol{k}) \int \mathrm{d}^3r \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\boldsymbol{r}} \xi(r). \tag{13.84d}$$

Here we set also $\xi(r) = \langle \phi(\boldsymbol{x}')\phi(\boldsymbol{x})\rangle$. Comparing the two expressions, we obtain the desired relation,

$$P(\boldsymbol{k}) = \int \mathrm{d}^3r \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\boldsymbol{r}} \xi(r). \tag{13.85}$$

Assuming an isotropic random field, $P(\boldsymbol{k}) = P(k)$, we can simplify the power spectrum introducing spherical coordinates and setting $\boldsymbol{k}\boldsymbol{r} = kr\cos\vartheta = krx$,

$$P(k) = 2\pi \int_0^\infty \mathrm{d}r\, r^2 \int_{-1}^1 \mathrm{d}x\, \mathrm{e}^{-\mathrm{i}krx} \xi(r) \tag{13.86}$$

$$= 4\pi \int_0^\infty \mathrm{d}r r^2 \frac{\sin(kr)}{kr} \xi(r) \tag{13.87}$$

$$= 4\pi \int_0^\infty \mathrm{d}r\, j_0(kr)\xi(r). \tag{13.88}$$

With $j_0(x) \equiv \sin(x)/x \simeq 1$ for $x \lesssim 1$ and $j_0(x) \simeq 0$ for $x \gtrsim 1$, we see at the scale k fluctuations up to $r \simeq 1/k$ dominate the integral. **more precise: correlation length**

Finally, we connect the power spectrum to the fluctuations of $\phi(x)$,

$$\langle\phi^2(\boldsymbol{x})\rangle = \int \frac{\mathrm{d}^3k'}{(2\pi)^3}\frac{\mathrm{d}^3k}{(2\pi)^3}\mathrm{e}^{\mathrm{i}(\boldsymbol{k}'-\boldsymbol{k})\boldsymbol{x}}\langle\phi_{\boldsymbol{k}'}\phi_{\boldsymbol{k}}\rangle = \int \frac{\mathrm{d}^3k'}{(2\pi)^3}P(\boldsymbol{k}). \tag{13.89}$$

We assume again isotropy $P(\boldsymbol{k}) = P(k)$, and introduce spherical coordinates in Fourier space,

$$\langle\phi^2(\boldsymbol{x})\rangle = \int \frac{4\pi k^2\mathrm{d}k}{(2\pi)^3}|\phi_k|^2 = \int \mathrm{d}k\, \underbrace{\frac{k^2}{2\pi^2}|\phi_k|^2}_{\equiv\mathcal{P}(k)} = \int \frac{\mathrm{d}k}{k}\,\Delta^2_\phi(k). \tag{13.90}$$

The functions $\mathcal{P}(k)$ and $\Delta^2_\phi(k)$ are the *linear* and the *logarithmic power spectrum*, respectively. The total area below the function $\Delta^2_\phi(k)$ plotted versus $\ln(k)$ gives $\langle\phi^2(\boldsymbol{x},t)\rangle$, as shown by the last part of Eq. (13.90). Thus $\Delta^2_\phi(k)$ is the variance, and $(\Delta^2_\phi(k))^{1/2}$ is the typical strength of the fluctuation $\phi(k)$.

Correlation functions We can express also $\xi(r)$ via $P(\boldsymbol{k})$,

Gaussian random fields Gaussian random variable – qm
Gaussian random fields – free qft
ergodic hypothesis

Problems

13.1 Dynamical stress tensor. Show that the definition of the dynamical stress tensor can be simplified to

$$T_{\mu\nu} = \frac{2}{\sqrt{|g|}}\frac{\delta S_{\mathrm{m}}}{\delta g^{\mu\nu}} = 2\frac{\partial\mathscr{L}}{\partial g^{\mu\nu}} - g_{\mu\nu}\mathscr{L}\,. \tag{13.91}$$

13.2 Variation $\delta g_{\mu\nu}$. The variation of S_{EH} w.r.t. $g_{\mu\nu}$ will lead to different signs in Eq. (8.42). Explain why one obtains the same Einstein equation.

13.3 Cosmological constant Λ ♣. a.) Compare the stress density $T_{\mu\nu} = \kappa\Lambda g_{\mu\nu}$ of the cosmological constant to the one of an ideal fluid and determine thereby its EoS $w = P_\Lambda/\rho_\Lambda$. b.) Confirm the EoS using $U = V\rho_\Lambda$ and thermodynamics. c.) Estimate a bound on ρ_Λ using that the observable universe with size $\sim 3000\,$Mpc looks flat.

13.4 Expansion of S_{EH}. Expand $g^{\mu\nu}$ and $\sqrt{|g|}$ up to $\mathcal{O}(\lambda^3)$ around Minkowski space. Show that the $\mathcal{O}(\lambda)$ term of $\mathscr{L}_{\mathrm{EH}}$ is a total derivative which can be dropped.

13.5 Riemann tensor. Derive the symmetry properties of the Riemann tensor and the number of its independent components for an arbitrary number of dimensions. (Hint: Use an inertial system.)

13.6 Helicity. Show that the unphysical degrees of freedom of an electromagnetic wave transform as helicity 0, and of a a gravitational wave as helicity 0 and 1.

13.7 GWs from a binary system. Consider a binary system of two stars with equal mass M on circular orbits. a.) Calculate the quadrupole moments I_{ab}. b.) Determine the amplitude of the gravitational wave $\bar{h}_{\alpha\beta}(t,\boldsymbol{x})$. c.) Estimate the strength for a Galactic neutron star-neutron star binary with a separation of $r = 0.1\,$AU.

13.8 GWs from a binary system. The energy flux $\mathcal{F}$ of a GW is $\mathcal{F} = \frac{c^3}{32\pi G}\omega^2(a^2+b^2)$, where a and b are the amplitudes of the two polarisation states. a.) Estimate the energy flux for the binary system in 8. b.) Estimate how much energy is dissipated if a GW crosses the interstellar or intergalactic medium: Which processes might be relevant? Use simple dimensional analysis for your estimate.

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