Compact Stars

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Preface

This project was done at the Institute of Physics at the Norwegian University of Science and Technology (NTNU). It is the Master's Thesis of the Master of Technology, technical physics, study-programme.

I would like to thank my supervisor Jens O. Andersen for guidance and advice.

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Ellen Egeland

Symbols and Definitions

Table 1: Constants and their values

Constants	*Values*
Electronvolts eV	$1.60219 \cdot 10^{-19}$J
Electron mass m_e	$9.10953 \cdot 10^{-31}$kg
Neutron mass m_N	$1.67493 \cdot 10^{-27}$kg
Newton's gravitational constant G	$6.67300 \cdot 10^{-11}$Nm^2/kg^2
Planck's constant h	$6.62516 \cdot 10^{-34}$ Js
Planck's reduced constant $\hbar = \frac{h}{2\pi}$	$1.05450 \cdot 10^{-34}$Js
Proton mass m_p	$1.67262 \cdot 10^{-27}$kg
Solar mass $M_\odot$	$1.98892 \cdot 10^{30}$kg
Solar radius $R_\odot$	$6.96000 \cdot 10^{8}$m
Speed of light c	$2.99792 \cdot 10^{8}$m/s

Abstract

Stars are formed in gas and dust clouds with a non-uniform matter distribution. The compact stars, such as white dwarfs and neutron stars, are past the phase of having fusion processes in their interior.

The pressure withstanding gravitational contraction comes from the Pauli exclusion principle. Due to the high density, the electrons or neutrons are so tightly packed that the degeneracy energy, a consequence of the fact that two fermions cannot be in the same single-particle state, is the dominating term. Because of this, we can use the Fermi gas equation of state for electrons and neutrons for white dwarfs and neutron stars respectively.

The compact stars can be approximated by having a temperature $T = 0$. It isn't actually zero, but it is a good approximation, because the energy of the highest occupied energy level is of a much larger magnitude than the thermal energy. Hence the fermions are in the ground state of the many-particle system. The density is then on the form $\rho = f(p)$, where $f(p)$ is an arbitrary function of the pressure. We have used 2 different models:

$\star$ ρ is a constant
$\star$ ρ is polytropic

A defining property of compact stars is their large densities. White dwarfs have densities of the order $\rho \sim 10^{10}\text{kg/m}^3$, whereas neutron stars have $\rho \sim 10^{18}\text{kg/m}^3$, i.e. nuclear densities. The densities for neutron stars demand the use of general relativity. Due to this, we have calculated radii and masses using the TOV-equations which incorporates general relativity.

Measurements conducted by the Wilkinson Microwave Anisotropy Probe indicate that the universe consists of approximately 4% ordinary matter, 23% dark matter and 73% dark energy. The leading theory for the dark energy is a cosmological constant, a homogenous vacuum density throughout the universe. This theory can be incorporated into the TOV-equations. We have solved this form of the TOV-equations both analytically (ρ is constant) and numerically (ρ is polytropic). This has not been done previously.

Contents

Preface ii
Symbols and definitions iii
Abstract iv

1 Introduction **1**
1.1 White dwarfs and neutron stars 1
1.2 The cosmological constant 5

2 The Tolman-Oppenheimer-Volkov (TOV) Equations **10**
2.1 Solving the TOV-equations with constant density 10
2.2 The upper mass limit for stars 12
2.3 The Schwarzschild radius 13
2.4 The Newtonian limit 14

3 Solving the TOV-equations With a Cosmological Constant **17**
3.1 Derivation of pressure with cosmological constant 17
3.2 Alternative derivation of pressure with cosmological constant 21
3.3 The upper limit for the mass 24

4 Numerical Solutions for White Dwarfs **28**
4.1 Dimensionless structure equations 28
4.2 Numerical integration 30
4.3 Choosing the values of α, β and $\bar{p}(0)$ 30
4.3.1 The relativistic case 30
4.3.2 The nonrelativistic case 31
4.4 Results of the numerical integration 32

5 Numerical Solutions for Neutron Stars **35**
5.1 Dimensionless structure equations 35
5.2 Choosing the values of α, β and $\bar{p}(0)$ 36
5.2.1 The nonrelativistic case 36
5.2.2 The relativistic case 36
5.3 Numerical result for neutron stars 37

6 The Equation of State for Arbitrary Relativity **38**
6.1 White dwarfs 38
6.1.1 Fermi gas model for electrons 38
6.1.2 Polytropic model 40
6.1.3 Results 41
6.2 Neutron stars 41
6.3 Maximum Mass 47
6.4 Neutron stars with cosmological constant 48

7 Nuclear Interactions **51**
7.1 Symmetric nuclear matter 53
7.2 Mass and radius for neutron star with symmetric nuclear matter 58
7.3 Nonsymmetric nuclear matter 61
7.3.1 Equation of state consistent with causality 68
7.3.2 Radius and mass of neutron star with pure neutron matter 70

8 Summary and Conclusion **73**

References **74**

A MATLAB-programmes **75**
A.1 Programmes for function fitting 75
A.2 Programmes for numerical integration 80
A.3 Programmes for plotting and miscellaneous 94

1 Introduction

This section provides a short description of compact stars, such as white dwarfs and neutron stars. In addition a description of dark matter and dark energy and the connection to the cosmical constant, proposed by Einstein, will be given.

1.1 White dwarfs and neutron stars

This section is based on the references [Hartle 2003], [Hemmer 2000], [Krane 1988] and [Shapiro and Teukolsky 1983] and will not be cited in the remainder of the section.

Stars form from gas clouds in space where the matter density is slightly higher than in its surroundings. The gravitational attraction in these unevenly distributed clouds cause matter to assemble in spheres. If these spheres consist of a sufficiently large amount of matter, it enables the gravitational attraction to release enough energy in the form of increasing temperature, and fusion processes take place in their cores. The fusion of hydrogen to helium in the proton-proton chain leads to energy being released. After the helium concentration gets so large that it interferes with this chain, the fusion ceases, and the star looses its outwards pressure caused by the radiation from the fusion. The star starts to collapse, gravitational energy is being released, and its size increases. The star is now a red giant.

The hydrogen burning is followed by the fusion of heavier nuclei, namely the triple-alpha process where oxygen is being produced. For the stars with the smallest mass, the instabilities produced by the triple-alpha process will give the cooling outer layers of the star enough kinetic energy to be ejected as planetary nebulae. The remainder is a white dwarf, which is a dense core mainly consisting of electrons, protons and neutrons. The neutrons and protons are bound in nuclei of mainly carbon, nitrogen and oxygen forming a lattice structure with an electron cloud surrounding the nuclei.

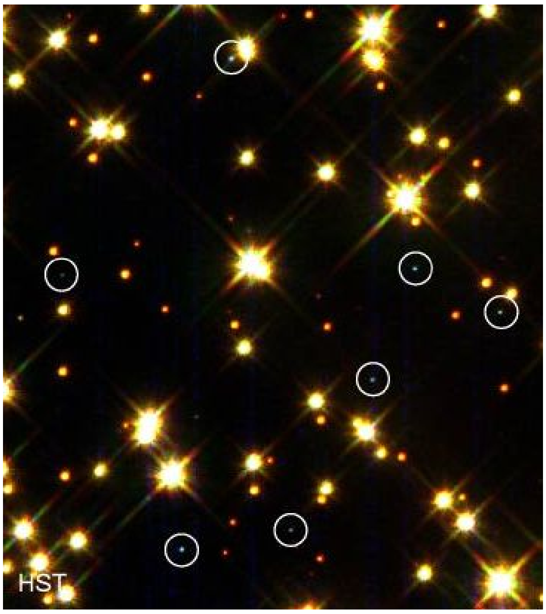


Figure 1: The stars marked by circles are white dwarfs cooling down to the ambient temperature. The cooling process takes longer than the present life-time of the universe.[1]

The mass of a white dwarf is usually approximately 0.5 - 0.6$M_\odot$ and its radius only 0.001$R_\odot$, just a little above the radius of the Earth, which causes the density to be very large ($\rho \sim 10^{10}$kg/m^3). A factor 10^8 less than nuclear matter density. The small size and the large density are characteristics distinguishing compact stars from normal stars.

The pressure supporting the star does not come from any fusion process in the centre of the star any more, but from the degenerate electrons. This pressure arises because of the Pauli principle which states that only one fermion can be in the same energy state (e.g. two fermions can have the same energy when they have opposite spin). This causes the electrons to have a certain amount of kinetic energy, giving rise to a pressure, which balance the gravitational attraction. This is another characteristic of compact stars.

The more massive stars, on the other hand, will start to contract again, thus heating the core enough to fuse heavier nuclei. The star continues the fusions in onion-like layers, with different fusion processes in each layer, until the core consists of iron. Further fusion will demand energy, not liberate it, hence the fusion stops. The star collapses and explodes in a supernova explosion as there is no longer any pressure from the fusion to balance gravity. The remaining mass is now a neutron star.

The neutron star is formed when gravity forces the electrons into the nuclei of the iron in the core, and through an inverse-β decay, reacts with protons to form neutrons. Neutron stars usually have a mass of 1.35 - 2.1$M_\odot$ and a radius of only 10 - 20km causing the density to be of the order of nuclear matter density ($\rho \sim 10^{18}$kg/m^3), and the radius decreases as the mass increases. Neutron stars are even smaller, and have a greater density than white dwarfs.

Figure 2: The Crab Pulsar, a magnetized neutron star spinning 30 times a second, lies in the inner region of the well-known Crab Nebula. It is more massive than the sun and has the density of an atomic nucleus. The spinning pulsar is the collapsed core of a massive star that exploded, while the nebula is the expanding remnant of the star's outer layers. This supernova explosion was witnessed in the year 1054.[2]

The pressure supporting the neutron stars are of the same origin as for the white dwarfs, namely the Quantum pressure (Pauli principle). Only, for neutron stars, the fermions providing this pressure, are neutrons (and a small portion of protons) instead of electrons. As an approximation, matter in compact stars can be thought of as completely degenerate. The pressure in these stars is mainly the Pauli pressure as the temperature is low compared to the temperature required to get thermal energy of the same order as the energy from the Pauli exclusion principle. The temperature in compact stars isn't actually zero, but it is a good approximation, because the energy of the highest occupied energy level is of a much larger magnitude than the energy caused by the temperature. So, the fermions are approximately in the ground-state of the many-particle system, hence the temperature is approximately $T = 0$, causing the equation of state effectively to be of the form

$$\rho = f(p). \tag{1}$$

Here ρ is the density and $f(p)$ is an arbitrary function of the pressure p.

White dwarfs can be observed directly in optical telescopes in their cooling period ($\sim$25billion years). Neutron stars, can be observed directly as pulsating radio sources (pulsars) or indirectly as periodic X-ray sources (X-ray pulsars).

1.2 The cosmological constant

This section is based on the references [6], [7], [8] and [Hartle 2003] and will not be cited in the remainder of the section.

The cosmological constant was first introduced by Albert Einstein, who needed a balancing term in the general relativistic equations describing the universe. Einstein assumed that the universe was static, and would therefore collapse without a balancing term because of the gravitational attraction. Later, we have observed that the universe is expanding, and Einstein called the introduction of a cosmological constant "his biggest blunder".

Even though there isn't a need of balancing the equations for the static universe any more, as we know the universe is dynamic, there are some indications that there is a cosmological constant. The Wilkinson Microwave Anisotropy Probe (WMAP) measures the cosmic microwave background radiation (CMB) of the universe.

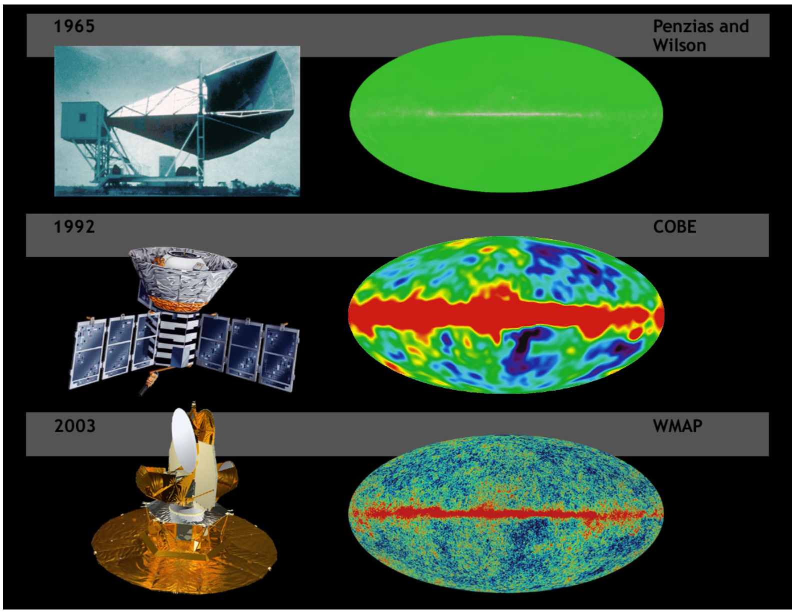


Figure 3: Penzias and Wilson discovered the remnant afterglow from the Big Bang and were awarded the Nobel Prize in 1978 for their discovery. COBE first discovered the patterns in the afterglow. WMAP will bring the patterns into much better focus to unveil a wealth of information about the history and fate of the universe. (TL) Penzias and Wilson microwave receiver - 1965 (TR) Simulation of the sky viewed by Penzias and Wilson's microwave receiver - 1965 (ML) COBE Spacecraft, Painting - 1992 (MR) COBE's view of early universe- 1992 (BL) WMAP Spacecraft, Computer Rendering - 2001 (BR) Simulated WMAP view of early universe. [4]

The measurements indicate that the universe is flat, hence there has to be a certain critical density. The first of the Friedman equations, which is an application of general relativity to cosmology, defines a density parameter:

$$\Omega = \frac{\rho}{\rho_c}, \tag{2}$$

Here ρ is the density of the universe and ρ_c is the critical density for which the geometry of the universe is flat. The value of Ω determines wether the universe is closed, open or flat. For Ω less than unity, the universe is open, and if it is larger than unity, the universe is closed. But this equation only holds for a universe without a cosmological constant.

The more general form, however, can be written as a sum of several contributions. A model for this is the Lambda-CDM model, where lambda is the cosmological constant and CDM denotes cold dark matter, both defined below. According to this model, there are important contributions to Ω from baryons, cold dark matter and dark energy. As measured by the WMAP, the space-time of the universe is nearly flat, thus implying the curvature parameter K to be close to zero. The first Friedman equation is often written in this form

$$\frac{H^2}{H_0^2} = \Omega_R a^{-4} + \Omega_M a^{-3} + \Omega_\Lambda - Kc^2 a^{-2}. \tag{3}$$

Here c is the speed of light, H_0 is the Hubble constant, and H is the Hubble parameter describing the rate of expansion of the universe. This is defined as

$$H = \frac{\dot{a}^2}{a^2}, \tag{4}$$

where a is the scale factor, a function of time which represents the relative expansion of the universe. Ω_R is the radiation density today, Ω_M is the matter (both baryonic and dark) density today and Ω_Λ is the vacuum density (or cosmological constant) today.

The CMB measured by WMAP indicates that the total amount of matter (both baryonic and dark matter) in the universe accounts for only approximately 27% of the critical density. By measuring the fluctuations in the CMB, the WMAP can determine the composition of the universe. In addition to the approximately 4% provided by atomic matter and approximately 23% from cold dark matter, approximately 73% comes from dark energy, of which little is known. Hence approximately 96% of the energy density in the universe is of a form that has never been directly detected in the laboratory.

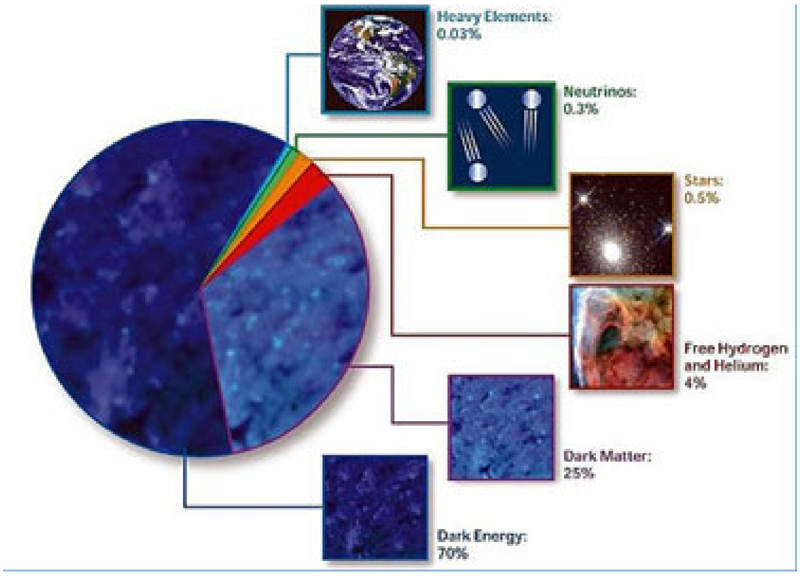


Figure 4: The composition of the universe calculated from the measurements of WMAP. [5]

The nature of dark energy is a matter of speculation, but it is known to be very homogenous, not very dense and to interact only through gravity. The two leading theories are quintessence and the cosmological constant. The cosmological constant describes a homogenous distribution of energy throughout the universe, and the quintessence describes a field of varying energy depending on the position and time. We will focus on the cosmological constant from now on.

The energy density connected to the cosmological constant is often called the vacuum energy as it is the energy of the vacuum. One of the problems with this model is that even though most quantum field theories predict an energy of the vacuum, the actual values vary widely, and can be as much as a factor of 10^{120} too large. This would need to be almost canceled by an equally large term of the opposite sign.

The dark energy is thought to be the reason for the acceleration of the expansion of the universe, because of its negative pressure. The dark energy has negative pressure because energy has to be lost from inside to do work on a container. A change in volume dV requires work equal to the energy $-pdV$ with p being the pressure. But the amount of energy in a box of vacuum energy must increase when the volume increases (dV is positive), because the energy is equal to $\rho_{\text{vac}}V$, where ρ_{vac} is the energy density of the cosmological constant. Therefore, p is negative and, in fact, $p = -\rho_{\text{vac}}$.

According to the Friedman equations, the pressure within a substance contributes to the gravitational attraction on other objects, just as its mass and density does. Hence, negative pressure causes repulsion. In dark energy, this effect is larger than the attraction caused by mass, and the overall effect is a repulsive force.

Dark matter, on the other hand, is matter interacting weakly with electromagnetic radiation, thus making it hard to observe. It is thought to be either baryonic matter found in the form of brown dwarf stars and MACHOs (massive astrophysical compact halo object), or non-baryonic matter. The total amount of baryonic dark matter can be calculated from the big bang nucleosynthesis and observations of the CMB, and both results in a much smaller value for baryonic dark matter than the total dark matter.

The more likely theory is that the dark matter mainly consists of one or a mix of the non-baryonic matter varities: Hot, warm or cold dark matter. The names reflects the energies of the particles, i.e. the hot particles move ultra-relativistically, the warm move relativistically and the cold move nonrelativistically. Examples of the different varieties are: Neutrinos (hot), gravitinos and photinos (warm) and supersymmetric particles such as WIMPs (Weakly Interacting Massive Particles, including neutralinos) (cold).

Figure 5: A composite image of the galaxy cluster CL0024+17 shows the gravitational lensing effect of a dark matter ring-like structure. It is the relatively weak distortions of the many distant faint blue galaxies all over the image that indicate the existence of the dark matter ring. The computationally modeled dark matter ring spans about five million light years and has been digitally superimposed to the image in diffuse blue.[3]

By using gravitational lensing, it is possible to calculate the abundance of dark matter in e.g. galaxy clusters. Gravitational lensing is based on general relativity to predict masses due to the bending of light caused by matter. Weak lensing looks at microscale distortions of galaxies, observed in vast galaxy surveys, due to foreground objects. Through statistical analysis, the mean distribution of dark matter can be found. This is how the dark matter ring in Fig. 5 was found.

2 The Tolman-Oppenheimer-Volkov (TOV) Equations

In this section the TOV-equations will be solved analytically for the case with constant matter density of the star. The limit for the maximal mass of stars and the Schwarzschild radius are found, and finally the Newtonian limit of the pressure is found and compared to the result for pure Newtonian theory.

The TOV-equations describe neutron stars better than the Newtonian equations of hydrostatic equilibrium, as the TOV-equations take general relativity into consideration. The TOV-equations for spherically symmetric stars are

$$\frac{dm}{dr} = 4\pi r^2 \rho \tag{5}$$

$$\frac{dp}{dr} = -\frac{G\rho m}{r^2} \frac{\left[1 + \frac{p}{\rho c^2}\right]\left[1 + \frac{4\pi r^3 p}{mc^2}\right]}{\left[1 - \frac{2Gm}{c^2 r}\right]}. \tag{6}$$

[Silbar and Reddy 2004] where $m = m(r)$ is the mass, r is the radius, $\rho = \rho(r)$ is the mass density and $p = p(r)$ is the pressure of the star. It is only equation (6) which differs from the corresponding equations from Newtonian theory of hydrostatical equilibrium

$$\frac{dp}{dr} = -\frac{G\rho m}{r^2} \tag{7}$$

by three additional dimensionless factors. Equation (5) is the same in both theories. When the term TOV-equations is used below, only one of the equations are different from their Newtonian counterparts, but the set of the two equations are normally called the TOV-equations so that is the convention we will stick to here.

We will in this section use natural units, where Newton's gravitational constant G and the speed of light c are equal to one.

2.1 Solving the TOV-equations with constant density

As we have constant density, the pressure p is independent of the density ρ. We need to get equation (6) on a different form to perform the integration:

$$\frac{dp}{dr} = -\frac{(\rho + p)(m + 4\pi r^3 p)}{r(r - 2m)}. \tag{8}$$

The general equation for the mass of a spherically symmetric object with constant density is

$$\int_0^m dm' = 4\pi\rho \int_0^r r'^2 dr'. \tag{9}$$

Solving the integral gives

$$m(r) = \frac{4}{3}\pi\rho r^3. \tag{10}$$

Inserting equation (10) into equation (8) yields

$$\frac{dp}{dr} = -\frac{4}{3}\pi r \frac{(\rho + p)(\rho + 3p)}{1 - \frac{8}{3}\pi\rho r^2}. \tag{11}$$

Integrating from a central pressure $p_c = p(r = 0)$, gives

$$\int_{p_c}^{p} \frac{dp'}{(\rho + p')(\rho + 3p')} = -\frac{4}{3}\pi \int_0^r \frac{r' dr'}{1 - \frac{8}{3}\pi\rho r'^2}, \tag{12}$$

where the integral on the right-hand side of the equation is

$$-\frac{4}{3}\pi \int_0^r \frac{r' dr'}{1 - \frac{8}{3}\pi\rho r'^2} = \frac{1}{4\rho} \int_1^{1-\frac{8}{3}\pi\rho r^2} \frac{du'}{u'}, \tag{13}$$

where we have made the substitution $u' = 1 - \frac{8}{3}\pi\rho r'^2$. The integration can then easily be carried out, and gives

$$\begin{aligned} \frac{1}{4\rho} \int_1^{1-\frac{8}{3}\pi\rho r^2} \frac{du'}{u'} &= \frac{1}{4\rho} \ln u \Big|_0^{1-\frac{8}{3}\pi\rho r^2} \\ &= \frac{1}{4\rho} \ln\left(1 - \frac{8}{3}\pi\rho r^2\right). \end{aligned} \tag{14}$$

The left-hand side of equation (12) is

$$\begin{aligned} \int_{p_c}^{p} \frac{dp'}{(\rho + p')(\rho + 3p')} &= \int_{p_c}^{p} \frac{dp'}{\rho^2 + 4\rho p + 3p^2} \\ &= \frac{1}{2\rho} \ln\left(\frac{3p + \rho}{p + \rho}\right)\Big|_{p_c}^{p} \\ &= \frac{1}{2\rho}\left[\ln\left(\frac{3p + \rho}{p + \rho}\right) - \ln\left(\frac{3p_c + \rho}{p_c + \rho}\right)\right]. \end{aligned} \tag{15}$$

Combining equation (14) and (15), gives

$$\begin{aligned}
\frac{1}{2\rho}\left[\ln\left(\frac{3p+\rho}{p+\rho}\right)-\ln\left(\frac{3p_c+\rho}{p_c+\rho}\right)\right] &= \frac{1}{4\rho}\ln\left(1-\frac{8}{3}\pi\rho r^2\right)\Rightarrow \\
\frac{3p+\rho}{p+\rho}\frac{p_c+\rho}{3p_c+\rho} &= \sqrt{1-\frac{8}{3}\pi\rho r^2}\Rightarrow \\
\frac{\rho+3p}{\rho+p} &= \frac{\rho+3p_c}{\rho+p_c}\sqrt{1-\frac{2m}{r}}, \qquad (16)
\end{aligned}$$

where we have substituted equation (10), $m(r)=\frac{4\pi\rho r^3}{3}$ to include the mass of the star.

The definition of the surface of the star is where the pressure is zero, i.e. $p(R)=0$. To find an expression for how the radius of the star depends on its mass and density, we insert the values $p=0$ and $r=R$ into equation (16), and obtain

$$R^2=\frac{3}{8\pi\rho}\left[1-\frac{(\rho+p_c)^2}{(\rho+3p_c)^2}\right], \qquad (17)$$

which we then substitute into equation (16) to eliminate the central pressure p_c:

$$p=\rho\frac{\sqrt{1-\frac{2Mr^2}{R^3}}-\sqrt{1-\frac{2M}{R}}}{3\sqrt{1-\frac{2M}{R}}-\sqrt{1-\frac{2Mr^2}{R^3}}}, \qquad (18)$$

where $M=\frac{4\pi\rho R^3}{3}$ is the mass of the star. We now have an equation for the pressure of a compact star with constant density.

2.2 The upper mass limit for stars

Inserting the value $r=0$ into equation (18) to get an expression for p_c in terms of R and M gives

$$\begin{aligned}
p_c &= \rho\frac{1-\sqrt{1-\frac{2M}{R}}}{3\frac{2M}{R}-1}\Rightarrow \\
\frac{2M}{R} &= 1-\left(\frac{p_c+\rho}{3p_c+\rho}\right)^2. \qquad (19)
\end{aligned}$$

Introducing the variable $x = \frac{\rho}{p_c}$ into equation (19) yields

$$\frac{2M}{R} = 1 - \left(\frac{1+x}{3+x}\right)^2 = 1 - [\lambda(x)]^2\,, \tag{20}$$

where $\lambda(x) = \frac{1+x}{3+x}$. Equation (20) reaches its maximum value when $\lambda(x)$ reaches its minimum, because of the negative sign of the $\lambda(x)$ term. x is a variable of the constant p_c and of ρ, both which can never be negative, hence x cannot be negative. To find the minimum of $\lambda(x)$, we find the derivative

$$\frac{d\lambda(x)}{dx} = \frac{2}{(3+x)^2}. \tag{21}$$

This is always positive, i.e. $\lambda(x)$ increase with increasing x, thus the minimum of $\lambda(x)$ must occur when $x = 0$. Inserting $x = 0$ in equation (20) yields

$$\begin{aligned} \frac{2M}{R} &= 1 - \frac{1}{9} \Rightarrow \\ \frac{M}{R} &= \frac{4}{9}. \end{aligned} \tag{22}$$

Equation (22) gives the value for the maximum mass of a star of given radius. The existence of such a limit is due to the TOV-equations incorporating relativity (both SR and GR). There is no corresponding limit in the purely Newtonian theory of hydrostatical equilibrium. The reason for this being a relativistic effect is treated in section 7.3.1.

2.3 The Schwarzschild radius

We insert $M = \frac{4\pi\rho R^3}{3}$ into equation (17), and get

$$(\rho + 3p_c)^2(1 - \frac{2M}{R}) = (\rho + p_c)^2. \tag{23}$$

We manipulate this equation a little:

$$2{p_c}^2\left(4 - \frac{9M}{R}\right) + p_c\left(4\rho - \frac{12M\rho}{R}\right) - \frac{2M\rho^2}{R} = 0 \tag{24}$$

and solving this for p_c yields

$$p_c = \frac{-4\rho(1 + \frac{3M}{R}) \pm \sqrt{\left[4\rho(1 - \frac{3M}{R})\right]^2 + \frac{16M\rho^2}{R}(4 - \frac{9M}{R})}}{4(4 - \frac{9M}{R})}. \tag{25}$$

To get a real value for p_c, the expression inside the square root of equation (25) must be positive, otherwise the central pressure becomes complex, and equation (25) makes no sense. To find the value where this happens, we calculate the limit, i.e. when $\left[4\rho(1-\frac{3M}{R})\right]^2 + \frac{16M\rho^2}{R}(4-\frac{9M}{R}) \to 0$:

$$\begin{aligned} \left[4\rho(1-\frac{3M}{R})\right]^2 + \frac{16M\rho^2}{R}(4-\frac{9M}{R}) &= 0 \Rightarrow \\ 1-\frac{2M}{R} &= 0. \end{aligned} \tag{26}$$

This is called the Schwarzschild radius, and this defines the event horizon of a black hole. From this radius, no information can escape.

2.4 The Newtonian limit

To find the Newtonian limit of equation (18), we use that $\frac{2M}{R} \ll 1$. This comes from the fact that the metric, which describes the geometry of space-time, must be nearly flat in the Newtonian limit. A flat metric describes the Euclidean space, and this is the form of the metric when there is no influence from general relativity, and in the Newtonian limit there are no general relativistic effects.

The general metric outside a spherically symmetric neutron star is of the form (as the star is static):

$$ds^2 = -g_{00}dt^2 + g_{rr}dr^2 + r^2 d\Omega^2 \tag{27}$$

[Schutz 1986], where $g_{00} = \left(1-\frac{2M}{R}\right)$ and $g_{rr} = \left(1-\frac{2M}{R}\right)^{-1}$ and M is the total mass of the star. For the metric to be nearly flat, g_{00} and g_{rr} must be approximately one:

$$\begin{aligned} \left(1-\frac{2M}{R}\right)^{\pm 1} &\approx 1 \Rightarrow \\ \frac{2M}{R} \to 0 & \end{aligned} \tag{28}$$

i.e. the metric is nearly flat when $M \ll R$.

When this approximation is applied to equation (18), we get

$$\begin{aligned} p &\approx \rho\frac{(1-\frac{Mr^2}{R^3})-(1-\frac{M}{R})}{3(1-\frac{M}{R})-(1-\frac{Mr^2}{R^3})} \\ &= \rho\frac{1-\frac{r^2}{R^2}}{\frac{2R}{M}+\frac{r^2}{R^2}-1} \end{aligned} \tag{29}$$

where $\frac{2R}{M}$ is much bigger than the other terms in the denominator, so we get

$$p \approx \frac{\rho M}{2R}\left(1-\frac{r^2}{R^2}\right). \tag{30}$$

Inserting the expression $M = \frac{4\pi\rho R^3}{3}$ yields

$$p = \frac{2\pi\rho^2}{3}\left(R^2 - r^2\right). \tag{31}$$

To get the units right, we must reinstate the factors of G and c. This is done by letting $M \to MG$, $R \to \frac{R^2}{c^2}$ and $r \to \frac{r^2}{c^2}$, which yields

$$p = \frac{2\pi G\rho^2}{3c^2}\left(R^2 - r^2\right). \tag{32}$$

This is the same expression we get when using a Newtonian theory as shown in the derivation below.

The equation of hydrostatical equilibrium, equation (7) is

$$\frac{dp}{dr} = -\frac{m(r)\rho(r)}{r^2}, \tag{33}$$

in natural units ($G = c = 1$). This was derived in the preliminary project to this master thesis along with the following derivation [Egeland 2006].

We assume constant density of the star, i.e. $\rho(r) = \rho$. Then we can insert equation (10) into equation (33), giving

$$\frac{dp}{dr} = -\frac{4\pi\rho^2 r}{3}. \tag{34}$$

For $r > R$, where R is the radius of the star, the pressure will be zero, but for $r < R$, we have

$$\begin{aligned} \int_{p_c}^{p} dp' &= -\frac{4\pi}{3}\rho^2 \int_0^r r' dr' \\ p_c - p &= \frac{2\pi\rho^2}{3} r^2 \end{aligned} \tag{35}$$

where the constant p_c is the pressure for $r = 0$ (central pressure):

$$\int_{p_c}^{0} dp = -\frac{4\pi}{3}\rho^2 \int_0^R r dr \tag{36}$$

$$p_c = \frac{2\pi\rho^2}{3} R^2 \tag{37}$$

and inserting this into equation (35), we obtain:

$$p = \frac{2\pi\rho^2}{3}\left(R^2 - r^2\right). \tag{38}$$

Reinstating the factors of G and c yields

$$p = \frac{2\pi G\rho^2}{3c^2}\left(R^2 - r^2\right). \tag{39}$$

This shows that taking the Newtonian limit of the solution to the TOV-equations for constant density yields the same result as when using pure Newtonian theory, which is what we would expect.

3 Solving the TOV-equations With a Cosmological Constant

In this section, the TOV-equations with a term including the cosmological constant, will be solved analytically for the case with constant matter density. The limit for the maximal mass of a neutron star is found and compared to the result where the cosmological constant is zero.

3.1 Derivation of pressure with cosmological constant

The cosmological constant is defined as

$$\Lambda = \frac{8\pi\rho_{\text{vac}}}{3} \tag{40}$$

in natural units ($G = c = 1$) and where ρ_{vac} is the matter density of dark energy in the universe. This definition is sometimes given without the factor of 3.

The TOV equation with a cosmological constant differs only by an extra correction term in comparison to the original TOV-equation, and this term is

$$\left[1 + \frac{4\pi r^3 p(r)}{m(r)c^2}\right] \to \left[1 + \frac{4\pi r^3 p(r)}{m(r)c^2} - \frac{\Lambda r^3}{2Gm(r)}\right] \tag{41}$$

[Silbar and Reddy 2004]. The TOV-equation with cosmological constant then becomes

$$\frac{dp}{dr} = -\frac{G\rho(r)m(r)}{r^2}\frac{\left[1 + \frac{p(r)}{\rho(r)c^2}\right]\left[1 + \frac{4\pi r^3 p(r)}{m(r)c^2} - \frac{\Lambda r^3}{2Gm(r)}\right]}{\left[1 - \frac{2Gm(r)}{c^2 r}\right]}. \tag{42}$$

We now use the assumption that the matter density is constant, i.e. $\rho(r) = \rho$, together with equation (10) for the mass, and natural units ($c = 1$ and $G = 1$), and get

$$\begin{aligned}\frac{dp}{dr} &= \frac{\frac{-4\pi\rho^2 r}{3}\left[1 + \frac{p}{\rho}\right]\left[1 + \frac{3p}{\rho} - \frac{3\Lambda}{8\pi\rho}\right]}{\left[1 - \frac{8\pi r^2\rho}{3}\right]} \\ &= \frac{-r}{2}\frac{(\rho + p)\left(\rho + p - \frac{3\Lambda}{8\pi}\right)}{\left(\frac{3}{8\pi} - r^2\rho\right)}.\end{aligned} \tag{43}$$

Defining the constant a

$$a^2 = \frac{3}{8\pi\rho}, \tag{44}$$

and inserting it into equation (43) yields

$$\frac{dp}{dr} = -\frac{(\rho + p)}{2\rho a^2}\frac{(\rho + 3p - \Lambda a^2\rho)}{\left(1 - \frac{r^2}{a^2}\right)}r. \tag{45}$$

Integrating from the central pressure p_c, gives

$$\int_{p_c}^{p} \frac{dp'}{(\rho + p')(\rho + 3p' - \Lambda a^2\rho)} = -\frac{1}{2\rho a^2}\int_0^r \frac{r'dr'}{1 - \frac{r'^2}{a^2}}. \tag{46}$$

Using the substitutions

$$\begin{aligned} r' &= a\sin\chi' && (47) \\ dr' &= a\cos\chi' d\chi', && (48) \end{aligned}$$

the integral on the right hand side of equation (46) becomes

$$\begin{aligned} I_R &= -\frac{1}{2\rho a^2}\int_0^r \frac{r'dr'}{1 - \frac{r'^2}{a^2}} \\ &= -\frac{1}{2\rho}\int_{\chi_c}^{\chi} \frac{\sin\chi'}{\cos\chi'}d\chi'. \end{aligned} \tag{49}$$

Performing the integration gives

$$I_R = \frac{1}{2\rho}\ln(\cos\chi')\Big|_{\chi_c}^{\chi}, \tag{50}$$

and using equation (47), yields

$$I_R = \frac{1}{4\rho}\ln\left(1 - \frac{r^2}{a^2}\right). \tag{51}$$

The integral on the left side of equation (46) is

$$\begin{aligned} I_L &= \int_{p_c}^{p} \frac{dp'}{(\rho + p')(\rho + 3p' - \Lambda a^2\rho)} \\ &= \int_{p_c}^{p} \frac{dp'}{3p'^2 + (4 - \Lambda a^2)\rho p' + (1 - \Lambda a^2)\rho^2}. \end{aligned} \tag{52}$$

We define the constants

$$
\begin{aligned}
A &= (4 - \Lambda a^2)\rho & (53) \\
B &= (1 - \Lambda a^2)\rho^2. & (54)
\end{aligned}
$$

Performing the integration on equation (52) gives three different cases depending on the sign of $A^2 - 3B$.

$$
I_L = \frac{1}{2\sqrt{A^2-3B}}\left[\ln\left(\frac{3p + A - \sqrt{A^2-3B}}{3p + A + \sqrt{A^2-3B}}\right) - \ln\left(\frac{3p_c + A - \sqrt{A^2-3B}}{3p_c + A + \sqrt{A^2-3B}}\right)\right] \tag{55}
$$

for $A^2 - 3B > 0$ which is the same case as for the ordinary TOV-equation.

$$
I_L = \frac{1}{\sqrt{3B-A^2}}\left[\arctan\left(\frac{3p + A}{\sqrt{3B-A^2}}\right) - \arctan\left(\frac{3p_c + A}{\sqrt{3B-A^2}}\right)\right] \tag{56}
$$

for $A^2 - 3B < 0$, and

$$
I_L = \frac{1}{3p_c + A} - \frac{1}{3p + A} \tag{57}
$$

for $A^2 - 3B = 0$.

To find which case to use, we differentiate $y = A^2 - 3B = (4 - \Lambda a^2)^2\rho^2 - (1 - \Lambda a^2)3\rho^2$ with respect to $x = \Lambda a^2$;

$$
\begin{aligned}
y(x) &= (4 - x)^2\rho^2 - (1 - x)3\rho^2 & (58) \\
\frac{dy}{dx} &= -5\rho^2 + 2\rho x & (59) \\
\frac{d^2y}{dx^2} &= 2\rho^2. & (60)
\end{aligned}
$$

Set equation (59) equal to 0 to find the extremal points of the function y, and get

$x = \Lambda a^2 = \frac{5}{2}$

so $\Lambda = \frac{20\pi\rho}{3}$ at the extremal point. Now, we want to find out if it is a minimum or a maximum. As equation (60) shows that the curvature of y is always positive, the point $x = \frac{5}{2}$ is a minimum. The function y has a positive minimum for positive values of the matter density ρ, i.e. the function is always positive, hence the first case ($A^2 - 3B > 0$) for the solution to the left hand side of equation (46) is applicable.

Combining the right and left-hand sides, equation (51) and (55), gives

$$\begin{aligned} \frac{1}{2\rho} \ln\left(1 - \frac{r^2}{a^2}\right) &= \frac{1}{C}\left[\ln\left(\frac{3p + A - C}{3p + A + C}\right) - \ln\left(\frac{3p_c + A - C}{3p_c + A + C}\right)\right] \Rightarrow \\ \frac{3p + A - C}{3p + A + C} &= \left(1 - \frac{r^2}{a^2}\right)^{\frac{C}{2\rho}} \frac{3p_c + A - C}{3p_c + A + C} \end{aligned} \tag{61}$$

where

$$C = \sqrt{A^2 - 3B}. \tag{62}$$

To eliminate p_c, we insert the value $p(r = R) = 0$, as the pressure being zero defines the surface of the star, into equation (61), and get

$$\frac{3p_c + A - C}{3p_c + A + C} = \frac{A - C}{A + C}\left(1 - \frac{R^2}{a^2}\right)^{-\frac{C}{2\rho}}. \tag{63}$$

Inserting equation (63) into equation(61) yields

$$p = \frac{C - A + (A - C)\left(\frac{a^2 - r^2}{a^2 - R^2}\right)^{\frac{C}{2\rho}}}{3\left[1 - \frac{A - C}{A + C}\left(\frac{a^2 - r^2}{a^2 - R^2}\right)^{\frac{C}{2\rho}}\right]}. \tag{64}$$

Setting the cosmological constant $\Lambda = 0$ in equation (64) should reproduce equation (18). Inserting $\Lambda = 0$ into equations (53), (54) and (62) gives

$$A = 2\rho \tag{65}$$

$$B = \rho^2 \tag{66}$$

$$C = \rho. \tag{67}$$

Inserting these values into equation (64) yields

$$
\begin{aligned}
p &= \frac{\rho\sqrt{\frac{a^2-r^2}{a^2-R^2}}-\rho}{3-\sqrt{\frac{a^2-r^2}{a^2-R^2}}} \\
&= \rho\frac{\sqrt{1-\frac{r^2}{a^2}}-\sqrt{1-\frac{R^2}{a^2}}}{3\sqrt{1-\frac{R^2}{a^2}}-\sqrt{1-\frac{r^2}{a^2}}}. \qquad (68)
\end{aligned}
$$

Inserting, the definition of a, equation (44) into equation (68), we get

$$
p = \rho\frac{\sqrt{1-\frac{2Mr^2}{R^3}}-\sqrt{1-\frac{2M}{R}}}{3\sqrt{1-\frac{2M}{R}}-\sqrt{1-\frac{2Mr^2}{R^3}}} \qquad (69)
$$

where $M = \frac{4\pi\rho R^3}{3}$. This equation is the same as equation (18), as it should be.

3.2 Alternative derivation of pressure with cosmological constant

This alternative way of deriving pressure for an incompressible star ($\rho(r) = \rho$) with cosmological constant, provides an easier way of finding the maximum mass of a neutron star. The only difference from the derivation in the previous section is the notation.

We begin with equation (45), and integrate from the surface of the star where $r = R$ and $p(R) = 0$.

$$
\int_0^p \frac{dp'}{(\rho+p')(\rho+3p'-\Lambda a^2\rho)} = -\frac{1}{2\rho a^2}\int_R^r \frac{r'dr'}{1-\frac{r'^2}{a^2}} \qquad (70)
$$

The integral on the right-hand side of the equation is as before

$$
\begin{aligned}
I_R &= \frac{1}{2\rho}\ln(\cos\chi)\Big|_{\chi_R}^{\chi} \\
&= \frac{1}{2\rho}\ln\left(\frac{\cos\chi}{\cos\chi_R}\right), \qquad (71)
\end{aligned}
$$

with

$$
\sin\chi_R = \frac{R}{a}. \qquad (72)
$$

The integral on the left hand side of equation (70) is

$$I_L = \int_0^p \frac{dp'}{(\rho + p')(k\rho + 3p')} \tag{73}$$

where

$$k = 1 - \Lambda a^2. \tag{74}$$

By fractional decomposition, we get

$$I_L = \frac{1}{\rho(k-3)} \int_0^p \left[\frac{1}{\rho + p'} - \frac{3}{k\rho + 3p'} \right] dp'. \tag{75}$$

Performing the integration gives

$$\begin{aligned} I_L &= \frac{1}{\rho(k-3)} \left[\ln\left(\rho + p'\right) - \ln\left(k\rho + 3p'\right) \right] \Big|_0^p \\ &= \frac{-1}{\rho(k-3)} \left[\ln\left(\frac{k\rho + 3p}{\rho + p} \right) + \ln\left(k\right) \right] \\ &= \frac{-1}{\rho(k-3)} \ln\left(\frac{k(k\rho + 3p)}{(\rho + p)} \right). \end{aligned} \tag{76}$$

Combining the right- and left-hand sides yields

$$\begin{aligned} \frac{-1}{k-3} \ln\left(\frac{k(k\rho + 3p)}{\rho + p} \right) &= \frac{1}{2} \ln\left(\frac{\cos\chi}{\cos\chi_R} \right) \Rightarrow \\ \frac{k(k\rho + 3p)}{\rho + p} &= \left(\frac{\cos\chi}{\cos\chi_R} \right)^{\frac{3-k}{2}}. \end{aligned}$$

This results in an expression for the pressure

$$p = \rho \frac{\cos\chi^{\frac{3-k}{2}} - k^2 \cos\chi_R^{\frac{3-k}{2}}}{3k \cos\chi_R^{\frac{3-k}{2}} - \cos\chi^{\frac{3-k}{2}}}. \tag{77}$$

This solution without the cosmological constant ($\Lambda = 0$ and $k = 1$) is

$$p = \rho \frac{\cos\chi - \cos\chi_R}{3\cos\chi_R - \cos\chi}. \tag{78}$$

Equation (77) plotted for different values of the cosmological constant Λ (i.e. k) is shown in Fig. 6.

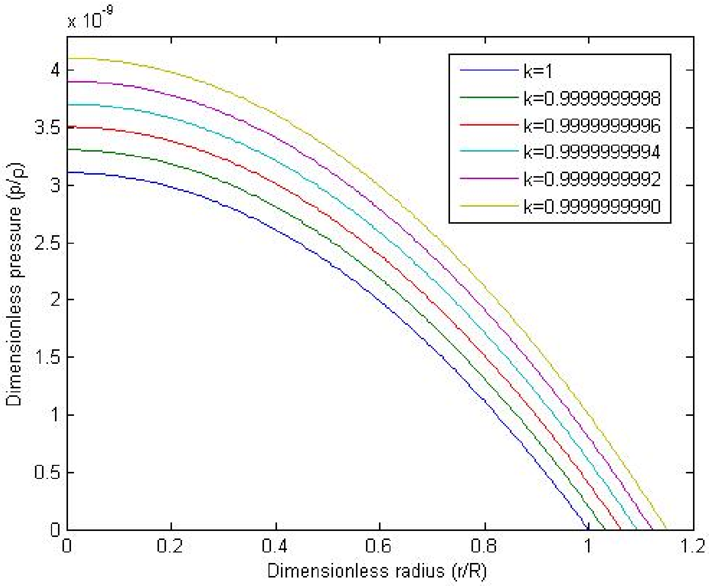


Figure 6: The dimensionless pressure as a function of dimensionless radius for different values of the cosmological constant.

Neutron stars usually have a mass of $1.35 - 2.1M_{\odot}$ and a radius of only 10-20km (where the stars with largest mass, have the smallest radius) causing the density to be of the order of nuclear matter density ($\sim 10^{18}$kg/m^3) [Shapiro and Teukolsky 1983]. The value of the density for the neutron star used in this plot is therefore $\rho = 10^{18} kg/m^3$.

Equation (40) and (44) inserted into equation (74) gives the definition of k:

$$k = 1 - \frac{\rho_{\rm vac}}{\rho}. \tag{79}$$

For $k = 0.999999999$, the vacuum density ρ_{vac} is 10^{-7}% of the density of the neutron star ρ. This is a very high value for the vacuum density, as it is calculated to be of the same order as the average matter density of the universe, which is approximately two hydrogen atoms per cubic metre ($\sim 10^{-27}$kg/m^3) [Wesley 1997]. As the density for a neutron star is of the order $\sim 10^{18}$kg/m^3 [Shapiro and Teukolsky 1983], this makes $\rho_{\rm vac}$ a factor of 10^{36} too big, i.e. much bigger than it could possibly be. Even at this much too large value for the cosmological constant Λ, it “only” results in a change in radius for the neutron star of approximately 15%. Conclusively the existence of a cosmological constant will not affect the pressure and radius of neutron stars, unless it is of an order 10^{36} greater than assumed which is highly unlikely.

3.3 The upper limit for the mass

The upper limit for the mass states that there can only be neutron stars below a certain mass. If the mass is greater than this limit, there will be a gravitational singularity, equilibrium will cease, and in some cases result in a black hole.

The origin of this limit is special relativity (SR). This causes pure Newtonian theories (like the hydrostatical equilibrium in section 2.4) to have no such limit. The theories incorporating special relativity, such as the Fermi gas model in section 6.3 provides a maximum mass both when using the hydrostatical equilibrium equations and the TOV-equations. The TOV-equations, however, do not demand the use of the Fermi gas model to give a maximum mass, as these equations incorporate both special- and general relativity.

Equation (10) gives

$$M = \frac{4\pi\rho R^3}{3} \tag{80}$$

where $M = m(R)$ is the mass of the star. Inserting equation (44) and (72) gives

$$\sin^2\chi_R = \frac{2M}{R}. \tag{81}$$

Inserting the value $p(r = 0) = p_c$ in equation (77) gives an expression for the central pressure of the star

$$p_c = \rho\frac{1 - k^2\cos^{\frac{3-k}{2}}\chi_R}{3k\cos^{\frac{3-k}{2}}\chi_R - 1}. \tag{82}$$

As the pressure p always has to be larger than zero for a stable star, we have the condition

$$\begin{aligned} 3k\cos^{\frac{3-k}{2}}\chi_R - 1 &> 0 \Rightarrow \\ \cos\chi_R &> \left(\frac{1}{3k}\right)^{\frac{2}{3-k}} \end{aligned} \tag{83}$$

from equation (82). Inserting the trigonometrical property $\sin^2\theta + \cos^2\theta = 1$ yields

$$\sin^2\chi_R < 1 - \left(\frac{1}{3k}\right)^{\frac{4}{3-k}}. \tag{84}$$

We then insert equation (72) into equation (81), and get

$$M = \frac{a}{2} \sin^3 \chi_R. \tag{85}$$

Inserting equation (44) into equation (85) gives

$$M = \frac{1}{2}\sqrt{\frac{3}{8\pi\rho}} \sin^3 \chi_R. \tag{86}$$

Using the condition equation (84) yields

$$M < \frac{1}{2}\sqrt{\frac{3}{8\pi\rho}} \left[1 - \left(\frac{1}{3k}\right)^{\frac{4}{3-k}}\right]^{\frac{3}{2}}. \tag{87}$$

For the case when the cosmological constant is zero $\Lambda = 0$ and $k = 1$, this simplifies to

$$M < \sqrt{\frac{3}{8\pi\rho}} \frac{8\sqrt{2}}{27}. \tag{88}$$

From equation (88), we see that the maximum mass is depending inversely on the density ρ. This can be explained physically by when the density increases, the particles in the star increase their kinetic energy (more relativistic), hence the central pressure increases, and can support a larger mass.

In Fig. 7 equation (87) is plotted as a function of k (i.e. the cosmological constant Λ) for different values of the density of a neutron star ρ. As k is defined as $k = 1 - \frac{\rho_{\text{vac}}}{\rho}$, and the densities always have to be positive, the possible interval for k is $0 \leq k \leq 1$. In Fig. 8, the density $\rho = 10^{18}$ is used when equation (87) is plotted as a function of k.

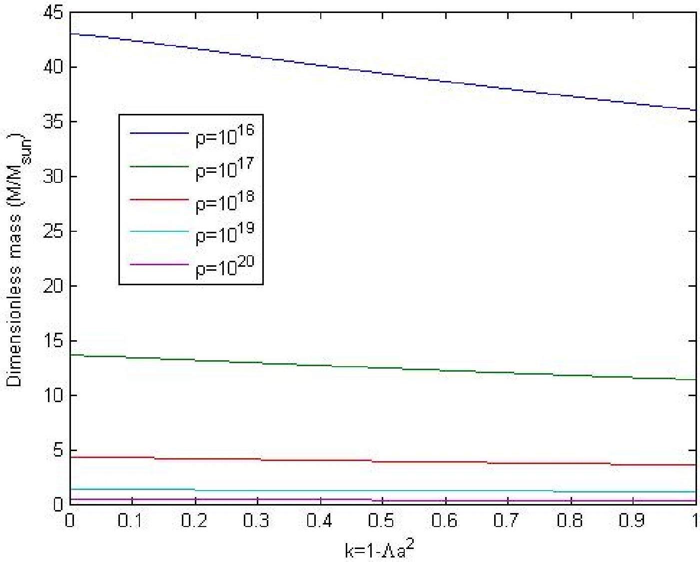


Figure 7: The dimensionless maximal mass as a function of k for different values of the density ρ of a neutron star.

Fig. 7 shows that the value of the constant k (i.e. the cosmological constant Λ) is most important to neutron stars with the smallest matter density. Increasing the value of the cosmological constant Λ causes the maximum mass to decrease. This effect is probably due to the dark energy having negative pressure, i.e. the pressure working inwards, causing objects to become smaller.

The plot also shows that the stars with the smallest density ρ has the biggest maximal mass. This is consistent with the result in section 2.2 where $\rho \to 0$ gives the maximal mass.

For nuclear density ($\rho \sim 10^{18}\text{kg/m}^3$), the value of the mass hardly varies with the value of k, i.e. the cosmological constant does not really affect the size of neutron stars as they have nuclear densities.

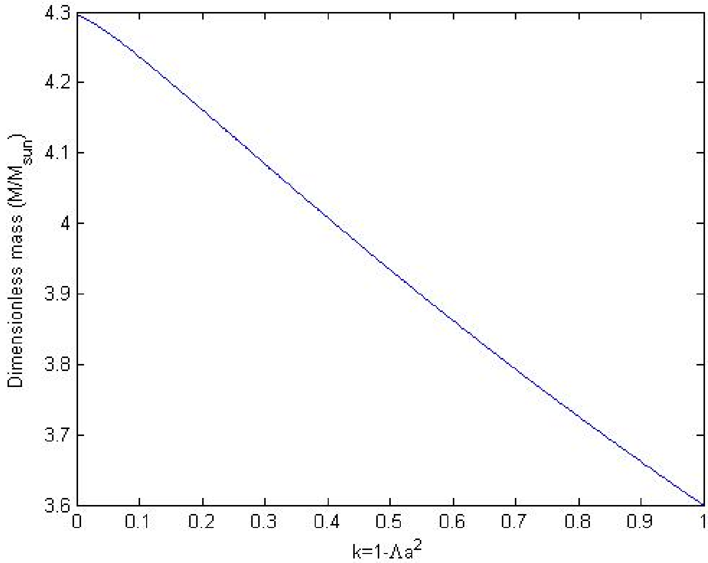


Figure 8: The dimensionless maximal mass as a function of k for $\rho = 10^{18} kg/m^3$.

For an average neutron star with density $\rho \sim 10^{18}\mathrm{kg/m^3}$, Fig. 8 shows a decrease of approximately 3% in maximal mass for a neutron star with the vacuum density ρ_{vac} being 1% of the density ρ of the neutron star. This value of ρ_{vac} is a factor 10^{43} higher than expected, hence the existence of a cosmological constant Λ will not influence the maximal mass of a neutron star in any observable way.

4 Numerical Solutions for White Dwarfs

In this section, the Newtonian theory of hydrostatical equilibrium is utilized to perform a numerical integration to find the masses and radii of white dwarfs in both the relativistic and the nonrelativistic case.

4.1 Dimensionless structure equations

The equation of hydrostatical equilibrium is

$$\frac{dp}{dr} = -\frac{Gm(r)\rho(r)}{r^2} \tag{89}$$

[Padmanabhan 2001]. The definition of the energy density $\epsilon(r)$ is

$$\rho(r) = \frac{\epsilon(r)}{c^2}. \tag{90}$$

This equation introduces special relativity (SR) into the theory, as it is equivalent to Einstein's famous equation for the relation between energy and mass, $E = mc^2$, of SR. Inserting equation (90) into equation (89) yields

$$\frac{dp}{dr} = -\frac{G\overline{M}(r)\epsilon(r)M_\odot}{c^2 r^2} \tag{91}$$

with

$$m(r) = \overline{M}(r)M_\odot, \tag{92}$$

where $M_\odot$ is the solar mass, and $\overline{M}(r)$ is a dimensionless number. Equation (91) can be written in the form:

$$\frac{dp}{dr} = -R_0\frac{\overline{M}(r)\epsilon(r)}{r^2}, \tag{93}$$

where $R_0 = \frac{GM_\odot}{c^2} = 1.47\text{km}$. In equation (93) p and ϵ carry dimensions of energy/(length)^3. Therefore, we define the dimensionless energy density $\bar{\epsilon}$, and pressure $\bar{p}$ by:

$$\begin{aligned} p &= \epsilon_0\bar{p} &(94)\\ \epsilon &= \epsilon_0\bar{\epsilon} &(95)\end{aligned}$$

where ϵ_0 has the dimension of energy density, energy/(length)^3, and can be chosen arbitrarily. We base this choice on the dimensionful numbers defining the problem. For a polytropic star, we can write

$$\bar{p} = \overline{K}\bar{\epsilon}^{\gamma}, \tag{96}$$

where

$$\overline{K} = K\bar{\epsilon_0}^{\gamma-1}. \tag{97}$$

$\overline{K}$ and K has different values for the relativistic and the non-relativistic case:

$$K_{rel} = \frac{\hbar c}{12\pi^2}\left(\frac{3\pi^2 Z}{Am_N c^2}\right)^{\frac{4}{3}}, \tag{98}$$

$$K_{nonrel} = \frac{\hbar^2}{15\pi^2 m_e}\left(\frac{3\pi^2 Z}{Am_N c^2}\right)^{\frac{5}{3}}, \tag{99}$$

[Silbar and Reddy 2004], where Z is the number of protons. This is derived in section 6.1.1 for a polytropic star obeying $p = K\epsilon^{\gamma}$ with the constant $\gamma = \frac{4}{3}$ in the relativistic case, and $\gamma = \frac{5}{3}$ in the nonrelativistic case.

Inserting equation (96) into equation (93) gives

$$\frac{d\bar{p}(r)}{dr} = -\frac{\alpha\bar{p}(r)^{\frac{1}{\gamma}}\overline{M}(r)}{r^2}, \tag{100}$$

where

$$\alpha = \frac{R_0}{\overline{K}^{\frac{1}{\gamma}}} = \frac{R_0}{(K\epsilon_0{}^{\gamma-1})^{\frac{1}{\gamma}}}. \tag{101}$$

R_0 has dimension of km, hence α is in km, and equation (100) has dimensions of km^{-1}. As ϵ_0 is still free, we can choose any convenient value for α. For a given value of α, ϵ_0 is given by equation (101):

$$\epsilon_0 = \left[\frac{1}{K}\left(\frac{R_0}{\alpha}\right)^{\gamma}\right]^{\frac{1}{\gamma-1}}. \tag{102}$$

We also wish to have the other coupled equation, equation (5)

$$\frac{dm(r)}{dr} = 4\pi r^2\rho(r) \tag{103}$$

in dimensionless form. Inserting eqs. (90), (92) and (94)-(95) yields:

$$\frac{d\overline{M}(r)}{dr} = \beta r^2 \left[\bar{p}(r)\right]^{\frac{1}{\gamma}}, \tag{104}$$

where

$$\beta = \frac{4\pi\epsilon_0}{M_\odot c^2 (K{\epsilon_0}^{\gamma-1})^{\frac{1}{\gamma}}}. \tag{105}$$

Equation (104) also has dimension km^{-1}.

4.2 Numerical integration

We will integrate the dimensionless equations (100) and (104) numerically from the initial values at the centre of the star. To do this we need the value of the central pressure and mass. $\overline{M}(0)$ has to be zero, and $\bar{p}(0)$ must be positive. The pressure will decrease towards zero and the mass will increase towards the total mass of the star. The radius of the star R and the mass $M = \overline{M}(R)$ will vary depending on the choice for $\bar{p}(0)$.

We wish that the constants α and β are not too different from each other for the purpose of (numerical) stability when solving the equations numerically. That can be arranged for both the relativistic and the nonrelativistic case.

To solve these coupled first order differential equations, we use a fixed step 4th order Runge-Kutta routine. The value of r at the centre was set equal to 0.0000001 instead of zero to avoid dividing by zero. It was solved using matlab, and the programme is listed in the appendix.

4.3 Choosing the values of α, β and $\bar{p}(0)$

4.3.1 The relativistic case

The relativistic regime includes the white dwarfs with the biggest mass. The big mass needs a bigger central pressure to support it, thus causing the squeezed electrons to be relativistic.

Some trial and error, gives the value

$\alpha = R_0 = 1.473\mathrm{km}$,

which by equation (102) fixes

$\epsilon_0 = 4.17 \frac{M_\odot c^2}{\mathrm{km}^3}$.

Equations (99) and (105) then gives the value

$\beta = 52.46\mathrm{km}^{-3}$,

which is approximately 30 times bigger than the value for α, but this is manageable with respect to the numerical integration.

We can estimate a value for $\bar{p}(0)$ by finding an average energy density of a star with radius $\sim 10^4$km and a mass $M_\odot$ by the ratio of its rest mass energy to its volume:

$$\langle\epsilon\rangle \approx \frac{M_\odot c^2}{R^3} = 10^{-12}\mathrm{km}^{-3}, \qquad (106)$$

which is much smaller than the value we have chosen for ϵ_0 here. In addition, the pressure p is approximately 2000 times smaller than the energy density ϵ [Silbar and Reddy 2004]. Thus choosing a value of $\bar{p}(0) \sim 10^{-15}$ yields reasonable results. Table 2 shows the results of our programme for R and M, and how they depend on $\bar{p}(0)$.

4.3.2 The nonrelativistic case

As the central pressure $\bar{p}(0)$ becomes smaller, the electrons are no longer relativistic. The smaller pressure can only support less massive stars than in the relativistic case, so the nonrelativistic white dwarfs are on the least massive end of the scale. These stars have larger radii than their relativistic counterparts.

When we go to the nonrelativistic limit, $\gamma = \frac{5}{3}$ in the polytropic equation of state, equation (96). The integration is performed in the same way as in the relativistic case, only the value of γ is changed.

After some experimentation, we choose:

$\alpha = 0.05\mathrm{km}$,

which by equation (102) fixes:

$\epsilon_0 = 0.01392 \frac{M_\odot c^2}{\mathrm{km}^3}$.

This value for ϵ_0 is smaller than in the relativistic case. The other constant comes from equation (105):

$\beta = 0.005924\text{km}^{-3}$,

which is smaller than the value of α in the nonrelativistic limit as opposed to the relativistic limit where it was the other way around. The values for the central pressure must be $\bar{p}(0) \leq 10^{-15}$ [Silbar and Reddy 2004] for the star to be nonrelativistic. Table 3 shows the results of our programme for R and $\overline{M}$, and how they depend on $\bar{p}(0)$.

4.4 Results of the numerical integration

For the relativistic case, we obtain the values of Table 2 for the radius and the mass of the star.

Table 2: Radius (in km) and mass (in $M_\odot$) for a white dwarf with relativistic Fermi electron gas.

Central Pressure $\bar{p}(0)$	Radius R (km)	Mass $\overline{M}$ ($M_\odot$)
10^{-14}	4962	1.24693455892
10^{-15}	8824	1.24693455892
10^{-16}	15691	1.24693455892

All these stars have the same mass. Hence increasing the central pressure does not allow the star to be more massive, just more compact. From the Lane-Emden formulation of the problem, one obtains the equation

$$M = 4\pi \left[\frac{(n+1)K}{4\pi G}\right]^{\frac{3}{2}} \rho_c{}^{\frac{(3-n)}{2n}} \xi_1{}^2 |\theta'(\xi_1)| \tag{107}$$

[Shapiro and Teukolsky 1983], for the mass of the star. The density $\rho_c \propto \bar{p}(0)$ and the constants $n = \frac{1}{\gamma - 1}$ and K from the polytrope equations (96) and (97) respectively. ξ_1 and $|\theta'(\xi_1)|$ are numerical constants depending on the choice of γ. ξ is the dimensionless radius, and $\theta(\xi)$ is the dimensionless density of the star. ξ_1 is the point corresponding to the surface of the star, i.e. where $|\theta(\xi_1)| = 0$.

This equation will in the relativistic case have $\gamma = \frac{4}{3}$, and thus be independent of the central pressure. The Lane-Emden approach gives the equation for the radial dependence of the mass:

$$M = 4\pi R^{\frac{(3-n)}{(1-n)}} \left[\frac{(n+1)K}{4\pi G} \right]^{\frac{n}{(n-1)}} \xi_1^{\frac{(3-n)}{1-n}} \xi_1^{2} |\theta'(\xi_1)| \tag{108}$$

[Shapiro and Teukolsky 1983]. From this equation, we see that the radius decrease with increasing mass. So, the most massive white dwarfs are also the smallest ones. The exception is for $n = 3$, the ultra-relativistic case, where the mass is independent of the radius.

Fig. 9 shows the dimensionless pressure and mass as a function of radius for a white dwarf.

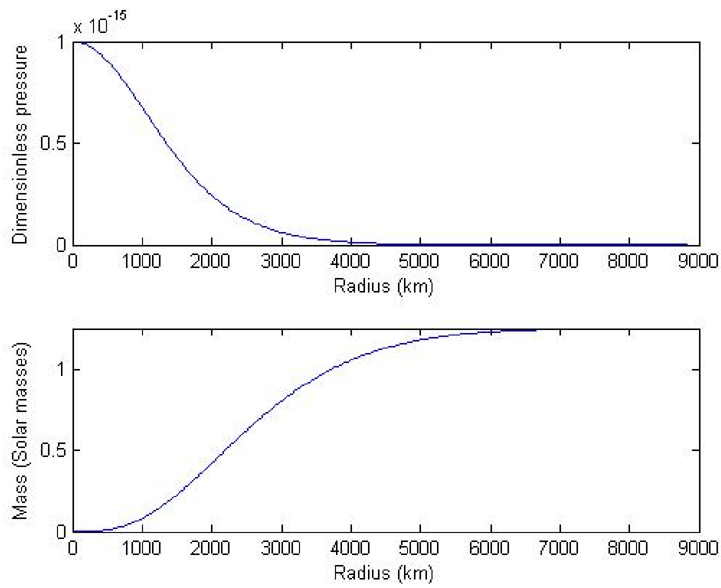


Figure 9: The dimensionless pressure and mass (in $M_\odot$) as a function of radius for a polytrope white dwarf with a relativistic Fermi electron gas with $\bar{p}(0) = 10^{-15}$.

In Fig. 9 the dimensionless pressure $\bar{p}(r)$ becomes small at around 5000km before going through zero at 8824km. So, this star has a very tall atmosphere.

For the nonrelativistic case, we obtain the values of Table 3 for the radius and mass of the star.

Table 3: Radius (in km) and mass (in $M_\odot$) for a white dwarf with nonrelativistic Fermi electron gas.

Central Pressure $\bar{p}(0)$	Radius R (km)	Mass $\overline{M}$ $(M_\odot)$
10^{-15}	10615	0.3942449554
10^{-16}	13363	0.1975905385
10^{-17}	16823	0.0990298554

Table 3 shows that in the nonrelativistic case the mass of the star depends on the central pressure, it increases with increasing pressure. It also shows that these stars are less massive than the relativistic stars, as expected.

Fig. 10 shows the dimensionless pressure as a function of radius for a white dwarf.

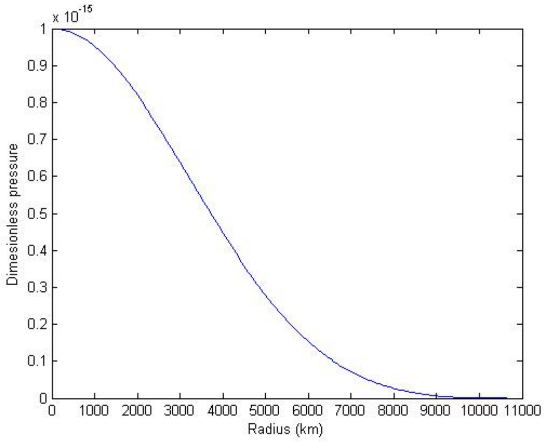


Figure 10: The dimensionless pressure as a function of radius for a polytrope white dwarf with a nonrelativistic Fermi electron gas with $\bar{p}(0) = 10^{-15}$.

In Fig. 10 the dimensionless pressure $\bar{p}(r)$ becomes small at around 10000km before going through zero at 10615km. Hence, this star has a much smaller atmosphere than the relativistic white dwarfs. The mass of the nonrelativistic white dwarf is smaller than that of the relativistic, but the radius is larger for the nonrelativistic case, i.e. the density is much smaller for the nonrelativistic white dwarf compared to its relativistic counterpart.

5 Numerical Solutions for Neutron Stars

In this section, the masses and radii for neutron stars are found numerically. The neutron star theory being used here differs from that of white dwarfs by using the TOV-equations which incorporate general relativity (GR) in addition to special relativity (SR) contributions, instead of the equation of hydrostatical equilibrium, equation (89). The three dimensionless factors in the TOV-equation represents the contributions from the theory of relativity.

$$\frac{dp}{dr} = -\frac{G\epsilon(r)M(r)}{c^2r^2}\left[1+\frac{p(r)}{\epsilon(r)}\right]\left[1+\frac{4\pi r^3p(r)}{M(r)c^2}\right]\left[1-\frac{2GM(r)}{c^2r}\right]^{-1}. \tag{109}$$

The first two square bracket factors is due to SR-effects of order $\frac{v^2}{c^2}$, where v is the speed of the particles. The third factor is due to GR [Shapiro and Teukolsky 1983].

The choice of equation of state will be based on a Fermi gas model for neutrons as opposed to electrons in section 4. This model is unrealistic for two reasons. First, important contributions to the energy density caused by the nucleon-nucleon interactions are ignored. Secondly, a neutron star contains not just neutrons, but also a fraction of protons and electrons preventing the neutrons decaying into protons and electrons by the weak interaction. These points will be dealt with in section 7.

5.1 Dimensionless structure equations

With the model described for a neutron star, we can use the results from the previous section on the results for white dwarfs, only changing the electron mass m_e to the neutron mass m_N in equation (100) and equation (104). This will give the result of Newtonian theory. To involve GR, we have to use the TOV-equation equation (109) together with equation (104). Using the definitions of the dimensionless variables for mass, pressure and energy density

$$\begin{aligned} M(r) &= M_{\odot}\overline{M}(r), && (110)\\ p &= \epsilon_0\bar{p}, && (111)\\ \epsilon &= \epsilon_0\bar{\epsilon}, && (112) \end{aligned}$$

the resulting dimensionless TOV-equation is

$$\frac{d\bar{p}(r)}{dr} = -\frac{\alpha\bar{p}(r)^{\frac{1}{\gamma}}\overline{M}(r)}{r^2}\left[1+\bar{K}^{\frac{1}{\gamma}}\bar{p}(r)\right]\left[1+\delta r^3\frac{\bar{p}(r)}{\overline{M}(r)}\right]\left[1-\frac{2R_0\overline{M}(r)}{r}\right]^{-1}. \tag{113}$$

Here $\delta = \frac{4\pi}{M_\odot c^2}\left[\frac{1}{K}\left(\frac{R_0}{\alpha}\right)^\gamma\right]^{\frac{1}{\gamma-1}}$ and the remaining constants are defined as in the previous section.

5.2 Choosing the values of α, β and $\bar{p}(0)$

5.2.1 The nonrelativistic case

As in the nonrelativistic case for white dwarfs, the polytropic index is $\gamma = \frac{5}{3}$. In this case equation (99) becomes

$$K_{\text{nonrel}} = \frac{\hbar^2}{15\pi^2 m_N}\left(\frac{3\pi^2 Z}{A m_N c^2}\right)^{\frac{5}{3}}. \tag{114}$$

When choosing $\alpha = 1\text{km}$, the scaling factor ϵ_0 is given by equation (102)

$\epsilon_0 = 1.603 \times 10^{37}\text{J/m}^3$,

which in turn by equations (114) and (105), gives the value for β:

$\beta = 0.7636\text{km}^{-3}$.

In this case the constants α and β are of numerically same order.

A typical neutron star has mass the size of the solar mass $M_\odot$ and a radius of 10km, which by equation (106) gives a value for the central pressure $\bar{p}(0) \sim 10^{-4}$ or less. The programme is essentially the same as for a nonrelativistic white dwarf, and the results are given in Table 4.

5.2.2 The relativistic case

The polytropic equation in this case has $\gamma = 1$, resulting in $p = \frac{\epsilon}{3}$, which is a well known result for a relativistic gas [Silbar and Reddy 2004]. Equations (96) and (97) causes the constants to be $K = \bar{K} = \frac{1}{3}$. We still use the same value for the scaling factor as in the non-relativistic case, $\epsilon_0 = 1.603 \times 10^{37}\text{J/m}^3$. If we then choose

$\alpha = 3R_0 = 4.428\text{km}$,

we find

$\beta = 3.374\text{km}^{-3}$.

The central pressure $\bar{p}(0)$ is expected to be greater than 10^{-4} as this was the value we chose for the nonrelativistic case. The programme for performing the numerical integration is listed in the appendix.

This numerical integration, however, doesn't give a value for the radius and mass for a neutron star. This is because of the pressure in the plot only falling monotonically towards zero, and never passing through zero. The reason for this model to fail is that the relativistic gas equation of state is not appropriate for such low pressures.

5.3 Numerical result for neutron stars

Table 4: Radius (in km) and mass (in $M_{\odot}$) for a neutron star with nonrelativistic Fermi gas equation of state.

Central Pressure $\bar{p}(0)$	Radius (Newton) R	Mass (Newton) $\overline{M}$	Radius (GR) R	Mass (GR) $\overline{M}$
10^{-4}	16.606	0.77463	15.313	0.63632
10^{-5}	20.906	0.38824	20.225	0.35830
10^{-6}	26.319	0.19458	25.971	0.18840

The general relativistic effects are small, but increase with increasing central pressure, as expected. The smaller mass neutron stars have the biggest radii as is also the case for white dwarfs, because the gravitational attraction is smaller and thus the star extends to larger radii.

6 The Equation of State for Arbitrary Relativity

In this section we will find an equation of state for a Fermi gas of electrons and neutrons for white dwarfs and neutron stars respectively, which works for both the relativistic and nonrelativistic cases.

6.1 White dwarfs

The Fermi gas model for electrons will be derived, and used together with the equation for polytropic stars to find an appropriate equation of state for arbitrary relativity for white dwarfs. This equation will be used to find the mass and radius of a white dwarf.

6.1.1 Fermi gas model for electrons

The number of states dn between the momentum k and $k + dk$ for free electrons is

$$dn = \frac{gd^3k}{(2\pi\hbar)^3} = \frac{4\pi gk^2dk}{(2\pi\hbar)^3}, \tag{115}$$

[Silbar and Reddy 2004], in three isotropic dimensions. g is the degeneracy, and is equal to 2, because there are two spin states for each electron energy level. Performing the integration gives

$$n = \frac{8\pi}{(2\pi\hbar)^3}\int_0^{k_\mathrm{F}} k^2dk = \frac{{k_\mathrm{F}}^3}{3\pi^2\hbar^3}, \tag{116}$$

with $k_\mathrm{F} = \frac{E_\mathrm{F}}{c}$ being the Fermi momentum (Fermi energy divided by the speed of light) for the electrons in the star we are considering.

The mass density of the star is given by

$$\rho = nm_N\frac{A}{Z} \tag{117}$$

if the electron mass m_e is neglected with respect to the nucleon mass m_N (which is $\sim 2000m_e$). $\frac{A}{Z}$ is the number of nucleons per electron (A is the number of nucleons and Z is the number of protons). Inserting equation (116) into equation (117), yields

$$\rho = m_N\frac{A}{Z}\frac{{k_\mathrm{F}}^3}{3\pi^2\hbar^3}. \tag{118}$$

As the energy density of this star is dominated by the nucleon masses, we have

$$\epsilon \approx \rho c^2. \tag{119}$$

The contribution to the energy density from the electrons is

$$\begin{aligned}
\epsilon_{elec}(k_{\mathrm{F}}) &= \frac{8\pi}{(2\pi\hbar)^3}\int_0^{k_F}\sqrt{k^2c^2+{m_e}^2c^4}\,k^2dk \\
&= \frac{{m_e}^4c^5}{8\pi^2\hbar^3}\left[(2x^3+x)\sqrt{1+x^2}-\operatorname{arcsinh}(x)\right],
\end{aligned} \tag{120}$$

[Silbar and Reddy 2004], where $x = \frac{k_{\mathrm{F}}}{m_e c}$. The total energy density is then

$$\epsilon = n m_N c^2 \frac{A}{Z} + \epsilon_{elec}(k_{\mathrm{F}}). \tag{121}$$

To get an equation of state, we need an expression for the pressure as well. We find this from the first law of thermodynamics $dU = dQ - pdV$ with the temperature fixed at $T = 0$, where dU is the change in internal energy, dQ is the change in heat and dV is the change in volume. Thus we get

$$p = -\left(\frac{\partial U}{\partial V}\right)_T = n^2\frac{d(\frac{\epsilon}{n})}{dn} = n\frac{d\epsilon}{dn} - \epsilon = n\mu - \epsilon, \tag{122}$$

as $dQ = 0$ because $dQ = TdS$, where $T = 0$ is temperature and dS is change in entropy. The quantity $\mu = \frac{d\epsilon}{dn}$ is called the chemical potential, and gives the energy required to add an extra electron to the star. Combining equation (122), equation (121) and equation (120) gives

$$\begin{aligned}
p(k_{\mathrm{F}}) &= \frac{8\pi}{3c(2\pi\hbar)^3}\int_0^{k_{\mathrm{F}}}\frac{1}{\sqrt{k^2c^2+m_e^2c^4}}k^4dk \\
&= \frac{m_e^4c^5}{24\pi^2\hbar^3}\left[(2x^3-3x)\sqrt{1+x^2}+3\operatorname{arcsinh}(x)\right]
\end{aligned} \tag{123}$$

Now we can make a plot of energy density ϵ versus pressure p, by using equation (120) and equation (123). The result is shown in Fig. 11

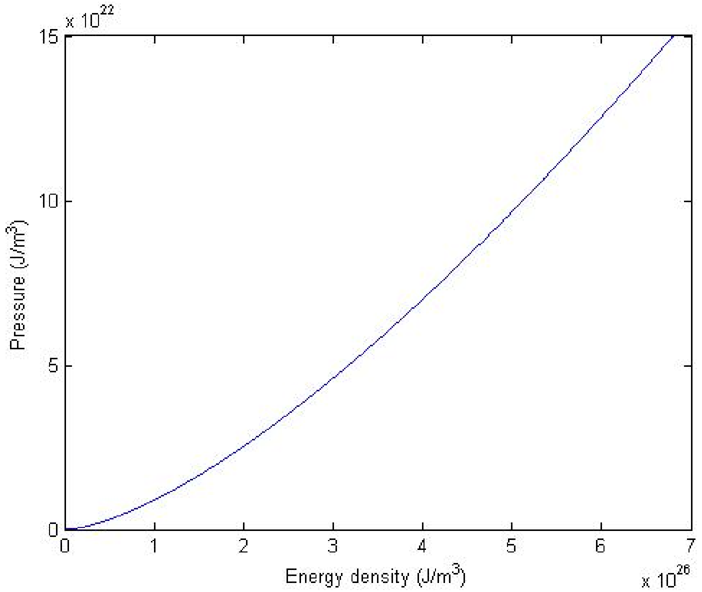


Figure 11: The energy density for a polytropic white dwarf consisting of a relativistic Fermi electron gas.

6.1.2 Polytropic model

In the relativistic case, equation (123) combined with equation (118) simplifies to

$$\begin{aligned} p(k_{\mathrm{F}}) &= \frac{{m_e}^4 c^5}{3\pi^2\hbar^3}\int_0^{\frac{k_{\mathrm{F}}}{m_e c}} u^3 du \\ &= \frac{\hbar c}{12\pi^2}\left(\frac{3\pi^2 Z\rho}{m_N A}\right)^{\frac{4}{3}} \\ &\approx K_{\mathrm{rel}}\epsilon^{\frac{4}{3}}, \end{aligned} \tag{124}$$

where

$$K_{\mathrm{rel}} = \frac{\hbar c}{12\pi^2}\left(\frac{3\pi^2 Z}{A m_N c^2}\right)^{\frac{4}{3}}. \tag{125}$$

This satisfies the polytropic equation $p = K\epsilon^{\gamma}$ with $\gamma = \frac{4}{3}$. In the non-relativistic case, however, there is another polytropic equation. In a similar way to the derivation of equation (124), we find

$$p(k_{\mathrm{F}}) = K_{\mathrm{nonrel}}\epsilon^{\frac{5}{3}} \tag{126}$$

where

$$K_{\mathrm{nonrel}} = \frac{\hbar^2}{15\pi^2 m_e}\left(\frac{3\pi^2 Z}{A m_N c^2}\right)^{\frac{5}{3}}. \tag{127}$$

Equation (124) describes the pressure as a function of the energy density for an ultra-relativistic white dwarf, and equation (126) describes the pressure as a function of the energy density for a nonrelativistic white dwarf. We rewrite these equations to be energy density as a function of pressure

$$\epsilon_{\mathrm{R}} = \left(\frac{p}{K_{\mathrm{R}}}\right)^{\frac{3}{4}}, \tag{128}$$

$$\epsilon_{\mathrm{NR}} = \left(\frac{p}{K_{\mathrm{NR}}}\right)^{\frac{3}{5}}. \tag{129}$$

To find an expression for the whole range of k_{F}, from nonrelativistic to ultra-relativistic, these equations must be combined. The way of doing this is to rewrite equation (128) and equation (129) like this

$$\bar{\epsilon}(\bar{p}) = A_{\mathrm{NR}}\bar{p}^{\frac{3}{5}} + A_{\mathrm{R}}\bar{p}^{\frac{3}{4}}, \tag{130}$$

where A_{NR} and A_{R} are constants. For low pressures (nonrelativistic), the first term is dominant, and for high pressures (ultra-relativistic), the second term is the dominant term. The values of the constants A_{NR} and A_{R} are fixed by using a programme for function-fitting on the result in Fig. 11. The programme is listed in the appendix.

6.1.3 Results

We chose to not give the results of the function fitting, and the masses and radii for white dwarfs because the matlab-programme demonstrated a numerical instability resulting in poor values. This problem does not occur in the numerical integration of neutron stars in the next section.

6.2 Neutron stars

As neutron stars in the ultra-relativistic case do not provide an answer for radius and mass, we want to find an equation suitable for arbitrary values of the relativity parameter $x = \frac{k_{\mathrm{F}}}{m_n c}$. The equation for a polytrope $p = K\epsilon^{\frac{1}{\gamma}}$

has the value $\gamma = \frac{5}{3}$ in the nonrelativistic case equal to the value of γ for the NR-case for white dwarfs. The value for the ultra-relativistic (UR) case is, however, different.

For white dwarfs, we derived the value of $\gamma = \frac{4}{3}$, but the corresponding value for neutron stars is $\gamma = 1$. The reason for this is the well-known result for an ultra-relativistic gas, derived in the preliminary project to this master's thesis, $p = \frac{\epsilon}{3}$ [Egeland 2006]. From this equation we observe that the pressure-dependence of the energy density is linear, resulting in $\gamma = 1$. This is applicable to neutron stars, and not white dwarfs because of a difference in equation (121). For white dwarfs, this equation is made of two terms, the first dominating over the second as the neutron mass is approximately 2000 times larger than the electron mass ($m_N \approx 2000 m_e$). The first term describes the rest mass energy of the nucleons, and the second term is the total energy density from the electrons (both rest mass and kinetic). For neutron stars, however, the first term is incorporated in the second term (which is equal to the term for white dwarfs with the exception of the electron mass being replaced by the neutron mass), i.e. there is only one term in equation (121) for pure neutron stars as there are no electrons present. This difference leads to a difference in the equation of state.

The equivalent to equation (130) for a neutron star is

$$\bar{\epsilon} = A_{\rm NR}\bar{p}^{\frac{3}{5}} + A_{\rm R}\bar{p}, \tag{131}$$

and the constant ϵ_0 defined as

$$\epsilon_0 = \frac{{m_n}^4 c^5}{(3\pi^2\hbar)^3} \approx 6.26031238 \cdot 10^{32} \mathrm{J/m}^3. \tag{132}$$

The result varies with the length of the interval of the relativity parameter x. For large values of this parameter, the function ϵ is dominated by the relativistic part of the pressure dependence, and for small values of x, it is dominated by the nonrelativistic part of the pressure dependence. The values of the fitting constants $A_{\rm NR}$ and $A_{\rm R}$ will therefore vary as the interval changes and the different parts of the pressure dependence is emphasized. A range of different intervals are chosen to show this effect, where $0 \leq x \leq x_{max}$.

Using the same programme as for white dwarfs to find the best fit of the constants A_{NR} and A_{R}, gives the results shown in Table 5.

Table 5: The nonrelativistic and relativistic constants for a neutron star with arbitrarily relativistic Fermi electron gas.

x_{max}	A_{NR}	A_{R}
100	1.145774	2.999737
200	0.8692956	2.999934
300	0.7393258	2.999971
400	0.6590228	2.999983
500	0.6027763	2.999989
600	0.5603698	2.999993
700	0.5269196	2.999995
800	0.4994799	2.999996
900	0.4765051	2.999997
1000	0.4568627	3.000000

Inserting these values in the same way as for white dwarfs yields the values shown in Table 6.

Table 6: Radius (in km) and mass (in $M_\odot$) for a neutron star with arbitrarily relativistic Fermi electron gas with classical newtonian mechanics and GR-theory.

x_{max}	Radius (Newton) R	Mass (Newton) $\overline{M}$	Radius (GR) R	Mass (GR) $\overline{M}$
100	30.02	3.622	24.42	2.071
200	38.48	5.548	29.89	2.945
300	44.43	7.044	33.51	3.548
400	49.17	8.300	36.29	4.014
500	53.17	9.396	38.59	4.396
600	56.67	10.38	40.57	4.720
700	59.80	11.27	42.31	5.002
800	62.65	12.09	43.89	5.253
900	65.27	12.86	45.32	5.478
1000	67.80	13.57	46.65	5.680

The definition of ϵ_0 and the chosen value $\alpha = R_0 = 1.476$km, gives the value $\beta = 0.03778\text{km}^{-3}$ from equation (101) and (105). We chose $\bar{p}(0) = 0.01$

which is clearly in the relativistic regime.

The article [Silbar and Reddy 2004] gives the values

$$A_{\mathrm{NR}} = 2.4216 \text{ and } A_{\mathrm{R}} = 2.8663 \tag{133}$$

for the constants of equation (131). We don't know the details in the calculation to get these values, e.g the length of the interval, and have not been able to reproduce them with neither Mathematica's intrinsic fitting function which the writers of [Silbar and Reddy 2004] claim to have used, nor a self-made programme in matlab. This programme is listed in the appendix.

By using the values from the article, we were able to reproduce the results for the radius and the mass of a neutron star within the expected numerical rounding differences. Our results for the radius and mass using the values (133) were:

$R = 15.06$km, $\overline{M} = 1.037M_\odot$ (Newtonian theory),

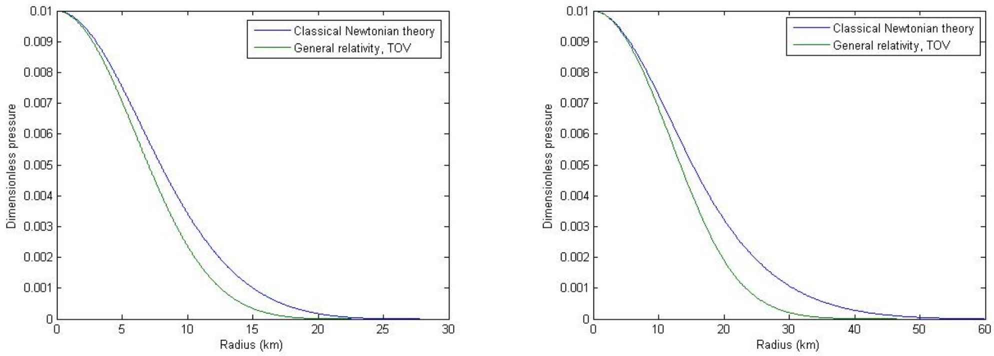
$R = 13.37$km, $\overline{M} = 0.6443M_\odot$ (General relativity, TOV).

The conclusion is that our programme for the calculation of the radius and pressure of a neutron star is correct, as it yields the same results as in [Silbar and Reddy 2004]:

$R = 15.0$km, $\overline{M} = 1.037M_\odot$ (Newtonian theory),

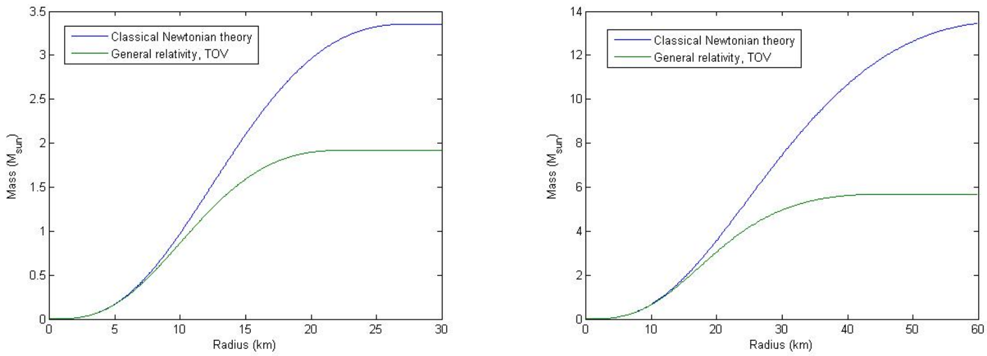
$R = 13.4$km, $\overline{M} = 0.717M_\odot$ (General relativity, TOV).

The pressure and mass as a function of the radius is plotted by a programme which is listed in the appendix.



(a) (b)

Figure 12: The dimensionless pressure $\bar{p}(r)$ as a function of the radius r in km for a pure neutron star with central pressure $\bar{p}(0) = 0.01$ using the Fermi equation of state for arbitrary relativity, equation (131), for (a) $x_{max} = 100$ and for (b) $x_{max} = 1000$.



(a) (b)

Figure 13: The mass $\bar{M}(r)$ in $M_{\odot}$ as a function of the radius r in km for a pure neutron star with central pressure $\bar{p}(0) = 0.01$ using the Fermi equation of state for arbitrary relativity, equation (131), for (a) $x_{max} = 100$ and for (b) $x_{max} = 1000$.

Our results plotted for two different intervals, $x_{max} = 100$ and $x_{max} = 1000$ with Newtonian theory versus GR theory. As expected, the GR effects are greatest for the interval with $x_{max} = 1000$. This is because this interval emphasizes the relativistic part of the equation of state, as large x is equivalent with large k_{F}, which in turn is equivalent with large energies (relativistic energies). The GR effects are most important for stars with large mass, i.e. stars with relativistic Fermi gas of neutrons, as the stars expand much in size and mass for a relativistic equation of state compared to a nonrelativistic equation of state.

By calculating the radius and mass for a range of initial pressure $\bar{p}(0)$, we get a plot of radius versus mass, shown in Fig. 14.

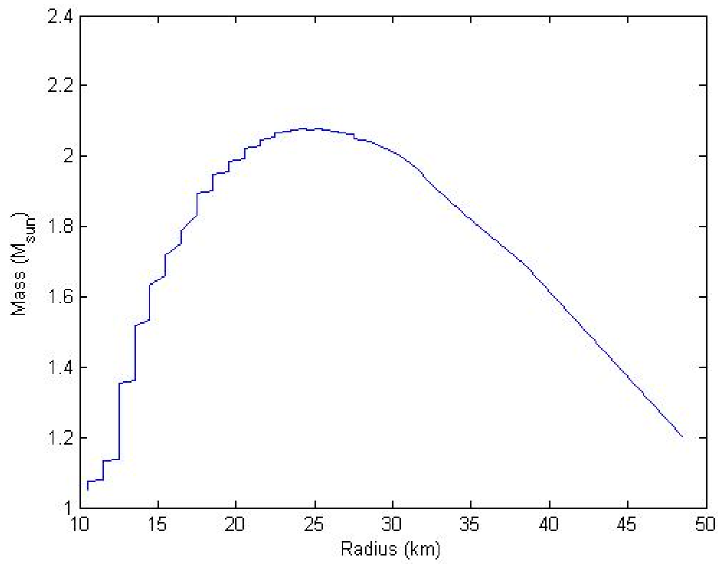


Figure 14: Mass (in $M_\odot$) as a function of radius (in km) for pure neutron stars using a Fermi gas equation of state with $x_{max} = 100$. The stars to the right of the maximum are stable, whereas the ones to the left are unstable against gravitational collapse.

In Fig. 14 the low mass stars with large radius are to the right in the graph and correspond to small values of initial pressure $\bar{p}(0)$. As the central pressure increases, the star is able to support a larger mass, hence the mass increases. A larger mass has bigger gravitational attraction, hence these stars have smaller radii. So, increasing the central pressure corresponds to "climbing the hill" in Fig. 14.

The star reaches the top of the hill at approximately $\bar{p}(0) = 0.03$, with a mass of $\overline{M} \approx 2.1 M_\odot$ and radius of $R \approx 25$km.

The solutions in Fig. 14 to the left side of the maximum, turn out be unstable against gravitational collapse into a black hole [Silbar and Reddy 2004].

6.3 Maximum Mass

The reason that Fig. 14 has a maximum mass can be given in a general way. One can argue that both white dwarfs and neutron stars have to have a limiting mass beyond which stable hydrostatic configurations are impossible.

In cold stars, the thermal component of the pressure is negligible as the temperature is much lower than the Fermi temperature of the star. Because of this, the variations in both pressure and energy density are caused only by changes in the density.

An increase in the density results in a proportional increase in the energy density as $\epsilon = \rho c^2$. This increase results in a corresponding increase in the gravitational attraction. To be able to balance this increase, the increment in pressure has to be large enough. But the rate of change of the pressure with respect to the energy density is related to the speed of sound (see section 7.3.1). In Newtonian theory the speed of sound does not have an upper limit. In special relativity, however, the speed of no propagating signals can exceed that of light. This limit is a bound on the pressure increment associated with changes in density.

As there is a bound on this increment in special relativistic theory, we can conclude that all cold compact objects (such as white dwarfs and neutron stars) will get in the situation where any increase in density will result in an increase in gravitational attraction that cannot be balanced by a corresponding increase in pressure, hence there is a limit to the mass of these objects. But this limit originates in relativistic theory, hence the purely Newtonian theories, such as hydrostatical equilibrium, does not provide a maximum mass.

When introducing general-relativistic corrections, the TOV-equations, they tend to "amplify" gravity. Thus giving a lower limit for the maximum mass than in the special relativistic case.

6.4 Neutron stars with cosmological constant

In this section, we will solve the TOV-equations numerically with a non-constant mass density. As in section 4.1, the definitions for the dimensionless mass, pressure and energy density are

$$\begin{aligned} m(r) &= \overline{M}(r)M_{\odot}, & (134)\\ p &= \epsilon_0 \bar{p}, & (135)\\ \epsilon &= \epsilon_0 \bar{\epsilon}. & (136) \end{aligned}$$

These equations, together with the equation for a polytropic star

$$\bar{p} = \bar{K}\bar{\epsilon}^{\gamma} \tag{137}$$

inserted into the TOV-equations, equations (5) and (42), yields the dimensionless equations

$$\begin{aligned} \frac{d\overline{M}}{dr} &= \beta r^2 \bar{p}^{\frac{1}{\gamma}} & (138)\\ \frac{d\bar{p}}{dr} &= -\frac{\alpha \overline{M}\bar{p}^{\frac{1}{\gamma}}}{r^2} \frac{\left[1 + \frac{R_0}{\alpha}\bar{p}^{\frac{\gamma-1}{\gamma}}\right]\left[1 + \frac{4\pi\epsilon_0\bar{p}r^3}{M_{\odot}c^2\overline{M}} - \frac{\Lambda r^3}{2R_0c^2\overline{M}}\right]}{\left[1 - \frac{2R_0\overline{M}}{r}\right]} & (139) \end{aligned}$$

with the constants defined as in the previous sections. The r-dependence of $\overline{M}$ and $\bar{p}$ is not written explicitly to simplify the notation slightly.

The values for α and β are the same as in the case with arbitrary relativity for neutron stars in section 6.2. These values were chosen as we also chose to use arbitrary relativity (as it is impossible to use an ultra-relativistc polytrope model for neutron stars, see section 5.2.2) to do the numerical integration.

We chose to use the values for the fitting constants A_{NR} and A_{R} with $x_{max} = 100$ and the initial value for the pressure $\bar{p}(0) = 0.01$. Equations (138) and (139) were solved using a matlab-programme listed in the appendix. The result is shown in Table 7

Table 7: Radius and mass for neutron star from TOV-equations with cosmological constant Λ. The values for the fitting constants A_{NR} and A_{R} are from the fitting with $x_{max} = 100$ and the initial pressure $\bar{p}(0) = 0.01$ which is in the relativistic regime where the cosmological constant should be of biggest importance.

Cosmological constant Λ (s^{-2})	Radius R (km)	Mass $\overline{M}$ ($M_\odot$)
0	24.559	2.0253
1	24.559	2.0253
10^{10}	24.559	2.0255
10^{11}	24.576	2.0270
10^{15}	17.216	8.2747

The resulting radii are exactly the same within our numerical precision for values of $0\mathrm{m}^{-2} \leq \Lambda \leq 10^{10}\mathrm{m}^{-2}$. The masses, however, change a little over this interval. The difference from the case with $\Lambda = 0\mathrm{m}^{-2}$ to the case with $\Lambda = 10^{10}\mathrm{m}^{-2}$ is only 0.008%. In Fig. 15 a plot of the dimensionless pressure and mass as a function of radius for a neutron star with $\Lambda = 10^{10}\mathrm{m}^{-2}$ is shown.

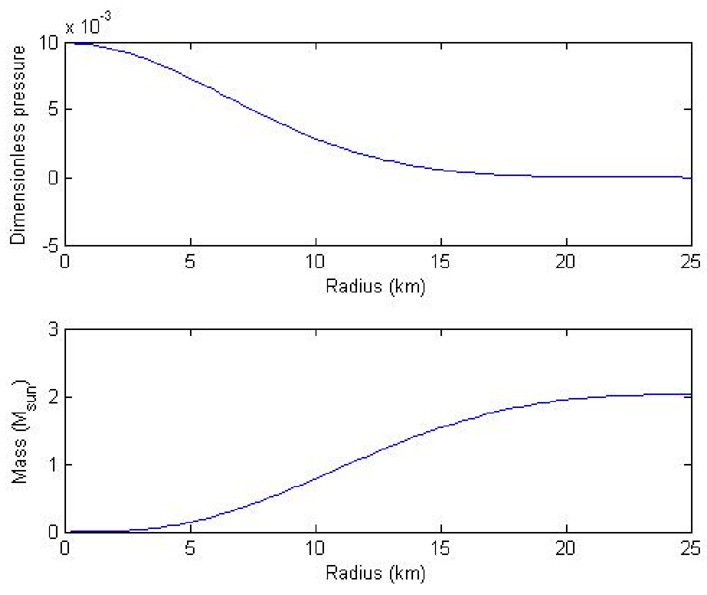


Figure 15: Dimensionless pressure and mass as a function of radius (in km) for a relativistic neutron star (TOV-equations) with cosmological constant $\Lambda \sim 10^{10}\mathrm{m}^{-2}$, i.e. $\rho_{vac} \sim 10^{36}\mathrm{kg/m}^3$.

The definition of Λ, equation (40), with reinstated units (i.e. $G \neq 0$ and $c \neq 0$) is

$$\Lambda = \frac{8\pi G}{3c^2}\rho_{\mathrm{vac}}, \tag{140}$$

with the cosmological constant Λ in units of m^{-2}. This causes the vacuum density to be

$$\rho_{\mathrm{vac}} = \frac{3c^2\Lambda}{8\pi G}. \tag{141}$$

For the difference in mass to be of only 0.008%, the cosmological constant had to be $\sim 10^{10}\mathrm{m}^{-2}$. Using equation (141), we find the corresponding vacuum density to be $\rho_{\mathrm{vac}} \sim 10^{36}\mathrm{kg/m}^3$. The value for the vacuum density is calculated to be of the order $\sim 10^{-27}\mathrm{kg/m}^3$ [Wesley 1997], i.e. the value for which our results only yields a difference in mass of 0.008% is a factor 10^{63} larger than the calculated value. From this we can conclude that the existence of a cosmological constant is not going to affect the radius nor the mass of neutron stars. This numerical calculation is more general than the result in section 3.2, as we do not have a constant density any more. This results in a more reliable model. Note that for the model with constant density, the effect of the cosmological constant Λ is of greater importance than for the non-constant density model.

7 Nuclear Interactions

In previous sections the masses and radii of compact stars have been calculated with the theory of an ideal gas leading to the Fermi gas equation of state. In this section, however, we will use a model that includes the nucleon-nucleon interactions to find the masses and radii of neutron stars. The effect of these interactions are negligible for white dwarfs, but must be taken into account for neutron stars as their large density ($\sim 10^8$kg/m^3 order larger than for white dwarfs) causes the nucleons to interact strongly.

The nucleon-nucleon interactions can be included in the equation of state by constructing a simple model for the nuclear potential that reproduces the general features of normal nuclear matter.

In this section, we will use natural units, where $c = 1$, and also MeV ($1.6 \cdot 10^{-13}$J) and fm (10^{-15}m) as the energy and distance units respectively. We will also neglect the mass difference between neutrons and protons, and label their masses as m_N.

The Bethe-Weizäcker mass formula is

$$BE = a_{\mathrm{V}} A + a_{\mathrm{S}} A^{\frac{2}{3}} + a_{\mathrm{C}} \frac{Z(Z-1)}{A^{\frac{1}{3}}} + a_{\mathrm{A}} \frac{(A-2Z)^2}{A} + \delta(A, Z), \tag{142}$$

[Krane 1988], where BE is the binding energy per nucleon, A the number of nucleons, the different a's are constants, and $\delta(A, Z)$ is a function defined below.

The Bethe-Weizäcker equation is semi-empirical, i.e. it is partly based on theory and partly on empirical measurements. The theory suggests a form the mass should take, and experiment provides the coefficients. The different terms have different origins; the first term, $a_{\mathrm{V}} A$, is the volume term, proportional to the number of nucleons A, i.e. proportional to the volume as $A \propto r^3$, where r is the radius of the nucleus. The second term, $a_{\mathrm{S}} A^{\frac{2}{3}}$, is the surface term, proportional to $A^{\frac{2}{3}} \propto r^2$, i.e. proportional to the surface of the nucleus. The third term, $a_{\mathrm{C}} \frac{Z(Z-1)}{A^{\frac{1}{3}}}$, is the Coulomb term and it describes the electrostatic repulsion between protons. It is proportional to $Z(Z-1)$ as there has to be more than one proton for the repulsion to exist, and the proportionality to $\frac{1}{A^{\frac{1}{3}}} \propto \frac{1}{r}$ is the radial dependence of the Coulomb potential for a point charge.

The fourth term, $a_\mathrm{A}\frac{(A-2Z)^2}{A}$, is the asymmetry term and describes that the Pauli exclusion principle leads to a higher energy for nuclei with more neutrons than protons as the neutrons then have to occupy higher energy states than they would have to if there were equally many neutrons and protons.

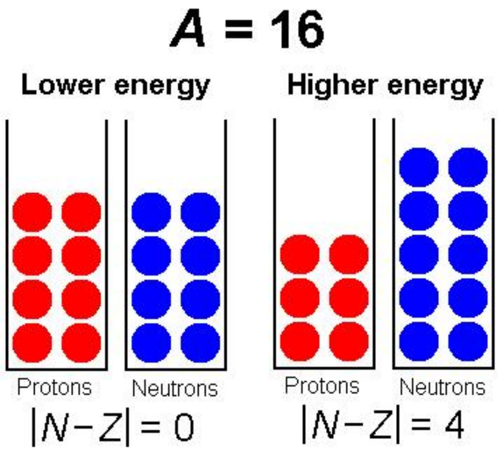


Figure 16: If there are more neutrons than protons in a nucleus, some of the neutrons will be higher in energy than the available states in the proton pool. If we could move some particles from the neutron pool to the proton pool, in other words change some neutrons into protons, we would decrease the energy.[9]

The fifth term, $\delta(A,Z)$, is the pairing term which reflects the pairwise nature of the nucleon-nucleon interaction. When there is an even number of both protons and neutrons, the binding energy gets a positive contribution, for an odd number of both protons and neutrons, the contribution is negative and for one odd and one even number of protons and neutrons, there is no contribution. The term can be written as

$$\delta(A,Z) = \begin{cases} +\delta_0 & Z, N \text{ even } (A \text{ even}) \\ 0 & A \text{ odd} \\ -\delta_0 & Z, N \text{ odd } (A \text{ even}) \end{cases} \tag{143}$$

where

$$\delta_0 = \frac{a_\mathrm{P}}{A^{\frac{1}{2}}}, \tag{144}$$

and a_P is a constant.

For normal symmetric ($N = Z$) nuclear matter, equation (142) gives an equilibrium number density of $n_0 = 0.16\mathrm{fm}^{-3}$. Equation (116) gives the corresponding value $k_{\mathrm{F}}^0 = 263\mathrm{MeV/c}$ for the Fermi momentum. The energy corresponding to the rest mass of nucleons is $m_N = 939\mathrm{MeV/c^2}$, hence normal nuclear matter can be treated nonrelativistically. At the equilibrium number density, the average binding energy per nucleon is $BE = -16\mathrm{MeV}$.

We want our model to reproduce the equilibrium density and the binding energy per nucleon. In addition we also want it to respect the nuclear compressibility K_0 as defined below, with a magnitude between 200 to 400MeV. Finally, the model should also allow for the asymmetry term, the fourth term in the Bethe-Weizäcker mass formula, equation (142), which when $Z = 0$ contributes with about 30MeV above the symmetric matter minimum of n_0.

7.1 Symmetric nuclear matter

We choose to consider the case when the nuclear matter is symmetric, i.e. $N = Z$. This is not correct for neutron stars, but the case with nonsymmetric nuclear matter is dealt with in the next section. We will here obtain an equation of state for nuclear matter when the proton and neutron number densities are equal, $n_n = n_p$. The total number density is $n = n_n + n_p$. We ignore the electrons present, since their contribution is small as we saw in section 6.1.1.

We want to relate n_0, BE and K_0 to the energy density for symmetric nuclear matter $\epsilon(n)$, where $n = n(k_{\mathrm{F}})$ is the nuclear density. The energy density will include the nuclear potential, $V(n)$, which we will model in terms of two simple functions with three parameters that are fitted to reproduce the values of n_0, BE and K_0.

The average energy per nucleon, $\frac{E}{A}$, for symmetric nuclear matter is related to the energy density by

$$\frac{E(n)}{A} = \frac{\epsilon(n)}{n}, \tag{145}$$

where the rest mass energy, m_N, is included. This equation has units of MeV. $\frac{E(n)}{A} - m_N$ has a minimum at $n = n_0$, with a depth $BE = -16\mathrm{MeV}$. This can be written as

$$\frac{d}{dn}\left(\frac{E(n)}{A}\right) = \frac{d}{dn}\left(\frac{\epsilon(n)}{n}\right) = 0 \tag{146}$$

at $n = n_0$. This equation is one constraint on the parameters of $V(n)$, another is the binding energy:

$$\frac{\epsilon(n)}{n} - m_N = BE \tag{147}$$

at $n = n_0$. The curvature of the curve is related to the nuclear compressibility by [Silbar and Reddy 2004]

$$K(n) = 9\frac{dp(n)}{dn}. \tag{148}$$

Inserting equation (122) into equation (148) gives

$$K(n) = 9\left[n^2\frac{d^2}{dn^2}\left(\frac{\epsilon}{n}\right) + 2n\frac{d}{dn}\left(\frac{\epsilon}{n}\right)\right], \tag{149}$$

where $K(n_0) = K_0$. The model for $\epsilon(n)$ we will use is

$$\frac{\epsilon(n)}{n} = m_N + \frac{3}{5}\frac{k_F^2}{2m_N} + \frac{A}{2}u + \frac{B}{\sigma+1}u^\sigma, \tag{150}$$

[Silbar and Reddy 2004], where $u = \frac{n}{n_0}$ and σ are dimensionless and A and B have units of MeV. The first term of this equation represents the rest mass energy and the second term the average kinetic energy per nucleon in the nonrelativistic case. At the equilibrium number density, $n = n_0$, we will denote the kinetic energy by $\langle E_F^0 \rangle$, which evaluates to 22.1MeV [Silbar and Reddy 2004]. Then, the second term can be written as $\langle E_F^0 \rangle u^{\frac{2}{3}}$. The first term, the mass, is clearly proportional to the number of nucleons A and the second term is proportional to $u^{\frac{2}{3}} \propto n^{\frac{2}{3}} \propto A^{\frac{2}{3}}$. So, these terms are easily recognized as the two first terms in the Bethe-Weizäcker equation, equation (142). The two last terms of equation (150) don't follow easily from equation (142). They are of a relativistic origin, and dominate for large values of k_{F} (i.e. for large $u = \frac{n}{n_0}$).

Inserting equation (150) into equation (146) gives the first equation for the parameters A, B and σ

$$\begin{aligned} \frac{2}{3}\frac{\langle E_F^0 \rangle}{n_0}u^{-\frac{1}{3}} + \frac{A}{2n_0} + \frac{B\sigma}{n_0(\sigma+1)}u^{\sigma-1} &= 0 \Rightarrow \\ \frac{2}{3}\langle E_F^0 \rangle + \frac{A}{2} + \frac{B\sigma}{\sigma+1} &= 0, \end{aligned} \tag{151}$$

as $u = 1$ at $n = n_0$. The second equation for the parameters comes from combining equation (150) with equation (147):

$$\begin{aligned} m_N + \langle E_F^0 \rangle u^{\frac{2}{3}} + \frac{A}{2}u + \frac{B}{\sigma+1}u^{\sigma} - m_N &= BE \Rightarrow \\ \langle E_F^0 \rangle + \frac{A}{2} + \frac{B}{\sigma+1} &= BE. \end{aligned} \tag{152}$$

Combining equation (150) and (149) gives the third equation for the parameters

$$\begin{aligned} \frac{K_0}{9} &= n^2 \left[\frac{B\sigma(\sigma-1)}{n_0(\sigma+1)} u^{\sigma-2} - \frac{2}{9} \frac{\langle E_F^0 \rangle}{n_0^2} u^{-\frac{4}{3}} \right] \\ &+ \frac{4}{3}\langle E_F^0 \rangle + A + \frac{2B\sigma}{(\sigma+1)} - \frac{2}{9}\langle E_F^0 \rangle + \frac{B\sigma(\sigma-1)}{(\sigma+1)} \\ &= \frac{10}{9}\langle E_F^0 \rangle + A + \frac{2B\sigma + B\sigma(\sigma-1)}{(\sigma+1)} \\ &= \frac{10}{9}\langle E_F^0 \rangle + A + B\sigma. \end{aligned} \tag{153}$$

Combining equation (151) and (152) yields

$$\begin{aligned} BE - \langle E_F^0 \rangle - \frac{B}{\sigma+1} &= -\frac{2}{3}\langle E_F^0 \rangle - \frac{B\sigma}{\sigma+1} \Rightarrow \\ B &= \frac{\sigma+1}{\sigma-1}\left(\frac{\langle E_F^0 \rangle}{3} - BE \right). \end{aligned} \tag{154}$$

Solving for the remaining constants in a similar way gives

$$\sigma = \frac{K_0 + 2\langle E_F^0 \rangle}{3\langle E_F^0 \rangle - 9BE} \tag{155}$$

$$A = BE - \frac{5}{3}\langle E_F^0 \rangle - B. \tag{156}$$

The values for the three constants were calculated by a matlab programme, listed in the appendix, and the results are shown in Table 8.

Table 8: A, B and σ for 200MeV< K_0 <400MeV

K_0	A	B	σ
200	-366.1	313.3	1.161
250	-193.3	140.5	1.399
300	-149.6	96.76	1.637
350	-129.6	76.81	1.874
400	-118.2	65.38	2.112

Here, $\sigma > 1$, which violates the basic principle of causality, and this will be dealt with in section 7.3.1. Using the values for the parameters obtained from $K_0 = 400$MeV in equation (150), we can make a plot of $\frac{E}{A} - m_N$. The result is shown in Fig. 17.

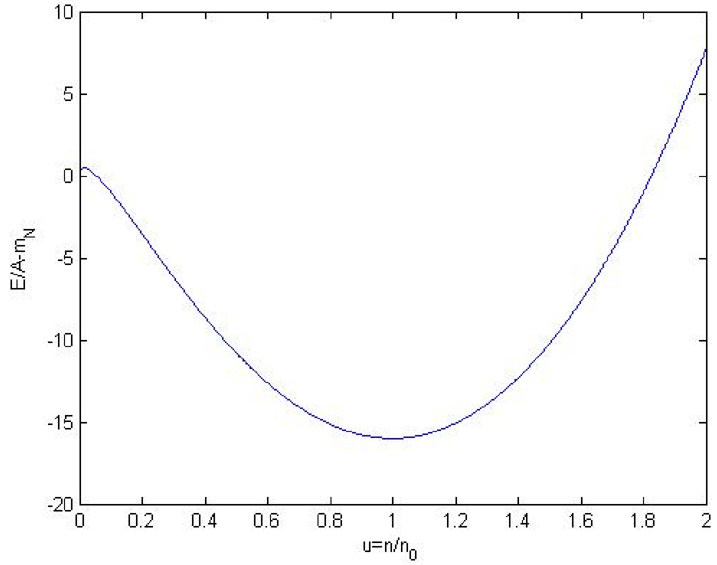


Figure 17: The average energy without the rest mass, as a function of $u = \frac{n}{n_0}$. The minimum is located at $n = n_0 = 0.16\text{fm}^{-3}$ with the depth being $BE = E/A - m_N - 16$MeV and the curvature corresponding to the nuclear compressibility $K_0 = 400$MeV.

We observe that in Fig. 17, the minimum of the binding energy is found at $u = 1$, i.e. $n = n_0$ and this occurs at -16MeV. So, the binding energy at the equilibrium number density is -16MeV, which is what we wanted. The curvature corresponds to $K_0 = 400$MeV, but this is not so easily seen from the figure. Another observation from this figure, is the little "bump" at small values of u, where the binding energy is positive. This occurs because at small enough densities, the nucleons do not form nuclei, hence their binding energy must be positive.

As we have an expression for $\epsilon(n)$ from equation (150), we can find the pressure using equation (122).

$$p(n) = n_0 \left[\frac{2}{3} \langle E_F^0 \rangle u^{\frac{5}{3}} + \frac{A}{2} u^2 + \frac{B\sigma}{\sigma + 1} u^{\sigma+1} \right]. \tag{157}$$

For the parameters when $K_0 = 400\text{MeV}$, the dependence of $p(n)$ on n is shown in Fig. 18.

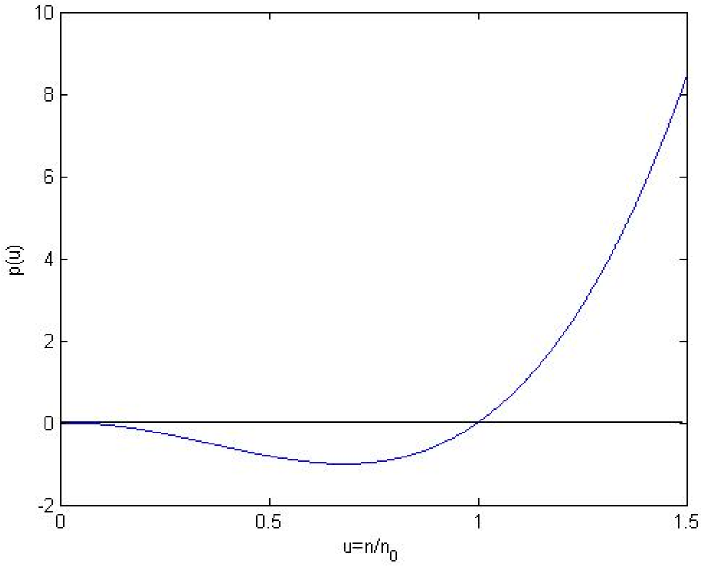


Figure 18: The pressure of symmetric nuclear matter as a function of $u = \frac{n}{n_0}$.

Fig. 18 shows that $p(u = 1) = 0$, i.e. the pressure is zero at the equilibrium number density. This is because when the nucleon number density reaches its equilibrium point, there is no pressure acting to change the number density as this point is the minimum of n. The pressure is also negative for $n < n_0$ for a similar reason, namely the pressure is acting inwards to increase the number density to reach the minimum (the equilibrium point $n = n_0$), whereas positive pressure is defined as acting outwards. Negative pressure is also a characteristic of the cosmological constant caused by dark energy in the universe.

7.2 Mass and radius for neutron star with symmetric nuclear matter

In this section, we will use the energy density, equation (150) and the pressure, equation (157) to find the constants A_{NR} and A_{R} as done in previous sections for an ideal Fermi gas. As the energy density and the pressure is parametrized by the variable k_{F} when equation (116) is inserted. We then have the equations:

$$\epsilon = \frac{k_{\mathrm{F}}^3}{(d\hbar c)^2}\left[m_N + \frac{3k_{\mathrm{F}}^2}{10m_N} + \frac{Ak_{\mathrm{F}}^3}{2n_0 d} + \frac{B}{\sigma+1}\left(\frac{k_{\mathrm{F}}^3}{n_0 d}\right)^{\sigma}\right], \tag{158}$$

$$p = n_0\hbar c\left[\frac{2}{3}\langle E_F^0\rangle\left(\frac{k_{\mathrm{F}}^3}{n_0 d}\right)^{\frac{5}{3}} + \frac{A}{2}\left(\frac{k_{\mathrm{F}}^3}{n_0 d}\right)^{2} + \frac{B\sigma}{\sigma+1}\left(\frac{k_{\mathrm{F}}^3}{n_0 d}\right)^{\sigma+1}\right], \tag{159}$$

where $d = 3\pi^2\hbar^3$ and inserted factors of $\hbar c = 197.3\mathrm{MeV}$, to get the units right, thus causing the energy density and pressure to have units $\mathrm{MeV/fm^3}$, i.e. $1.6\cdot 10^{11}\mathrm{J/m^3}$.

Equations (158) and (159) give the pressure as a function of energy density shown in Fig. 19 for $K_0 = 200\mathrm{MeV}$.

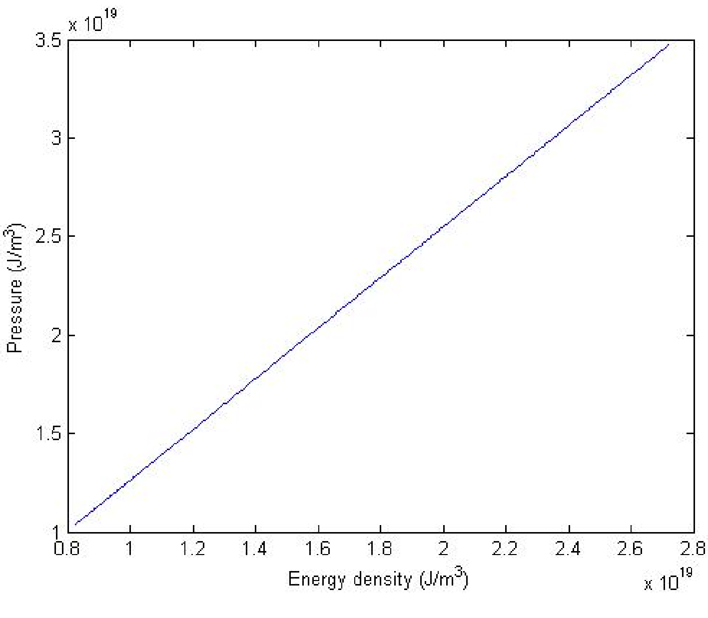


Figure 19: The energy density as a function of pressure for $K_0 = 200\mathrm{MeV}$ with $1.68m_N c < k_{\mathrm{F}} < 2m_N c$ as the plot is only valid for $n > n_0$.

The plot in Fig. 19 is only valid for $n > n_0$, as the pressure below this value is negative (i.e. acting inwards). At the values plotted in Fig. 19, we have $p \approx \epsilon/3$, as it should for an ultra-relativistic nucleon gas.

Using the matlab-programme for fitting the constants A_{NR} and A_{R} yields the results shown in Table 9.

Table 9: The constants A_{NR} and A_{R} for 200MeV< K_0 <400MeV with $x_{max} = 1000$.

K_0	A_{NR}	A_{R}
200	3.3744	0.85788
250	1.0392	0.71437
300	1.1318	0.61086
350	2.2592	0.53362
400	1.1038	0.47348

The values from Table 9 used in the matlab-programme for performing the numerical integration yields the results shown in Table 10.

Table 10: The radius (in km) and mass (in $M_{\odot}$) of a neutron star with general relativity (TOV-equation) and pressure $\bar{p}(0) = 0.01$ for 200MeV< K_0 <400MeV.

K_0	Radius R	Mass $\overline{M}$
200	10.169	0.38327
250	27.319	3.12905
300	25.708	2.79470
350	14.653	0.86426
400	26.273	2.95735

Fig. 20 shows the pressure and mass as a function of radius for a neutron star using the TOV-equations.

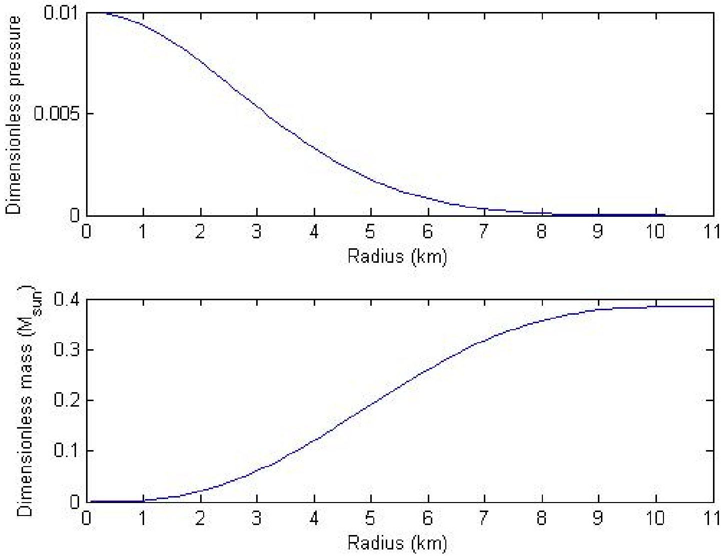


Figure 20: The dimensionless mass and pressure as a function of radius for a neutron star using general relativity (TOV-equation) with $K_0 = 200\text{MeV}$ and $\bar{p}(0) = 0.01$.

This result will not be emphasized any more as we are more interested in the case with $N \ll Z$, i.e. much more neutrons than protons. This case is treated in the next section.

7.3 Nonsymmetric nuclear matter

We will now find the equation of state for nonsymmetric nuclear matter. This is done by introducing a parameter α defined by

$$n_n = \frac{1+\alpha}{2}n, \tag{160}$$

$$n_p = \frac{1-\alpha}{2}n. \tag{161}$$

The constant α in equations (160)-(161) should not be confused with the constant defined by equation (101). For matter consisting of only neutrons, we have α=1 (pure neutron matter). Combining equations (160) and (161) yields

$$\alpha = \frac{n_n - n_p}{n} = \frac{N-Z}{A}, \tag{162}$$

where N is the number of neutrons, Z is the number of protons and A is the number of nucleons (i.e. the sum of neutrons and protons). The fourth (asymmetry) term in the Bethe-Weizäcker mass formula, equation (142), can be rewritten by using the definition $A = Z + N$ and equation (162):

$$a_{\mathrm{A}}\frac{(A-2Z)^2}{A} = a_{\mathrm{A}}\frac{(N-Z)^2}{A} = Constant \cdot \alpha^2, \tag{163}$$

i.e. the symmetry-breaking interaction is proportional to α^2.

We will now consider the difference in energy density between the symmetric case discussed above (with $\alpha = 0$) and the nonsymmetric case. The kinetic energy term (the second term) in equation (150) has contributions from both protons and neutrons for the nonsymmetric case.

$$\begin{aligned}\epsilon_{\mathrm{KE}}(n,\alpha) &= \frac{3}{5}\frac{k^2_{\mathrm{F,n}}}{2m_N}n_n + \frac{3}{5}\frac{k^2_{\mathrm{F,p}}}{2m_N}n_p \\ &= \langle E_{F,n}\rangle u^{\frac{2}{3}} n_n + \langle E_{F,p}\rangle u^{\frac{2}{3}} n_p \end{aligned} \tag{164}$$

$\langle E_{F,n}\rangle$ and $\langle E_{F,p}\rangle$ is the kinetic energy for symmetric nuclear matter corresponding to $k_{\mathrm{F,n}}$ and $k_{\mathrm{F,p}}$ respectively.

$$\langle E_F\rangle = \frac{3}{5}\frac{\hbar^2}{2m_N}\left(\frac{3\pi^2 n}{2}\right)^{\frac{2}{3}}. \tag{165}$$

From equation (150) at $n = n_0$, we have $\langle E_F \rangle = \langle E_F^0 \rangle$.

Inserting equation (165) into equation (164) yields

$$\epsilon_{KE}(n, \alpha) = \frac{2^{\frac{2}{3}} \langle E_F \rangle}{n^{\frac{2}{3}}} \left(n_n^{\frac{5}{3}} + n_p^{\frac{5}{3}} \right). \tag{166}$$

Then inserting equations (160) and (161) gives

$$\epsilon_{KE}(n, \alpha) = \frac{n \langle E_F \rangle}{2} \left[(1 + \alpha)^{\frac{5}{3}} + (1 - \alpha)^{\frac{5}{3}} \right]. \tag{167}$$

The excess kinetic energy for nonsymmetric nuclear matter ($\alpha \neq 0$) is

$$\Delta\epsilon_{KE}(n, \alpha) = \epsilon_{KE}(n, 0) - \epsilon_{KE}(n, \alpha). \tag{168}$$

Inserting equation (167) gives

$$\Delta\epsilon_{KE}(n, \alpha) = n \langle E_F \rangle \left(\frac{1}{2} \left[(1 + \alpha)^{\frac{5}{3}} + (1 - \alpha)^{\frac{5}{3}} \right] - 1 \right). \tag{169}$$

For pure neutron matter ($\alpha = 1$) this reduces to

$$\Delta\epsilon_{KE}(n, \alpha = 1) = n \langle E_F \rangle \left(2^{\frac{2}{3}} - 1 \right). \tag{170}$$

Using the binomial expansion $(1 + x)^y = \sum_{n=0}^{\infty} \binom{y}{n} x^n$ to the second order of α on equation (169) yields

$$\begin{aligned} \Delta\epsilon_{KE}(n, \alpha) &\approx n \langle E_F \rangle \left(\frac{1}{2} \left[1 + \frac{5}{3}\alpha + \frac{5}{9}\alpha^2 + 1 - \frac{5}{3}\alpha + \frac{5}{9}\alpha^2 \right] - 1 \right) \\ &\approx n \langle E_F \rangle \frac{5}{9} \alpha^2. \end{aligned} \tag{171}$$

Inserting the well-known dependence of the Fermi energy on the mean value of energy, $\langle E_F \rangle = \frac{5}{3} E_F$, gives

$$\Delta\epsilon_{KE}(n, \alpha) \approx n E_F \frac{\alpha^2}{3}. \tag{172}$$

Keeping terms of order α^2 will be good enough for our purposes. The energy per particle $(\frac{\epsilon}{n})$ for pure neutron matter at normal density is $\frac{\Delta\epsilon_{KE}(n_0, 1)}{n_0} \approx$ 13MeV, and this is more than a third of the total bulk symmetry energy of 30MeV. The potential energy contribution thus is approximately 20MeV. Assuming that the approximation of α^2 is sufficient for this potential energy, we can write the total energy per particle as

$$\frac{E(n,\alpha)}{A} = \frac{E(n,0)}{A} + \frac{\alpha^2 S(n)}{A}. \tag{173}$$

As the symmetry breaking is proportional to α^2, we have a reflection of the pairwise nature of the nuclear interactions.

We assume that $S(u)$, $u = \frac{n}{n_0}$, is of the form

$$S(u) = (2^{\frac{2}{3}} - 1)\langle E_F^0 \rangle (u^{\frac{2}{3}} - F(u)) + S_0 F(u). \tag{174}$$

$S_0 = 30$MeV is the bulk symmetry energy parameter. As $S(u)$ has to satisfy $S(u=1) = S_0$, the function $F(u)$ has to satisfy $F(u=1) = 1$, and $S(u=0) = 0$ causing $F(u=0) = 0$. Besides these two constraints, the function $F(u)$ can be chosen arbitrarily. We choose the simple form

$$F(u) = u^{\gamma}, \tag{175}$$

where γ is a constant $0 < \gamma \leq 2$, not to be confused with the constant in the polytropic equation, equation (96). Fig. 21 shows the energy per particle for pure neutron matter ($\alpha = 1$) as a function of u for $K_0 = 400$MeV and $S_0 = 30$MeV.

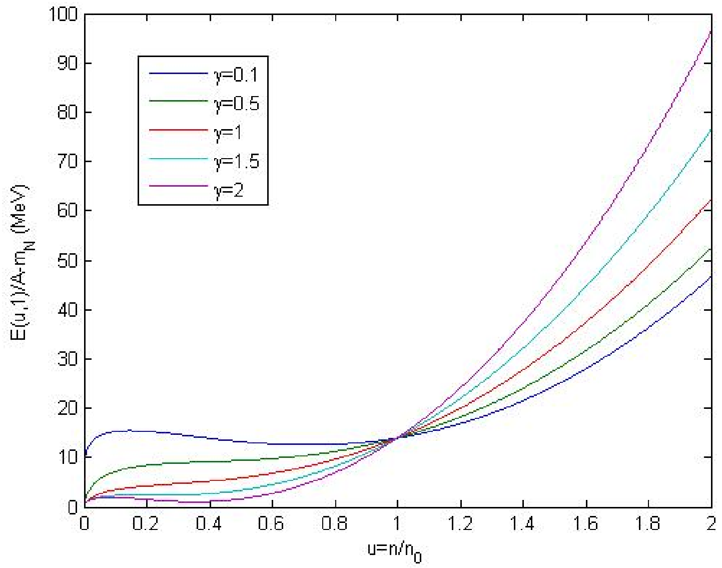


Figure 21: Energy per particle less its rest mass for pure neutron matter as a function of u with $K_0 = 400$MeV and $S_0 = 30$MeV for different values of γ.

As opposed to the plot in Fig. 17 with $\alpha = 0$, $E(u, 1) \geq 0$ and is monotonically increasing. The plots for $\alpha \geq 1$ looks almost quadratic in u. At large values of u, the dominating term is u^σ, with $\sigma = 2.112$ (for the case with $K_0 = 400$MeV). We might, however, expect a linear increase instead. But this will be discussed further in the next section.

Given the kinetic term of the energy density, equation (167), inserted into the total energy density, equation (150) gives

$$\frac{\epsilon(n,\alpha)}{n} = m_N + \frac{\langle E_F \rangle}{2}\left[(1+\alpha)^{\frac{5}{3}} + (1-\alpha)^{\frac{5}{3}}\right] + \frac{A}{2}u + \frac{B}{\sigma+1}u^\sigma \qquad (176)$$

with $\langle E_F \rangle$ defined as the kinetic energy for symmetric nuclear matter corresponding to k_{F}.

Equation (122) becomes

$$\begin{aligned} p &= n^2 \frac{d(\frac{\epsilon}{n})}{dn} \\ &= n\frac{d\epsilon}{dn} - \epsilon. \end{aligned} \qquad (177)$$

Inserting the definition $u = \frac{n}{n_0}$ gives

$$p = u\frac{d\epsilon}{du} - \epsilon. \tag{178}$$

Using $\frac{E}{A} = \frac{\epsilon}{n}$, equations (173)-(175), we can write

$$\epsilon(u,\alpha) = \epsilon(u,0) + \alpha^2 n_0 \left[(2^{\frac{2}{3}} - 1)\langle E_F^0\rangle(u^{\frac{5}{3}} - u^{\gamma+1}) + S_0 u^{\gamma+1}\right], \tag{179}$$

where $\epsilon(u,0)$ is defined in equation (150). Inserting equation (179) into equation (178) yields

$$\begin{aligned}
p &= u\frac{d}{du}\{\epsilon(u,0)\} - \epsilon(u,0) \\
&+ \alpha^2 u n_0 \left[\left(2^{\frac{2}{3}} - 1\right)\langle E_F^0\rangle \left\{\frac{5}{3}u^{\frac{2}{3}} - (\gamma+1)u^{\gamma}\right\} + S_0(\gamma+1)u^{\gamma}\right] \\
&- \alpha^2 n_0 \left[\left(2^{\frac{2}{3}} - 1\right)\langle E_F^0\rangle \left\{u^{\frac{5}{3}} - u^{\gamma+1}\right\} + S_0 u^{\gamma+1}\right] \\
&= u\frac{d}{du}\{\epsilon(u,0)\} - \epsilon(u,0) \\
&+ n_0\alpha^2 \left[\frac{2^{\frac{2}{3}} - 1}{3}\langle E_F^0\rangle \left(2u^{\frac{5}{3}} - 3\gamma u^{\gamma+1}\right) + S_0\gamma u^{\gamma+1}\right].
\end{aligned} \tag{180}$$

$p(u,0) = u\frac{d}{du}\{\epsilon(u,0)\} - \epsilon(u,0)$ is defined by equation (157). This causes equation (180) to become

$$p = p(u,0) + n_0\alpha^2 \left[\frac{2^{\frac{2}{3}} - 1}{3}\langle E_F^0\rangle \left(2u^{\frac{5}{3}} - 3\gamma u^{\gamma+1}\right) + S_0\gamma u^{\gamma+1}\right]. \tag{181}$$

Fig. 22 shows the energy density ϵ, equation (179), and pressure p, equation (181), as a function of $u = \frac{n}{n_0}$.

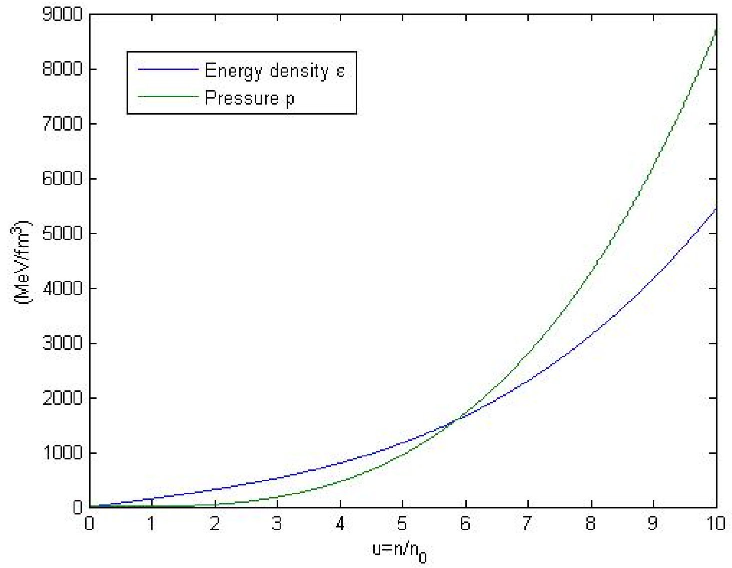


Figure 22: Energy density and pressure for pure neutron matter ($\alpha = 1$) as a function of u with $K_0 = 400$MeV and $S_0 = 30$MeV for $\gamma = 1$.

The plot in Fig. 22 ranges from 0 to 10 times normal nuclear density. Both the energy density and pressure increase monotonically, but the pressure becomes bigger than the energy density at approximately $u = 6$. This isn't consistent with the ultra-relativistic (UR) nucleon gas where $p = \frac{\epsilon}{3}$. The reason why our model differs from the result from UR nucleon gas is because we assumed that the neutrons are never relativistic because of their large mass.

Now, we want to find the equation of state for nonsymmetric nuclear matter to be able to solve the TOV-equations to find the radius and mass of the neutron star. Fig. 23 shows this calculated dependence (the blue points). The pressure is smooth, non-negative and monotonically increasing as a function of energy density. It almost looks like a quadratic dependence, suggesting that we can try to fit it with a simple polytropic equation. Hence the red line is the best-fit line for the function

$$p(\epsilon) = \kappa_0 \epsilon^{\Gamma} \tag{182}$$

where κ_0 and Γ are constants. These are found with the matlab-programme for fitting an arbitrary function, listed in the appendix.

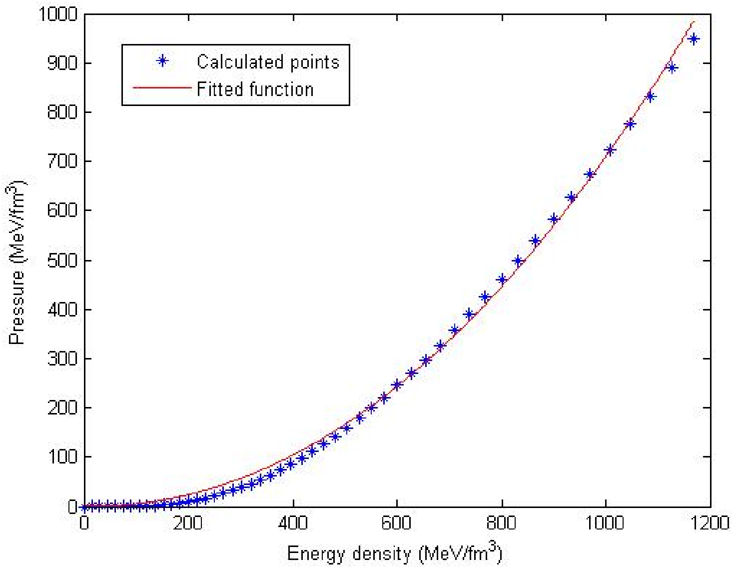


Figure 23: Pressure as a function of energy density (equation of state) for pure neutron matter ($\alpha = 1$) with $K_0 = 400$MeV and $S_0 = 30$MeV for $\gamma = 1$.

In the article [Silbar and Reddy 2004], the constant Γ isn't fitted to the data points. A value of $\Gamma = 2.1$ is used (determined by "we simply guessed" [Silbar and Reddy 2004]), and only κ_0 is fitted. Whereas we fit both constants. The result for the constants in Fig. 23 (with 10,000 data points) is

$\kappa_0 = 3.235 \cdot 10^{-4}$ and $\Gamma = 2.114$,

where κ_0 has appropriate units so that p and ϵ are in MeV/fm^3. The values for κ_0 and Γ with varying the parameters K_0, α and γ is shown in Table 11.

Table 11: κ_0 and Γ for $200MeV \leq K_0 \leq 400MeV$, $0.5 \leq \alpha \leq 1$ and $0 < \gamma \leq 1$.

K_0 (MeV)	α	γ	κ_0 ($\cdot 10^{-4}$)	Γ
200	0.5	0.1	0.1518	2.430
200	0.5	0.5	0.1862	2.390
200	0.5	1	0.3583	2.272
200	1	0.1	0.9874	2.189
200	1	0.5	1.107	2.167
200	1	1	1.619	2.098
400	0.5	0.1	1.253	2.257
400	0.5	0.5	1.325	2.243
400	0.5	1	1.751	2.187
400	1	0.1	2.627	2.158
400	1	0.5	2.717	2.149
400	1	1	3.235	2.114

The results in Table 11 are not consistent with causality, as the parameter $\sigma \geq 1$. Therefore, we will not use these results further to find mass and radius, but rather find an equation of state which is consistent with causality.

7.3.1 Equation of state consistent with causality

For our model to be consistent with special relativity, causality cannot be violated, i.e. nothing can go faster than the speed of sound. From the elementary formula for the square of the speed of sound c_s in terms of the bulk modulus, we can show that

$$\left(\frac{c_s}{c}\right)^2 = \frac{dp}{d\epsilon} = \frac{dp/dn}{d\epsilon/dn} \tag{183}$$

[Silbar and Reddy 2004]. When the density becomes sufficiently large $u \to \infty$, the speed of sound c_s exceeds that of light, and violates causality. For our model of nuclear interactions, the dominant terms of ϵ and p is of the order $u^{\sigma+1}$. From equation (179), we get

$$\lim_{u\to\infty} \frac{d\epsilon}{dp} = Bu^{\sigma} \tag{184}$$

when $\gamma \leq 1$. From equation (181), we correspondingly get

$$\lim_{u\to\infty} \frac{dp}{dn} = B\sigma u^{\sigma} \tag{185}$$

also for $\gamma \leq 1$, which we will assume for the remainder of this section. Equations (184) and (185) inserted into equation (183) yields

$$\lim_{u \to \infty} \left(\frac{c_s}{c} \right)^2 = \sigma. \tag{186}$$

From Table 8, we observe that σ is greater than one for all 200MeV$\leq K_0 \leq$400MeV, i.e. causality is violated. To respect causality, we must assure that both $\epsilon(u)$ and $p(u)$ grow no faster than u^2 (or generally $u^{\gamma+1}$). A way of obtaining this, is to introduce a fourth parameter C so that for symmetric nuclear matter ($\alpha = 0$), we have

$$V_{\mathrm{nuc}} = \frac{A}{2} + \frac{B}{\sigma + 1} \frac{u^\sigma}{1 + C u^{\sigma - 1}}. \tag{187}$$

The parameters A and B are not independent of each other in the nuclear potential, so the constraint equations for A, B and σ, equations (154)-(156), will be more complicated. We can choose C small enough so that the effect on the denominator will be non-negligible only for very large values of u. As we can choose a small C freely, the values of the other parameters should not change much. These values can be fitted to incorporate the effects of the introduction of the extra parameter C by just trial and error. Thus readjusting A, B and σ values to put the minimum $E/A - m_N$ at the right position (n_0) and depth (BE), and calculating the corresponding value of the compressibility K_0.

We choose the value $C = 0.2$, and the parameter values for $K_0 = 400$MeV in Table 8 for further calculations. By trying only with B and σ, we could fit n_0 and BE with only small changes

$B = 65.38 \to 73.23$MeV
$\sigma = 2.112 \to 1.937$.

These new values of B and σ causing A and K_0 to become

$A = -118.2 \to -126.1$MeV
$K_0 = 400 \to 363.2$MeV.

The value of the nuclear compressibility K_0 is within 200MeV$\leq K_0 \leq$400MeV, so we still have a reasonable nuclear model.

As in the previous section, we can find a fit to equation (182), and the resulting constants are

$\kappa_0 = 2.2706 \cdot 10^{-4}$
$\Gamma = 2.1473$

for pure neutron matter ($\alpha = 1$). This result is quite similar to the corresponding result in Table 11 with $K_0 = 400$MeV, $\alpha = 1$ and $\gamma = 1$. To solve the TOV-equations, it is more useful to express the energy density ϵ as a function of the pressure p

$$\epsilon(p) = \left(\frac{p}{\kappa_0}\right)^{\frac{1}{\Gamma}}. \tag{188}$$

7.3.2 Radius and mass of neutron star with pure neutron matter

Before we perform the numerical integration of the TOV-equations, it is useful to get equation (188) in terms of the dimensionless $\bar{p}$ and $\bar{\epsilon}$. This is done by going from the units MeV/fm^3 to J/m^3, then to M$_\odot$/km^3, and finally to $\bar{p}$ and $\bar{\epsilon}$. Inserting equations (94) and (95) into (188) yields

$$\begin{aligned} \bar{\epsilon}(p) &= \left(\frac{\epsilon_0^{\frac{1}{\Gamma}-1}}{\kappa_0^{\frac{1}{\Gamma}}}\right)\bar{p}^{\frac{1}{\Gamma}} \\ &= A_0\bar{p}^{\frac{1}{\Gamma}}. \end{aligned} \tag{189}$$

Now, we have defined

$$\epsilon_0 = \frac{m_N^4 c^5}{3\pi^2\hbar^3} = 3404\text{MeV/fm}^3. \tag{190}$$

This causes A_0 from equation (189) to become

$A_0 = 0.6454.$

The α from equation (101) is now $\alpha = A_0 R_0 = 0.9526$km, causing β from equation (105) to become $\beta = 0.02490/\text{km}^3$. By substituting equation (189) into the TOV-equations, we can solve the TOV-equations numerically as before. The result is shown in Table 12.

Table 12: The radius (in km) and mass (in $M_\odot$) for different values of initial pressure $\bar{p}(0)$.

Pressure $\bar{p}(0)$	Radius R (km)	Mass $\bar{M}$ ($M_\odot$)
10^{-1}	16.17	4.935
10^{-2}	25.78	4.711
10^{-3}	29.74	2.131
10^{-4}	29.51	0.680
10^{-5}	27.86	0.193

Fig. 24 shows the dimensionless pressure and mass as a function of radius for the case with $\bar{p}(0) = 0.01$.

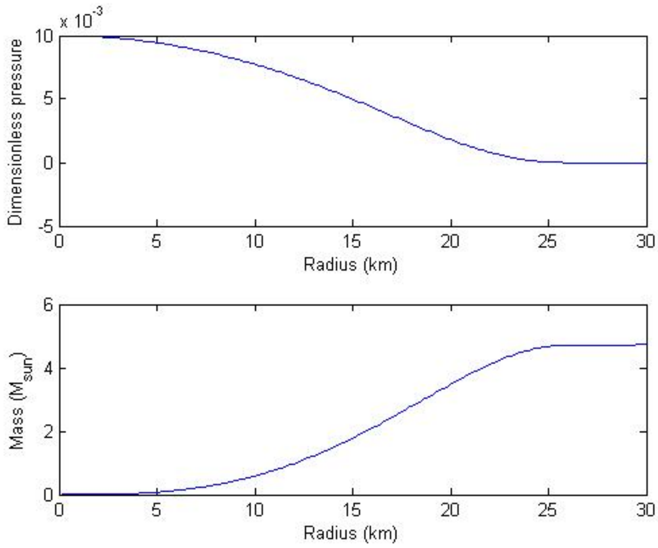


Figure 24: Dimensionless pressure and mass (in $M_\odot$) as a function of radius (in km) with initial pressure $\bar{p}(0) = 0.01$.

The plot in Fig. 24 can be compared to Fig. 12 and Fig. 13. We observe that the left hand side of both figures are most similar to Fig. 24, and the plots considered are the ones using the TOV-equations, as this is what we have done in this section as well and it is a more accurate model incorporating general relativity. The plots on the left-hand side have $x_{max} = 100$, whereas the plots on the right hand side have $x_{max} = 1000$. There are, however, a little difference from the result in this section; the radius and the mass of the star are slightly larger than for the case with $x_{max} = 100$. This suggests that using a value of approximately $x_{max} = 200$, when doing the fitting of the constants $A_{\rm NR}$ and $A_{\rm R}$ in section 6.2, would be optimal.

As done in section 6.2 it can be interesting to solve the TOV-equations for a range of initial pressures $\bar{p}(0)$, and plot the radius R versus the mass $\overline{M}$ in a plot which now includes nucleon-nucleon interactions. This is done in Fig. 25

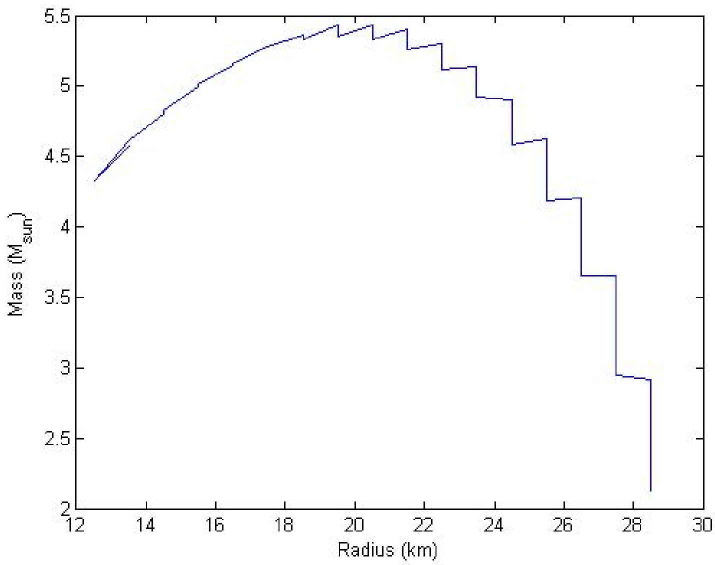


Figure 25: Mass (in $M_\odot$) as a function of radius (in km) for pure neutron stars ($\alpha = 1$) including nucleon-nucleon interactions. The stars to the right of the maximum are stable, whereas the ones to the left are unstable against gravitational collapse.

By comparing with the Fermi gas model predictions of R versus $\overline{M}$ in Fig. 14, we observe that the effect of the nuclear potential is enormous. The star with maximum mass is now $\overline{M} \approx 5.5M_\odot$, not $\overline{M} \approx 2.1M_\odot$. The corresponding radius of this star is $R \approx 21km$, which is a little smaller than the Fermi gas model radius of $R \approx 25km$. The maximum mass is so large because of the large value for the nuclear compressibility $K_0 = 363$MeV as a more incompressible matter can support more mass. To get a smaller value for the maximum mass, we would only have to fit to a smaller value of K_0.

8 Summary and Conclusion

The models used to find the analytical solutions are not very realistic as they use the assumption of constant density, i.e. the density does not depend on the pressure. To get a more realistic model, the polytropic form of pressure dependency was chosen, as it has a solid foundation in statistical mechanics.

The polytropic equation was used together with quantum mechanical theory (Fermi gas) to find an expression for the equation of state for the matter in compact stars. The resulting equation incorporates special relativity. This equation of state provided results for mass and radii of stars with different central pressures (i.e. central densities) by using the classical theory of hydrostatical equilibrium and the more accurate results from general relativity of the TOV-equations.

The effects of the GR-theory were significant, especially for neutron stars as their gravitational fields are much stronger than those of the white dwarfs.

To refine the model further, nucleon-nucleon interactions were taken into account. This was done by modeling a nuclear potential and obtaining a new and improved equation of state. The procedure of finding masses and radii for this more sophisticated model was similar to the procedure with the ordinary Fermi gas equation of state.

The effect of incorporating the nucleon-nucleon interactions into the theory was enormous. This was only done for neutron stars as they have nuclear densities, as opposed to the white dwarfs having much smaller densities. The densities of neutron stars cause the neutrons to interact more strongly than the electrons of the white dwarfs.

Lastly, the theories incorporated the existence of dark energy in the form of a cosmological constant depending on the density of vacuum (dark energy), defined as

$$\Lambda = \frac{8\pi\rho_{\text{vac}}}{3}.$$

This was done both analytically, assuming constant density, and numerically using the Fermi equation of state together with the TOV-equations. These calculations have not been carried out before, and the results indicate that the existence of a nonzero Λ will not affect the mass and radii of neutron stars.

References

[Egeland 2006] Egeland, E., "Stars and White Dwarfs", Trondheim, Norway, 2006.

[Hemmer 2000] Hemmer, P.C., "Kvantemekanikk", Tapir akademiske forlag, Trondheim, Norway, 2000.

[Hartle 2003] Hartle, James B., "Gravity - an Introduction to Einstein's General Relativity", Addison Wesley, San Fransisco, 2003.

[Krane 1988] Krane, K.S., "Introductory Nuclear Physics", John Wiley & Sons, United States of America, 1988.

[Padmanabhan 2001] Padmanabhan T., "Theoretical Astrophysics" *Volume II: Stars and Stellar Systems*, Cambridge University Press, Cambridge, UK, 2001.

[Schutz 1986] Schutz, B.F., "A first course in general relativity", Cambridge University Press, 1986.

[Shapiro and Teukolsky 1983] Shapiro, S.L. and Teukolsky S.A., "Black Holes, White Dwarfs and Neutron Stars", Wiley Interscience, 1983.

[Silbar and Reddy 2004] Silbar, R.R. and Reddy S., "Neutron stars for undergraduates", Am. J. Phys. 72 (7), 2004.

[Wesley 1997] Guth, Alan H., "The Inflationary Universe", Addison Wesley, New York, p. 22, 1997.

[1] http://antwrp.gsfc.nasa.gov/apod/ap971102.html

[2] http://antwrp.gsfc.nasa.gov/apod/ap050326.html

[3] http://antwrp.gsfc.nasa.gov/apod/ap070516.html

[4] http://map.gsfc.nasa.gov/m_ig/030644/030644.html

[5] http://en.wikipedia.org/wiki/Dark_energy

[6] http://astro.berkeley.edu/m̃white/darkmatter/dm.html

[7] http://wmap.gsfc.nasa.gov/m_uni/uni_101matter.html

[8] http://super.colorado.edu/m̃ichaele/Lambda/lambda.html

[9] http://en.wikipedia.org/wiki/Image:Semf_asymmetric_term.PNG

The picture at the front page was found at:
http://antwrp.gsfc.nasa.gov/apod/ap000711.html

A MATLAB-programmes

A.1 Programmes for function fitting

```
%*****************************************
% Uses fminsearch to least squares fit
% on the function in fitfunction.m
%*****************************************
function a = fit()
clear;close
%The input data to be fitted
x=splittingtablex(energyandpressure());
y=splittingtabley(energyandpressure());
%Plot of the fitting function
xmin=min(x);
xmax=max(x);
npts=1001;
dx=(xmax-xmin)/(npts-1);
xplot=xmin:dx:xmax;
%Set ifit to 0 and don't continue on to the fit until
%I set it to 1
ifit=0;
while ifit==0
    disp(' Enter an initial guess for the function ')
    a=input('parameters [a1,a2,...] in vector form [...]- ')
    %Plot the data and the initial function guess
    yplot=function(a,xplot);
    plot(x,y,'b*',xplot,yplot,'r-')
    xlabel('x')
    ylabel('y')
    ifit=input('Enter 0 to guess again, 1 to try to fit with this guess')
end
```

```
%*****************************************************************
% Do the fit with leastsq.m which uses fitfunction.m. fminsearch
% finds the minimum of the function specified, here leastsq.m.
%*****************************************************************
option=optimset('MaxFunEvals',1000000,'MaxIter',1000000);
a=fminsearch(@leastsq,a,option,x,y)
%Plot the data and the final function fit
yplot=fitfunction(a,xplot);
%Plot the final fit and the data
plot(x,y,'b*',xplot,yplot,'r-');
xlabel('x')
ylabel('y')
title('Final Fit and Data')
```

```
%**********************************************************
%This function evaluates the fitting function f(x,a),
%where a is a vector of fitting parameters a(1) and a(2).
%The function returns a vector f=f(x,a1,a2,...).
%**********************************************************
function f = fitfunction(a,x)
n=size(x);
%Fitting functions:
%White dwarfs
for i=1:n(2)
    f(i)=a(1)*(x(i)^(3/5)) + a(2)*(x(i)^(3/4));
end
%Neutron stars
%for i=1:n(2)
    %f(i)=a(1)*(x(i)^(3/5))+a(2)*x(i);
%end
%Neutron star with nonsymmetric nuclear matter
%for i=1:n(2)
    %f(i)=a(1)*x(i)^(a(2));
%end
```

```
%*********************************************************
% leastsq.m can be passed to fminsearch to do a
% non-linear least squares fit of fitfunction(a,x)
% to the data set (x,y).
% a is a vector of variable parameters; x and y
% are the arrays of data points
%*********************************************************
function s=leastsq(a,x,y)
% Find s, the sum of the squares of the differences
% between the fitting function and the data.
s=sum((y-fitfunction(a,x)).^2);
```

```
%**********************************************************
% This function calculates the data points for the
% equation of state, i.e. the energy density dependence
% of the pressure for Fermi gas. The enrgy density and
% pressure are both parametrized in k.
%**********************************************************
function svar = energyandpressure()
%White dwarf fitting constants
konst=5.477e19;
konst1=(1.440559478939809e23)/konst;
konst2=(8.829110575801710e25)/konst;
konst3=(6.002331162249205e21)/konst;
%Neutron star fitting constants
%konst=6.2603e32;
%konst1=2.058113267386856e35/konst;
%konst2=6.860377557956187e34/konst;
%The interval of k-values
k=[0: .0005 :.9995,1: .6 :600];
n=size(k);
%Neutron star (without electrons)
%for i=1:n(2)
    %svar(1,i)=(((2*(k(i)^3)-(3*k(i)))
    %           *sqrt(1+k(i)^2)+3*asinh(k(i)))/3);
    %svar(2,i)=((2*(k(i)^3)+k(i))
    %           *sqrt(1+k(i)^2)-asinh(k(i)));
%end
%White dwarf (with electrons, the third term in svar(2,i))
for i=1:n(2)
    svar(1,i)=konst3*((2*(k(i)^3)-(3*k(i)))
              *sqrt(1+k(i)^2)+3*asinh(k(i)));
    svar(2,i)=konst1*((2*(k(i)^3)+k(i))
              *sqrt(1+k(i)^2)-asinh(k(i)))+konst2*(k(i)^3);
end
```

```
%**********************************************************
% This function calculates the data points for the
% equation of state, i.e. the energy density dependence
% of the pressure for symmetric nuclear matter.
% The energy density and pressure are both parametrized
% in k.
%**********************************************************
function svar = energyandpressuresymmetric()
%**********************************************************
%Misc constants
A=-366.1; %MeV;
B=313.3; %MeV;
sigma=1.161;
n0=0.16; %nucleons/fm^3;
EF=22.1; %MeV;
konst=0.03377372788078;
mN=938; %MeV;
hc=197.3; %MeVfm
const=1.602176e11; %conversion factor to SI-units
%**********************************************************
k=[1.68: .001 :2];
n=size(k);
%Neutron star (without electrons)
for i=1:n(2)
    svar(1,i)=const*n0*hc*(((2*EF)/3)*((konst*mN^3*k(i)^3)
    /(hc^3*n0))^(5/3)+(A/2)*((konst*mN^3*k(i)^3)
    /(n0*hc^3))^2+((B*sigma)/(sigma+1))*((konst*mN^3*k(i)^3)
    /(n0*hc^3))^(sigma+1));
    svar(2,i)=(const/hc^2)*(konst*mN^3*k(i)^3)*
    (mN+((3*mN*k(i)^2)/(10))+((A*konst*mN^3*k(i)^3)
    /(2*n0*hc^3))+((B)/(sigma+1)*((konst*mN^3*k(i)^3)
    /(n0*hc^3))^sigma));
end
```

A.2 Programmes for numerical integration

```
%*****************************************************
% The routine for calculating the masses and radii
% for compact stars, i.e. a solver for two coupled
% partial derivative equations. It uses a 4th order
% Runge-Kutta routine to do this. Its input arguments
% are the central pressure: y, the constants alpha: a,
% beta: b and the number of iterations being performed.
%*****************************************************
function tabell = diffsolver(y,a,b,n)
% The initial values for the radius: x and the mass: z.
x=1e-10;
z=1e-5;
% Step length
h=.02;
% The 4th order Runge-Kutta routine
kk=0;
kl=0;
km=0;
kn=0;
lk=0;
ll=0;
lm=0;
ln=0;
```

```
    for i=1:n
    kk=h*F(x,y,z,a);
    lk=h*G(x,y,z,b);
    kl=h*F(x+h/2,y+kk/2,z+lk/2,a);
    ll=h*G(x+h/2,y+kk/2,z+lk/2,b);
    km=h*F(x+h/2,y+kl/2,z+ll/2,a);
    lm=h*G(x+h/2,y+kl/2,z+ll/2,b);
    kn=h*F(x+h,y+km,z+lm,a);
    ln=h*G(x+h,y+km,z+lm,b);
    y=y+(kk/6)+(kl/3)+(km/3)+(kn/6);
    z=z+(lk/6)+(ll/3)+(lm/3)+(ln/6);
    tabell(1,i)=x;
    tabell(2,i)=y;
    tabell(3,i)=z;
    x = x+h;
    end
%Plotting the function
d=size(tabell);
e=zeros(d(2)/100);
f=zeros(d(2)/100);
g=zeros(d(2)/100);
for i=1:d(2)/100
    e(i)=tabell(1,100*i);
    f(i)=tabell(2,100*i);
    g(i)=tabell(3,100*i);
end
subplot(2,1,1), plot(e,f);%, axis([0 15 0 1e-2]);
subplot(2,1,2), plot(e,g);%, axis([0 15 0 5]);
```

```
%Finding the zeros of the function
[dimj,dimi]=size(tabell);
posverdi1=0;
negverdi1=0;
posverdi2=0;
negverdi2=0;
svar=zeros(2);
i=1;
while tabell(2,i) > 0
    posverdi1 = tabell(1,i);
    posverdi2 = tabell(3,i);
    if i < (dimi-1)
        negverdi1 = tabell(1,i+1);
        negverdi2 = tabell(3,i+1);
    else
        disp('This function does not have a zero point.')
    end
    i = i + 1;
end
svar(1,1) = (posverdi1 + negverdi1)/2;
svar(1,2) = (posverdi2 + negverdi2)/2;
for n=1:dimi
    if tabell(2,n)==0
    svar(1,1) = tabell(1,n);
    svar(1,2) = tabell(3,n);
    end
end
format long
disp(svar)
```

```
%**********************************************************
% diffsolver.m uses this function to calculate the next
% step in the Runge-Kutta routine. The same code is used
% for many different cases by changing which lines in the
% code are commented out, and the values of the constants.
% This is the equivalent of dp/dr.
%**********************************************************
function f=F(x,y,z,u)
%The values for the fitting constants for white dwarfs
%a1=-3.45432844877789e5;
%a2=0.47013474067981e5;
%The fitting function for white dwarfs
%f=((-u)*z*(a1*(y^(3/5))+a2*(y^(3/4))))/(x^2);
%The fitting constants for neutron stars
%Fermi gas
%a1=1.145774;
%a2=2.999737;
%Nuclear matter
a1=0.1518e-4;
a2=2.430;
epsilon_0=3404.1;
%The fitting function for neutron stars
%f=((-u)*z*(a1*(y^(3/5))+a2*y))/(x^2);
%The nuclear fitting function for neutron stars
%f=((-u)*z*((y^(1/a2))/(a1^(1/a2))))/(x^2);
%The not fitting equation
%f=(((-u)*z*(y^(3/4)))/(x^2));
```

```
%GR-corrections:
%K=9.397694715284140e-26;
%d=7.0299e-47*(((1/K)*((1.473/u)^(5/3)))^(3/2));
%epsilon_0=6.26031e32;
%epsilon_0=(1/K*(1.473/u)^(gamma))^(1/(gamma-1));
d=epsilon_0*7.0299323e-47;
% Terms of the equation dp/dr
f1=1+((a1*(y^(2/5))));
f2=1+d*(((x^3)*y)/z);
f3=(1-((2.946*z)/x))^(-1);
%TOV-equation, not fitting equation
%f=(((-u)*z*(y^(3/5)))/(x^2))*f1*f2*f3;
%TOV-equation, fitting equation
%f=(((-u)*z*(a1*(y^(3/5))+a2*y))/(x^2))*f1*f2*f3;
%TOV-equation, nuclear fitting equation
f=(((-u)*z*((y^(1/a2)*epsilon_0^((1/a2)-1))
  /(a1^(1/a2))))/(x^2))*f1*f2*f3;
```

```
%**********************************************************
% diffsolver.m uses this function to calculate the next
% step in the Runge-Kutta routine. The same code is used
% for many different cases by changing which lines in the
% code are commented out. This is the equivalent of
% dm/dr.
%**********************************************************
function g=G(x,y,z,v)
%The fitting constants for white dwarfs
%a1=-3.45432844877789e5;
%a2=0.47013474067981e5;
%The fitting function for white dwarfs
%g=v*(x^2)*((a1*(y^(3/5))+a2*(y^(3/4))));
%The values for the fitting constants for neutron stars
%Fermi gas
%a1=1.145774;
%a2=2.999737;
%Nuclear matter
a1=0.1518e-4;
a2=2.430;
epsilon_0=3404.1;
%The fitting function for neutron stars
%g=v*(x^2)*((a1*(y^(3/5))+a2*y));
%The nuclear fitting equation for neutron stars
g=v*(x^2)*(((y^(1/a2)*epsilon_0^((1/a2)-1))/(a1^(1/a2))));
%The not fitting equation
%g=v*(x^2)*((y^(3/4)));
```

```
%*********************************************************
% The routine for calculating the masses and radii
% for compact stars, i.e. a solver for two coupled
% partial derivative equations. It uses a 4th order
% Runge-Kutta routine to do this. Its input arguments
% are the central pressure: y, the constants alpha: a,
% beta: b and the number of iterations being performed.
% This is similar to diffsolver, but includes a non-
% zero cosmological constant in the matlab function
% FF.m.
%*********************************************************
function tabell = diffsolver_with_lambda(y,a,b,n)
% The initial values for the radius: x and the mass: z.
x=1e-5;
z=1e-10;
% Step length
h=.0025;
% The 4th order Runge-Kutta routine
kk=0;
kl=0;
km=0;
kn=0;
lk=0;
ll=0;
lm=0;
ln=0;
```

```
    for i=1:n
    kk=h*FF(x,y,z,a);
    lk=h*G(x,y,z,b);
    kl=h*FF(x+h/2,y+kk/2,z+lk/2,a);
    ll=h*G(x+h/2,y+kk/2,z+lk/2,b);
    km=h*FF(x+h/2,y+kl/2,z+ll/2,a);
    lm=h*G(x+h/2,y+kl/2,z+ll/2,b);
    kn=h*FF(x+h,y+km,z+lm,a);
    ln=h*G(x+h,y+km,z+lm,b);
    y=y+(kk/6)+(kl/3)+(km/3)+(kn/6);
    z=z+(lk/6)+(ll/3)+(lm/3)+(ln/6);
    tabell(1,i)=x;
    tabell(2,i)=y;
    tabell(3,i)=z;
    x = x+h;
    end
%Plotting the function
d=size(tabell);
e=zeros(d(2)/100);
f=zeros(d(2)/100);
g=zeros(d(2)/100);
for i=1:d(2)/100
    e(i)=tabell(1,100*i);
    f(i)=tabell(2,100*i);
    g(i)=tabell(3,100*i);
end
subplot(2,1,1), plot(e,f);%, axis([0 30 0 1e-5]);
subplot(2,1,2), plot(e,g);%, axis([0 30 0 4]);
```

```
%Finding the zeros of the function
[dimj,dimi]=size(tabell);
posverdi1=0;
negverdi1=0;
posverdi2=0;
negverdi2=0;
svar=zeros(2);
i=1;
while tabell(2,i) > 0
    posverdi1 = tabell(1,i);
    posverdi2 = tabell(3,i);
    if i < (dimi-1)
        negverdi1 = tabell(1,i+1);
        negverdi2 = tabell(3,i+1);
    else
        disp('This function does not have a zero point.')
    end
    i = i + 1;
end
svar(1,1) = (posverdi1 + negverdi1)/2;
svar(1,2) = (posverdi2 + negverdi2)/2;
for n=1:dimi
    if tabell(2,n)==0
    svar(1,1) = tabell(1,n);
    svar(1,2) = tabell(3,n);
    end
end
format long
disp(svar)
```

```
%************************************************************
% diffsolver_with_lambda.m uses this function to
% calculate the next step in the Runge-Kutta routine.
% It contains a nonzero cosmological constants lambda.
% This is the equivalent of dp/dr.
%************************************************************
function f=FF(x,y,z,u)
% Constants
gamma=5/3;
K=9.397694715284140e-26;
lambda=1e17;
epsilon_0=6.2603e32;
d=epsilon_0*7.0299323e-47;
e=lambda/(2.946*8.988e16);
% Terms of the equation dp/dr
f1=1+((y^((gamma-1)/gamma))*(1.914^(1/gamma)));
f2=1+d*(((x^3)*y)/z)-e*(x^3/z);
f3=(1-((2.946*z)/x))^(-1);
% TOV-equation with cosmological constant lambda
% f=(((-u)*z*(y^(1/gamma)))/(x^2))*f1*f2*f3;
% The fitting constants for neutron stars
% Fermi gas
a1=1.145774;
a2=2.999737;
% TOV-equation, fitting equation with cosmological
% constant lambda
f=(((-u)*z*(a1*(y^(3/5))+a2*y))/(x^2))*f1*f2*f3;
```

```
%*********************************************************
% Makes a vector with masses and radii to be plotted.
%*********************************************************
function svar = snurreplotnuclear(p)
%Nuclear matter model
%a=0.95264796447526;
%b=0.02490205767726661;
%Fermi gas model
a=1.476;
b=0.03778;
n=size(p);
for i=1:n(2)
   svar(1,i)=splittingtablex(
              plotradiusmassnuclear(p(i),a,b,100));
   svar(2,i)=splittingtabley(
              plotradiusmassnuclear(p(i),a,b,100));
end
```

```
%**********************************************************
% The routine for calculating the masses and radii
% for compact stars, i.e. a solver for two coupled
% partial derivative equations. It uses a 4th order
% Runge-Kutta routine to do this. Its input arguments
% are the central pressure: y, the constants alpha: a,
% beta: b and the number of iterations being performed.
% The difference from diffsolver.m is only that this
% function does not plot the results, because they are
% used by snurreplotnuclear.m instead.
%**********************************************************
function svar = plotradiusmassnuclear(y,a,b,n)
x=1e-5;
z=1e-15;
h=1;
kk=0;
kl=0;
km=0;
kn=0;
lk=0;
ll=0;
lm=0;
ln=0;
    for i=1:n
    kk=h*F(x,y,z,a);
    lk=h*G(x,y,z,b);
    kl=h*F(x+h/2,y+kk/2,z+lk/2,a);
    ll=h*G(x+h/2,y+kk/2,z+lk/2,b);
    km=h*F(x+h/2,y+kl/2,z+ll/2,a);
    lm=h*G(x+h/2,y+kl/2,z+ll/2,b);
    kn=h*F(x+h,y+km,z+lm,a);
    ln=h*G(x+h,y+km,z+lm,b);
```

```
    y=y+(kk/6)+(kl/3)+(km/3)+(kn/6);
    z=z+(lk/6)+(ll/3)+(lm/3)+(ln/6);
    tabell(1,i)=x;
    tabell(2,i)=y;
    tabell(3,i)=z;
    x = x+h;
    end
%Finding the zeros of the function
[dimj,dimi]=size(tabell);
posverdi1=0;
negverdi1=0;
posverdi2=0;
negverdi2=0;
svar=[0:0];
i=1;
while tabell(2,i) > 0
    posverdi1 = tabell(1,i);
    posverdi2 = tabell(3,i);
    if i < (dimi-1)
        negverdi1 = tabell(1,i+1);
        negverdi2 = tabell(3,i+1);
    else
        disp('This function does not have a zero point.')
    end
    i = i + 1;
end
svar(1,1) = (posverdi1 + negverdi1)/2;
svar(2,1) = (posverdi2 + negverdi2)/2;
for n=1:dimi
    if tabell(2,n)==0
    svar(1,1) = tabell(1,n);
    svar(2,1) = tabell(3,n);
    end
end
format long
```

```matlab
%*************************************************
% Makes a vector with values for pressure used by
% plotradiusmassnuclear.m
%*************************************************
function p = pressurevector()
%Fermi gas pressure
p=[1e-4:5e-4:1e-2,1e-2:1e-3:5e-2,5e-2:1e-2:2];
%Nuclear matter pressure
%p=[1e-3:1e-4:1e-2,1e-2:1e-3:5e-2,5e-2:1e-2:.17];
```

A.3 Programmes for plotting and miscellaneous

```
%****************************************
% Extracts the first row of a matrix
% into a vector.
%****************************************
function svar = splittingtablex(tabell)
n=size(tabell);
for i=1:n(2)
    svar(i)=tabell(1,i);
end
```

```
%*****************************************
% Extracts the second row of a matrix
% into a vector.
%*****************************************
function svar = splittingtabley(tabell)
n=size(tabell);
for i=1:n(2)
    svar(i)=tabell(2,i);
end
```

```
%**************************************************
% Calculates the maximal mass as a function of k,
% the constant involving the cosmological constant.
% The input parameter is the density.
%**************************************************
function svar = calculatecosmologicalmaxmass(rho)
k = [0: 0.001 :1];
% Constants
Msun = 1.98892e30;
c = 2.9979e8;
G = 6.67e-11;
a = sqrt((3*c^6)/(8*pi*G^3*rho));
svar(1,:) = k;
n=size(k);
for i=1:n(2)
    svar(2,i)= (0.5*a*(1-((1/3*k(i))
    ^(4/(3-k(i)))))^(1.5))/Msun;
end
```

```
%**************************************************
% Calculates the pressure as a function of the
% dimensionless radius. The input parameter is k,
% the constant involving the cosmological constant.
%**************************************************
function svar = calculatecosmologicalpressure(k,q)
% Radius
z = [0: 100 :q];
% Constants
rho = 1e18;
c = 2.9979e8;
G = 6.67e-11;
a = sqrt((3*c^2)/(8*pi*G*rho));
R = 20000;
x = asin(z./(R*a));
chi = cos(asin(1/a));
svar(1,:) = z./R;
n=size(z);
for i=1:n(2)
    svar(2,i)=(((cos(x(i)))^((3-k)/(k))-k*k*chi^((3-k)
    /(k)))/(3*k*chi^((3-k)/(k))-(cos(x(i)))^((3-k)/(k))));
end
```

```
%**************************************************
% Plots the average energy per nucleon less its
% rest mass for symmetric nuclear matter.
%**************************************************
function svar = symmetricnuclearplot()
% Constants
EF=22.1;      %in MeV
A=-126.0631; %in MeV
B=73.2298;   %in MeV
sigma=1.9372;
% The particle density
u=[0:.001:2];
% Energy per particle less its rest mass
svar=EF*u.^(2/3)+(A.*u)/2+(B*(u.^(sigma)))/(sigma+1);
plot(u,svar);
```

```
%**************************************************
% Plots the pressure for symmetric nuclear matter.
%**************************************************
function svar = symmetricnuclearplotpress()
% Constants
EF=22.1;    %in MeV
n0=0.16;    %in /fm^(-3)
A=-118.2;   %in MeV
B=65.38;    %in MeV
sigma=2.112;
% The particle density
u=[0:.001:1.5];
% The pressure
svar=(2*n0*EF*u.^(5/3))/3+(n0*A*u.^2)/2+((B*sigma*n0)
      /(sigma+1))*u.^(sigma+1);
plot(u,svar), axis([0 1.5 -2 10]);
```

```
%*******************************************
% Solving the equations for the parameters
% A, B and sigma. The input is the value
% for K_0, the nuclear compressibility.
%*******************************************
function [ans] = parametereqn(k)
% Constants
BE=-16;  %in MeV
EF=22.1; %in MeV
% The parameterequations
ans(1)=(k+2*EF)/(3*EF-9*BE);
ans(2)=((ans(1)+1)/(ans(1)-1))*((EF/3)-BE);
ans(3)=BE-(5/3)*EF-ans(2);
```

```
%****************************************************
% Calculates the energy density and pressure for
% nonsymmetric nuclear matter. The input parameter
% n is gamma, the exponential of the chosen function
% F(u)=u^gamma.
%****************************************************
function svar = nuclearplot(n)
% Constants
EF=22.1;
A=-126.1;
B=73.23;
sigma=1.937;
alpha=1;
s0=30;
n0=0.16;
mN=938;
% The particle density
u=[0:.0005:5];
a=size(u);
for i=1:a(2)
    % Energy density
    svar(1,i)=n0*u(i)*(mN+EF*u(i)^(2/3)+(A*u(i))/2+(B*(u(i)^(sigma)))
          /(sigma+1)+(alpha^2)*(2^(2/3)-1)*EF*(u(i)^(2/3)-u(i)^n)
          + s0.*u(i)^n);
    % Pressure
    svar(2,i)=(2*n0*EF*u(i)^(5/3))/3+(n0*A*u(i)^2)/2+((B*sigma*n0)
          /(sigma+1))*u(i)^(sigma+1)+n0*alpha^2*(((2^(2/3)-1)/3)
          *EF*(2*u(i)^(5/3)-3*u(i)^2)+s0*u(i)^2);
end
```